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It was proved by the Merton school that the quantity of motion in such a case is equal to the quantity of a uniform motion at the speed achieved halfway through the accelerated motion; in modern formulation, $s = at^2/2$ (Merton rule). Discussions like this certainly influenced Galileo indirectly and may have influenced the founding of coordinate geometry in the 17th century. Another important development in the scholastic "calculations" was the summation of infinite series.

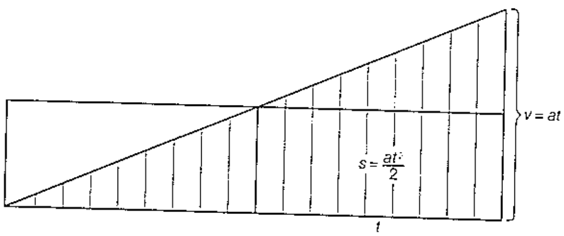

Figure 5: Uniformly accelerated motion; s = speed, a = acceleration, t = time, and v = velocity.

Basing his work on translated Greek sources, about 1464 the German mathematician and astronomer Regiomontanus wrote the first book (printed in 1533) in the West on plane and spherical trigonometry independent of astronomy. He also published tables of sines and tangents that were in constant use for more than two centuries.

The Renaissance. Italian artists and merchants influenced the mathematics of the late Middle Ages and the Renaissance in several ways. In the 15th century a group of Tuscan artists, including Filippo Brunelleschi, Leon Battista Alberti, and Leonardo da Vinci, incorporated linear perspective into their practice and teaching, about a century before the subject was formally treated by mathematicians. Italian *maestri d'abbaco* tried, albeit unsuccessfully, to solve nontrivial cubic equations. In fact, the first general solution was found by Scipione del Ferro at the beginning of the 16th century and rediscovered by Niccolò Tartaglia several years later. The solution was published by Gerolamo Cardano in his *Ars magna* (*Ars Magna or the Rules of Algebra*) in 1545, together with Lodovico Ferrari's solution of the quartic equation.

By 1380 an algebraic symbolism had been developed in Italy in which letters were used for the unknown, for its square, and for constants. The symbols used today for the unknown (for example, x), the square root sign, and the signs + and − came into general use in southern Germany beginning about 1450. They were used by Regiomontanus and by Fridericus Gerhart and received an impetus about 1486 at the University of Leipzig from Johann Widman. The idea of distinguishing between known and unknown quantities in algebra was first consistently applied by François Viète, with vowels for unknown and consonants for known quantities. Viète found some relations between the coefficients of an equation and its roots. This was suggestive of the idea, explicitly stated by Albert Girard in 1629 and proved by Carl Friedrich Gauss in 1799, that an equation of degree n has n roots. Complex numbers, which are implicit in such ideas, were gradually accepted about the time of Rafael Bombelli (died 1572), who used them in connection with the cubic.

Apollonius' *Conics* and the investigations of areas (quadratures) and of volumes (cubatures) by Archimedes formed part of the humanistic learning of the 16th century. These studies strongly influenced the later developments of analytic geometry, the infinitesimal calculus, and the theory of functions, subjects that were developed in the 17th century.

(Mc.F.)

Mathematics in the 17th and 18th centuries

THE 17TH CENTURY

The 17th century, the period of the scientific revolution, witnessed the consolidation of Copernican heliocentric astronomy and the establishment of inertial physics in the work of Johannes Kepler, Galileo, René Descartes, and Isaac Newton. This period was also one of intense activity and innovation in mathematics. Advances in numerical calculation, the development of symbolic algebra and analytic geometry, and the invention of the differential and integral calculus resulted in a major expansion of the subject areas of mathematics. By the end of the 17th century a program of research based in analysis had replaced classical Greek geometry at the centre of advanced mathematics. In the next century this program would continue to develop in close association with physics, more particularly mechanics and theoretical astronomy. The extensive use of analytic methods, the incorporation of applied subjects, and the adoption of a pragmatic attitude to questions of logical rigour distinguished the new mathematics from traditional geometry.

Development of analysis

Institutional background. Until the middle of the 17th century, mathematicians worked alone or in small groups, publishing their work in books or communicating with other researchers by letter. At a time when people were often slow to publish, "invisible colleges," networks of scientists who corresponded privately, played an important role in coordinating and stimulating mathematical research. Marin Mersenne in Paris acted as a clearinghouse for new results, informing his many correspondents—including Pierre de Fermat, Descartes, Blaise Pascal, Gilles Personne de Roberval, and Galileo—of challenge problems and novel solutions. Later in the century John Collins, librarian of London's Royal Society, performed a similar function among British mathematicians.

In 1660 the Royal Society of London was founded, to be followed in 1666 by the French Academy of Sciences, in 1700 by the Berlin Academy, and in 1724 by the St. Petersburg Academy. The official publications sponsored by the academies, as well as independent journals such as the *Acta Eruditorum* (founded in 1682), made possible the open and prompt communication of research findings. Although universities in the 17th century provided some support for mathematics, they became increasingly ineffective as state-supported academies assumed direction of advanced research.

Numerical calculation. The development of new methods of numerical calculation was a response to the increased practical demands of numerical computation, particularly in trigonometry, navigation, and astronomy. New ideas spread quickly across Europe and resulted by 1630 in a major revolution in numerical practice.

Simon Stevin of Holland, in his short pamphlet *La Disme* (1585), introduced decimal fractions to Europe and showed how to extend the principles of Hindu-Arabic arithmetic to calculation with these numbers. Stevin emphasized the utility of decimal arithmetic "for all accounts that are encountered in the affairs of men," and he explained in an appendix how it could be applied to surveying, stereometry, astronomy, and mensuration. His idea was to extend the base-10 positional principle to numbers with fractional parts, with a corresponding extension of notation to cover these cases. In his system the number 237.578 was denoted

237 ⓪ 5 ① 7 ② 8 ③,

in which the digits to the left of the zero are the integral part of the number. To the right of the zero are the digits of the fractional part, with each digit succeeded by a circled number that indicates the negative power to which 10 is raised. Stevin showed how the usual arithmetic of whole numbers could be extended to decimal fractions using rules that determined the positioning of the negative powers of 10.

In addition to its practical utility, *La Disme* was significant for the way it undermined the dominant style of classical Greek geometry in theoretical mathematics. Stevin's proposal required a rejection of the distinction in Euclidean geometry between magnitude, which is continuous, and number, which is a multitude of indivisible units. For Euclid, unity, or one, was a special sort of thing, not number but the origin, or principle, of number. The introduction of decimal fractions seemed to imply that the unit could be subdivided and that arbitrary continuous magnitude could be represented numerically; it implicitly supposed the concept of a general positive real number.

Logarithms

Tables of logarithms were first published in 1614 by the Scottish laird John Napier in his treatise *Description of the Marvelous Canon of Logarithms*. This work was followed (posthumously) five years later by another in which Napier set forth the principles used in the construction of his tables. The basic idea behind logarithms is that addition and subtraction are easier to perform than multiplication and division, which, as Napier observed, require a "tedious expenditure of time" and are subject to "slippery errors." By the law of exponents, $a^n a^m = a^{n+m}$; that is, in the multiplication of numbers, the exponents are related additively. By correlating the geometric sequence of numbers a, a^2, a^3, . . . (a is called the base) and the arithmetic sequence 1, 2, 3, . . . and interpolating to fractional values, it is possible to reduce the problem of multiplication and division to one of addition and subtraction. To do this Napier chose a base that was very close to 1, differing from it by only $1/10^7$. The resulting geometric sequence therefore yielded a dense set of values, suitable for constructing a table.

In his work of 1619 Napier presented an interesting kinematic model to generate the geometric and arithmetic sequences used in the construction of his tables. Assume two particles move along separate lines from given initial points. The particles begin moving at the same instant with the same velocity. The first particle continues to move with a speed that is decreasing, proportional at each instant to the distance remaining between it and some given fixed point on the line. The second particle moves with a constant speed equal to its initial velocity. Given any increment of time, the distances traveled by the first particle in successive increments form a geometrically decreasing sequence. The corresponding distances traveled by the second particle form an arithmetically increasing sequence. Napier was able to use this model to derive theorems yielding precise limits to approximate values in the two sequences.

Napier's kinematic model indicated how skilled mathematicians had become by the early 17th century in analyzing nonuniform motion. Kinematic ideas, which appeared frequently in mathematics of the period, provided a clear and visualizable means for the generation of geometric magnitude. The conception of a curve traced by a particle moving through space later played a significant role in the development of the calculus.

Napier's ideas were taken up and revised by the English mathematician Henry Briggs, the first Savilian Professor of Geometry at Oxford. In 1624 Briggs published an extensive table of common logarithms, or logarithms to the base 10. Because the base was no longer close to 1, the table could not be obtained as simply as Napier's, and Briggs therefore devised techniques involving the calculus of finite differences to facilitate calculation of the entries. He also devised interpolation procedures of great computational efficiency to obtain intermediate values.

In Switzerland the instrument maker Joost Bürgi arrived at the idea for logarithms independently of Napier, although he did not publish his results until 1620. Four years later a table of logarithms prepared by Kepler appeared in Marburg. Both Bürgi and Kepler were astronomical observers, and Kepler included logarithmic tables in his famous *Tabulae Rudolphinae* (1627; "Rudolphine Tables"), astronomical tabulations of planetary motion derived using the assumption of elliptical orbits about the Sun.

Analytic geometry. The invention of analytic geometry was, next to the differential and integral calculus, the most important mathematical development of the 17th century. Originating in the work of the French mathematicians Viète, Fermat, and Descartes, it had by the middle of the century established itself as a major program of mathematical research.

Two tendencies in contemporary mathematics stimulated the rise of analytic geometry. The first was an increased interest in curves, resulting in part from the recovery and

Influence of Greek geometers

Latin translation of the classical treatises of Apollonius, Archimedes, and Pappus, and in part from the increasing importance of curves in such applied fields as astronomy, mechanics, optics, and stereometry. The second was the emergence a century earlier of an established algebraic practice in the work of the Italian and German algebraists and its subsequent shaping by Viète into a powerful mathematical tool at the end of the century.

Viète was a prominent representative of the humanist movement in mathematics that set itself the project of restoring and furthering the achievements of the Classical Greek geometers. In his *In artem analyticem isagoge* (1591; "Introduction to the Analytic Arts") Viète, as part of his program of rediscovering the method of analysis used by the ancient Greek mathematicians, proposed new algebraic methods that employed variables, constants, and equations, but he saw this as an advancement over the ancient method, a view he arrived at by comparing the geometric analysis contained in Book VII of Pappus' *Collection* with the arithmetic analysis of Diophantus' *Arithmetica*. Pappus had employed an analytic method for the discovery of theorems and the construction of problems; in analysis, by contrast to synthesis, one proceeds from what is sought until one arrives at something known. In approaching an arithmetic problem by laying down an equation among known and unknown magnitudes and then solving for the unknown, one was, Viète reasoned, following an "analytic" procedure.

Viète's notation

Viète introduced the concept of algebraic variable, which he denoted using a capital vowel (A, E, I, O, U), as well as the concept of parameter (an unspecified constant quantity), denoted by a capital consonant (B, C, D, and so on). In his system the equation $5BA^2 - 2CA + A^3 = D$ would appear as $B5$ in A quad $- C$ plano 2 in $A + A$ cub aequatur D solido.

Viète retained the classical principle of homogeneity, according to which terms added together must all be of the same dimension. In the above equation, for example, each of the terms has the dimension of a solid or cube; thus the constant C, which denotes a plane, is combined with A to form a quantity having the dimension of a solid.

It should be noted that in Viète's scheme the symbol A is part of the expression for the object obtained by operating on the magnitude denoted by A. Thus operations on the quantities denoted by the variables are reflected in the algebraic notation itself. This innovation, considered by historians of mathematics to be a major conceptual advance in algebra, facilitated the study of the symbolic solution of algebraic equations and led to the creation of the first conscious theory of equations.

After Viète's death the analytic art was applied to the study of curves by his countrymen Fermat and Descartes. Both men were motivated by the same goal, to apply the new algebraic techniques to Apollonius' theory of loci as preserved in Pappus' *Collection*. The most celebrated of these problems consisted of finding the curve or locus traced by a point whose distances from several fixed lines satisfied a given relation.

Fermat adopted Viète's notation in his paper "Ad Locos Planos et Solidos Isagoge" (1636; "Introduction to Plane and Solid Loci"). The title of the paper refers to the ancient classification of curves as plane (straight lines, circles), solid (ellipses, parabolas, and hyperbolas), or linear (curves defined kinematically or by a locus condition). Fermat considered an equation among two variables. One of the variables represented a line measured horizontally from a given initial point, while the other represented a second line positioned at the end of the first line and inclined at a fixed angle to the horizontal. As the first variable varied in magnitude, the second took on a value determined by the equation, and the endpoint of the second line traced out a curve in space. By means of this construction Fermat was able to formulate the fundamental principle of analytic geometry:

> Whenever two unknown quantities are found in final equality, there results a locus fixed in place, and the endpoint of one of these unknown quantities describes a straight line or a curve.

The principle implied a correspondence between two different classes of mathematical objects: geometric curves and algebraic equations. In the paper of 1636 Fermat showed that, if the equation is a quadratic, then the curve is a conic section—that is, an ellipse, parabola, or hyperbola. He also showed that the determination of the curve given by an equation is simplified by a transformation involving a change of variables to an equation in standard form.

Descartes's *La Géométrie* appeared in 1637 as an appendix to his famous *Discourse on Method*, the treatise that presented the foundation of his philosophical system. Although supposedly an example from mathematics of his rational method, *La Géométrie* was a technical treatise understandable independently of philosophy. It was destined to become one of the most influential books in the history of mathematics.

In the opening sections of *La Géométrie* Descartes introduced two innovations. In place of Viète's notation he initiated the modern practice of denoting variables by letters at the end of the alphabet (x, y, z) and parameters by letters at the beginning of the alphabet (a, b, c) and of using exponential notation to indicate powers of x (x^2, x^3, . . .). More significant conceptually, he set aside Viète's principle of homogeneity, showing by means of a simple construction how to represent multiplication and division of lines by lines; thus all magnitudes (lines, areas, and volumes) could be represented independently of their dimension in the same way.

Descartes's goal in *La Géométrie* was to achieve the construction of solutions to geometric problems by means of instruments that were acceptable generalizations of ruler and compass. Algebra was a tool to be used in this program:

> If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem necessary for its construction—to those that are unknown as well as to those that are known. Then, making no distinction in any way between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.

In the problem of Apollonius, for example, one sought to find the locus of points whose distances from a collection of fixed lines satisfied a given relation. One used this relation to derive an equation, and then, using a geometric procedure involving acceptable instruments of construction, one obtained points on the curve given by the roots of the equation.

Descartes described instruments more general than the compass for drawing "geometric" curves. He stipulated that the parts of the instrument be linked together so that the ratio of the motions of the parts could be knowable. This restriction excluded "mechanical" curves generated by kinematic processes. The Archimedean spiral, for example, was generated by a point moving on a line as the line rotated uniformly about the origin. The ratio of the circumference to the diameter did not permit exact determination:

> the ratios between straight and curved lines are not known, and I even believe cannot be discovered by men, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact.

Descartes concluded that a geometric or nonmechanical curve was one whose equation $f(x, y) = 0$ was a polynomial of finite degree in two variables. He wished to restrict mathematics to the consideration of such curves.

Descartes's emphasis on construction reflected his classical orientation. His conservatism with respect to what curves were acceptable in mathematics further distinguished him as a traditional thinker. At the time of his death, in 1650, he had been overtaken by events, as research moved away from questions of construction to problems of finding areas (then called problems of quadrature) and tangents. The geometric objects that were then of growing interest were precisely the mechanical curves that Descartes had wished to banish from mathematics.

Following the important results achieved in the 16th century by Gerolamo Cardano and the Italian algebraists, the theory of algebraic equations reached an impasse. The ideas needed to investigate equations of degree higher than four were slow to develop. The immediate historical influence of Viète, Fermat, and Descartes was to furnish algebraic methods for the investigation of curves. A vigorous school of research became established in Leiden around Frans van Schooten, a Dutch mathematician who edited and published in 1649 a Latin translation of *La Géométrie*. Van Schooten published a second two-volume translation of the same work in 1659–1661 that also contained mathematical appendices by three of his disciples, Johan de Witt, Johan Hudde, and Hendrick van Heuraet. The Leiden group of mathematicians, which also included Christiaan Huygens, was in large part responsible for the rapid development of Cartesian geometry in the middle of the century.

The calculus. The historian Carl Boyer called the calculus "the most effective instrument for scientific investigation that mathematics has ever produced." As the mathematics of variability and change, the calculus was the characteristic product of the scientific revolution. The subject was properly the invention of two mathematicians, the German Gottfried Wilhelm Leibniz and the Englishman Isaac Newton. Both men published their researches in the 1680s, Leibniz in 1684 in the recently founded journal *Acta Eruditorum* and Newton in 1687 in his great treatise, the *Principia*. Although a bitter dispute over priority developed later among followers of the two men, it is now clear that they each arrived at the calculus independently.

The calculus developed from techniques to solve two types of problems, the determination of areas and volumes and the calculation of tangents to curves. In classical geometry Archimedes had advanced furthest in this part of mathematics, having used the method of exhaustion to establish rigorously various results on areas and volumes and having derived for some curves (*e.g.*, the spiral) significant results concerning tangents. In the early 17th century there was a sharp revival of interest in both classes of problems.

The precalculus period. In his treatise *Geometria Indivisibilibus Continuorum* (1635; "Geometry of Continuous Indivisibles") Bonaventura Cavalieri, a professor of mathematics at the University of Bologna, formulated a systematic method for the determination of areas and volumes. As had Archimedes, Cavalieri regarded a plane figure as being composed of a collection of indivisible lines, "all the lines" of the plane figure. The collection was generated by a fixed line moving through space parallel to itself. Cavalieri showed that these collections could be interpreted as magnitudes obeying the rules of Euclidean ratio theory. In proposition 4 of Book II, he derived the result that is written today as Cavalieri

$$\int_0^1 x^2 dx = \frac{1}{3}:$$

> Let there be given a parallelogram in which a diagonal is drawn; then "all the squares" of the parallelogram will be triple "all the squares" of each of the triangles determined by the diagonal.

Cavalieri showed that this proposition could be interpreted in different ways—as asserting, for example, that the volume of a cone is one-third the volume of the circumscribed cylinder or that the area under a segment of a parabola is one-third the area of the associated rectangle. In a later treatise he generalized the result by proving

$$\int_0^1 x^n dx = \frac{1}{(n+1)}$$

for $n = 3$ to $n = 9$. To establish these results, he introduced transformations among the variables of the problem, using a result equivalent to the binomial theorem for integral exponents. The ideas involved went beyond anything that had appeared in the classical Archimedean theory of content.

Although Cavalieri was successful in formulating a systematic method based on general concepts, his ideas were not easy to apply. The derivation of very simple results required intricate geometric considerations, and the turgid style of the *Geometria Indivisibilibus* was a barrier to its reception.

John Wallis presented a quite different approach to the theory of quadratures in his *Arithmetica Infinitorum* (1655; *The Arithmetic of Infinitesimals*). Wallis, a successor to Henry Briggs as the Savilian Professor of Geometry at Oxford, was a champion of the new methods of arithmetic algebra that he had learned from his teacher William Oughtred. Wallis expressed the area under a curve as the sum of an infinite series and used clever and unrigorous inductions to determine its value. To calculate the area under the parabola,

$$\int_0^1 x^2dx,$$

he considered the successive sums

$$\frac{0+1}{1+1}=\frac{1}{3}+\frac{1}{6}, \frac{0+1+4}{4+4+4}=\frac{1}{3}+\frac{1}{12}, \frac{0+1+4+9}{9+9+9+9}=\frac{1}{3}+\frac{1}{18}$$

and inferred by "induction" the general relation

$$\frac{0^2+1^2+2^2\ldots+n^2}{n^2+n^2+n^2\ldots+n^2}=\frac{1}{3}+\frac{1}{6n}.$$

By letting the number of terms be infinite, he obtained ⅓ as the limiting value of the expression. With more complicated curves he achieved very impressive results, including the infinite expression now known as Wallis' product:

$$\frac{4}{\pi}=\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots.$$

Research on the determination of tangents, the other subject leading to the calculus, proceeded along different lines. In *La Géométrie* Descartes had presented a method that could in principle be applied to any algebraic or "geometric" curve—*i.e.*, any curve whose equation was a polynomial of finite degree in two variables. The method depended upon finding the normal, the line perpendicular to the tangent, using the algebraic condition that it be the unique radius to intersect the curve in only one point. Descartes's method was simplified by Hudde, a member of the Leiden group of mathematicians, and was published in 1659 in van Schooten's edition of *La Géométrie*.

Non-algebraic curves

A class of curves of growing interest in the 17th century were those generated kinematically by a point moving through space. The famous cycloidal curve, for example, was traced by a point on the perimeter of a wheel that rolled on a line without slipping or sliding. These curves were nonalgebraic and hence could not be treated by Descartes's method. Gilles Personne de Roberval, professor at the Collège Royale in Paris, devised a method borrowed from dynamics to determine their tangents. In his analysis of projectile motion Galileo had shown that the instantaneous velocity of a particle is compounded of two separate motions: a constant horizontal motion and an increasing vertical motion due to gravity. If the motion of the generating point of a kinematic curve is likewise regarded as the sum of two velocities, then the tangent will lie in the direction of their sum. Roberval applied this idea to several different kinematic curves, obtaining results that were often ingenious and elegant.

In an essay of 1636 circulated among French mathematicians, Fermat presented a method of tangents adapted from a procedure he had devised to determine maxima and minima and used it to find tangents to several algebraic curves of the form $y=x^n$. His account was short and contained no explanation of the mathematical basis of the new method. It is possible to see in his procedure an argument involving infinitesimals, and Fermat has sometimes been proclaimed the discoverer of the differential calculus. Modern historical study, however, suggests that he was working with concepts introduced by Viète and that his method was based on finite algebraic ideas.

Isaac Barrow, the Lucasian Professor of Mathematics at the University of Cambridge, published in 1670 his *Geometrical Lectures*, a treatise that more than any other anticipated the unifying ideas of the calculus. In it he adopted a purely geometric form of exposition to show how the determinations of areas and tangents are inverse problems. He began with a curve and considered the slope of its tangent corresponding to each value of the abscissa. He then defined an auxiliary curve by the condition that its ordinate be equal to this slope and showed that the area under the auxiliary curve corresponding to a given abscissa is equal to the rectangle whose sides are unity and the ordinate of the original curve. When reformulated analytically, this result expresses the inverse character of differentiation and integration, the fundamental theorem of the calculus. Although Barrow's decision to proceed geometrically prevented him from taking the final step to a true calculus, his lectures influenced both Newton and Leibniz.

Newton and Leibniz. The essential insight of Newton and Leibniz was to use Cartesian algebra to synthesize the earlier results and to develop algorithms that could be applied uniformly to a wide class of problems. The formative period of Newton's researches was from 1665 to 1670, while Leibniz worked a few years later, in the 1670s. Their contributions differ in origin, development, and influence, and it is necessary to consider each man separately.

The son of an English farmer, Newton became in 1669 the Lucasian Professor of Mathematics at the University of Cambridge. Newton's earliest researches in mathematics grew in 1665 from his study of van Schooten's edition of *La Géométrie* and Wallis' *Arithmetica Infinitorum*. Using the Cartesian equation of the curve, he reformulated Wallis' results, introducing for this purpose infinite sums in the powers of an unknown x, now known as infinite series. Possibly under the influence of Barrow, he used infinitesimals to establish for various curves the inverse relationship of tangents and areas. The operations of differentiation and integration emerged in his work as analytic processes that could be applied generally to investigate curves.

Fluents and fluxions

Unusually sensitive to questions of rigour, Newton at a fairly early stage tried to establish his new method on a sound foundation using ideas from kinematics. A variable was regarded as a "fluent," a magnitude that flows with time; its derivative or rate of change with respect to time was called a "fluxion," denoted by the given variable with a dot above it. The basic problem of the calculus was to investigate relations among fluents and their fluxions. Newton finished a treatise on the method of fluxions as early as 1671, although it was not published until 1736. In the 18th century this method became the preferred approach to the calculus among British mathematicians, especially after the appearance in 1742 of Colin Maclaurin's influential *Treatise of Fluxions*.

Newton first published the calculus in Book I of his great *Philosophiae Naturalis Principia Mathematica* (1687; *Mathematical Principles of Natural Philosophy*). Originating as a treatise on the dynamics of particles, the *Principia* presented an inertial physics that combined Galileo's mechanics and Kepler's planetary astronomy. It was written in the early 1680s at a time when Newton was reacting against Descartes's science and mathematics. Setting aside the analytic method of fluxions, Newton introduced in 11 introductory lemmas his calculus of first and last ratios, a geometric theory of limits that provided the mathematical basis of his dynamics.

Newton's use of the calculus in the *Principia* is illustrated by proposition 11 of Book I: if the orbit of a particle moving under a centripetal force is an ellipse with the centre of force at one focus, then the force is inversely proportional to the square of the distance from the centre. Because the planets were known by Kepler's laws to move in ellipses with the Sun at one focus, this result supported his inverse square law of gravitation. To establish the proposition, Newton derived an approximate measure for the force by using small lines defined in terms of the radius (the line from the force centre to the particle) and the tangent to the curve at a point. This result expressed geometrically the proportionality of force to vector acceleration. Using properties of the ellipse known from classical geometry, Newton calculated the limit of this measure and showed that it was equal to a constant times 1 over the square of the radius.

Newton avoided analytical processes in the *Principia* by expressing magnitudes and ratios directly in terms of geometric quantities, both finite and infinitesimal. His decision to eschew analysis constituted a striking rejection of the algebraic methods that had been important in his own early researches on the calculus. Although the *Principia* was of inestimable value for later mechanics, it would be reworked by researchers on the Continent and expressed in the mathematical idiom of the Leibnizian calculus.

Leibniz's interest in mathematics was aroused in 1672 during a visit to Paris, where the Dutch mathematician Christiaan Huygens introduced him to his work on the theory of curves. Under Huygens' tutelage Leibniz immersed himself for the next several years in the study of mathematics. He investigated relationships among the summing and differencing of finite and infinite sequences of numbers. Having read Barrow's geometric lectures, he devised a transformation rule to calculate quadratures, obtaining the famous infinite series for $\pi/4$:

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots.$$

Notation

Leibniz was interested in questions of logic and notation, of how to construct a *characteristica universalis* for rational investigation. After considerable experimentation he arrived by the late 1670s at an algorithm based on the symbols d and $\int$. He first published his research on differential calculus in 1684 in an article in the *Acta Eruditorum*, "Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, qua nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro illi Calculi Genus" ("A New Method for Maxima and Minima as Well as Tangents, Which Is Impeded Neither by Fractional nor by Irrational Quantities, and a Remarkable Type of Calculus for This"). In this article he introduced the differential dx satisfying the rules $d(x + y) = dx + dy$ and $d(xy) = xdy + ydx$ and illustrated his calculus with a few examples. Two years later he published a second article, "On a Deeply Hidden Geometry," in which he introduced and explained the symbol $\int$ for integration. He stressed the power of his calculus to investigate transcendental curves, the very class of "mechanical" objects Descartes had believed lay beyond the power of analysis, and derived a simple analytical formula for the cycloid.

Leibniz continued to publish results on the new calculus in the *Acta Eruditorum* and began to explore his ideas in extensive correspondence with other scholars. Within a few years he had attracted a group of researchers to promulgate his methods, including the brothers Johann Bernoulli and Jakob Bernoulli in Basel and the priest Pierre Varignon and Guillaume-François-Antoine de L'Hospital in Paris. In 1700 he convinced Frederick William I of Prussia to establish the Brandenburg Society of Sciences (later renamed the Berlin Academy of Sciences), with himself appointed president for life.

Leibniz's vigorous espousal of the new calculus, the didactic spirit of his writings, and his ability to attract a community of researchers contributed to his enormous influence on subsequent mathematics. In contrast, Newton's slowness to publish and his personal reticence resulted in a reduced presence within European mathematics. Although the British school in the 18th century included capable researchers, Abraham de Moivre, James Stirling, Brook Taylor, and Maclaurin among them, they failed to establish a program of research comparable to that established by Leibniz's followers on the Continent. There is a certain tragedy in Newton's isolation and his reluctance to acknowledge the superiority of continental analysis. As the historian Michael Mahoney observed:

> Whatever the revolutionary influence of the *Principia*, mathematics would have looked much the same if Newton had never existed. In that endeavour he belonged to a community, and he was far from indispensable to it.

THE 18TH CENTURY

Institutional background. After 1700 a movement to found learned societies on the model of Paris and London spread throughout Europe and the American colonies. The academy was the predominant institution of science until it was displaced by the university in the 19th century. The leading mathematicians of the period, such as Leonhard Euler, Jean Le Rond d'Alembert, and Joseph-Louis Lagrange, pursued academic careers at St. Petersburg, Paris, and London.

French Academy of Sciences

The French Academy of Sciences (Paris) provides an informative study of the 18th-century learned society. The academy was divided into six sections, three for the mathematical and three for the physical sciences. The mathematical sections were for geometry, astronomy, and mechanics, the physical sections for chemistry, anatomy, and botany. Membership in the academy was divided by section, with each section contributing three *pensionnaires*, two associates, and two adjuncts. There was also a group of free associates, distinguished men of science from the provinces, and foreign associates, eminent international figures in the field. A larger group of 70 corresponding members had partial privileges, including the right to communicate reports to the academy. The administrative core consisted of a permanent secretary, treasurer, president, and vice president. In a given year the average total membership in the academy was 153.

Prominent characteristics of the academy included its small and elite membership, made up heavily of men from the middle class, and its emphasis on the mathematical sciences. In addition to holding regular meetings and publishing memoirs, the academy organized scientific expeditions and administered prize competitions on important mathematical and scientific questions.

The historian Roger Hahn noted that the academy in the 18th century allowed "the coupling of relative doctrinal freedom on scientific questions with rigorous evaluations by peers," an important characteristic of modern professional science. Academic mathematics and science did, however, foster a stronger individualistic ethos than is usual today. A determined individual such as Euler or Lagrange could emphasize a given program of research through his own work, the publications of the academy, and the setting of the prize competitions. The academy as an institution may have been more conducive to the solitary patterns of research in a theoretical subject like mathematics than it was to the experimental sciences. The separation of research from teaching is perhaps the most striking characteristic that distinguished the academy from the model of university-based science that developed in the 19th century.

Analysis and mechanics. The scientific revolution had bequeathed to mathematics a major program of research in analysis and mechanics. The period from 1700 to 1800, "the century of analysis," witnessed the consolidation of the calculus and its extensive application to mechanics. With expansion came specialization, as different parts of the subject acquired their own identity: ordinary and partial differential equations, calculus of variations, infinite series, and differential geometry. The applications of analysis were also varied, including the theory of the vibrating string, particle dynamics, the theory of rigid bodies, the mechanics of flexible and elastic media, and the theory of compressible and incompressible fluids. Analysis and mechanics developed in close association, with problems in one giving rise to concepts and techniques in the other, and all the leading mathematicians of the period made important contributions to mechanics.

Applications of analysis

The close relationship between mathematics and mechanics in the 18th century had roots extending deep into Enlightenment thought. In the organizational chart of knowledge at the beginning of the preliminary discourse to the *Encyclopédie*, Jean Le Rond d'Alembert distinguished between "pure" mathematics (geometry, arithmetic, algebra, calculus) and "mixed" mathematics (mechanics, geometric astronomy, optics, art of conjecturing). Mathematics generally was classified as a "science of nature" and separated from logic, a "science of man." The modern disciplinary division between physics and mathematics and the association of the latter with logic had not yet developed.

Mathematical mechanics itself as it was practiced in the 18th century differed in important respects from later physics. The goal of modern physics is to explore the ultimate particulate structure of matter and to arrive at fundamental laws of nature to explain physical phenomena. The character of applied investigation in the 18th century was rather different. The material parts of a given system and their interrelationship were idealized for the purposes of analysis. A material object could be treated as a point-mass, as a rigid body, as a continuously deformable medium, and so on. The intent was to obtain a mathematical description of the macroscopic behaviour of the system rather than to ascertain the ultimate physical basis of the phenomena. In this respect the 18th-century viewpoint is closer to modern mathematical engineering than it is to physics.

Mathematical research in the 18th century was coordinated by the Paris, Berlin, and St. Petersburg academies, as well as by several smaller provincial scientific academies and societies. Although England and Scotland were important centres early in the century, with Maclaurin's death in 1746 the British flame was all but extinguished.

History of analysis. The history of analysis in the 18th century can be followed in the official memoirs of the academies and in independently published expository treatises. In the first decades of the century the calculus was cultivated in an atmosphere of intellectual excitement, as mathematicians applied the new methods to a range of problems in the geometry of curves. The brothers Johann and Jakob Bernoulli showed that the shape of a smooth wire along which a particle descends in the least time is the cycloid, a transcendental curve much studied in the previous century. Working in a spirit of keen rivalry, the two brothers arrived at ideas that would later develop into the calculus of variations. In his study of the rectification of the lemniscate, a ribbon-shaped curve discovered by Jakob Bernoulli in 1694, Giulio Carlo Fagnano (1682–1766) introduced ingenious analytic transformations that laid the foundation for the theory of elliptic integrals. Nikolaus I Bernoulli (1687–1759), the nephew of Johann and Jakob, proved the equality of mixed second-order partial derivatives and made important contributions to differential equations by the construction of orthogonal trajectories to families of curves. Pierre Varignon (1654–1722), Johann Bernoulli, and Jakob Hermann (1678–1733) continued to develop analytic dynamics as they adapted Leibniz's calculus to the inertial mechanics of Newton's *Principia*.

Geometric conceptions and problems predominated in the early calculus. This emphasis on the curve as the object of study provided coherence to what was otherwise a disparate collection of analytic techniques. With its continued development, the calculus gradually became removed from its origins in the geometry of curves, and a movement emerged to establish the subject on a purely analytic basis. In a series of textbooks published in the middle of the century, the Swiss mathematician Leonhard Euler systematically accomplished the separation of the calculus from geometry. In his *Introductio in Analysin Infinitorum* (1748; *Introduction to the Analysis of the Infinite*) he made the notion of function the central organizing concept of analysis.

Foundations of the calculus

Euler's analytic approach is illustrated by his introduction of the sine and cosine functions. Trigonometry tables had existed since antiquity, and the relations between sines and cosines were commonly used in mathematical astronomy. In the early calculus mathematicians had derived in their study of periodic mechanical phenomena the differential equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

and they were able to interpret its solution geometrically in terms of lines and angles in the circle. Euler was the first to introduce the sine and cosine functions as quantities whose relation to other quantities could be studied independently of any geometric diagram.

Euler's analytic approach to the calculus received support from his younger contemporary Joseph-Louis Lagrange, who, following Euler's death in 1783, replaced him as the leader of European mathematics. In 1755 the 19-year-old Lagrange wrote to Euler to announce the discovery of a new algorithm in the calculus of variations, a subject to which Euler had devoted an important treatise 11 years earlier. Euler had used geometric ideas extensively and had emphasized the need for analytic methods. Lagrange's idea was to introduce the new symbol δ into the calculus and to experiment formally until he had devised an algorithm to obtain the variational equations. Mathematically quite distinct from Euler's procedure, his method required no reference to the geometric configuration. Euler immediately adopted Lagrange's idea, and in the next several years the two men systematically revised the subject using the new techniques.

In 1766 Lagrange was invited by the Prussian king, Frederick the Great, to become mathematics director of the Berlin Academy. During the next two decades he wrote important memoirs on nearly all of the major areas of mathematics. In 1788 he published his famous *Mécanique analytique*, a treatise that used variational ideas to present mechanics from a unified analytic viewpoint. In the preface Lagrange wrote:

> One will find no Figures in this Work. The methods that I present require neither constructions nor geometrical or mechanical reasonings, but only algebraic operations, subject to a regular and uniform course. Those who admire Analysis, will with pleasure see Mechanics become a new branch of it, and will be grateful to me for having extended its domain.

Following the death of Frederick the Great, Lagrange traveled to Paris to become a *pensionnaire* of the Academy of Sciences. With the establishment of the École Polytechnique in 1794 he was asked to deliver the lectures on mathematics. There was a concern in European mathematics at the time to place the calculus on a sound basis, and Lagrange used the occasion to develop his ideas for an algebraic foundation of the subject. The lectures were published in 1797 under the title *Théorie des fonctions analytiques* ("Theory of Analytical Functions"), a treatise whose contents were summarized in its longer title "Containing the Principles of the Differential Calculus Disengaged from All Consideration of Infinitesimals, Vanishing Limits, or Fluxions and Reduced to the Algebraic Analysis of Finite Quantities." Lagrange published a second treatise on the subject in 1801, a work that appeared in a revised and expanded form in 1806.

The range of subjects presented and the consistency of style distinguished Lagrange's didactic writings from other contemporary expositions of the calculus. He began with Euler's notion of a function as an analytic expression composed of variables and constants. He defined the "derived function," or derivative $f'(x)$ of $f(x)$, to be the coefficient of i in the Taylor expansion of $f(x+i)$. Assuming the general possibility of such expansions, he attempted a rather complete theory of the differential and integral calculus, including extensive applications to geometry and mechanics. Lagrange's lectures represented the most advanced development of the 18th-century analytic conception of the calculus.

The theo… functi… of a r… variab…

Beginning with Baron Cauchy in the 1820s, later mathematicians used the concept of limit to establish the calculus on an arithmetic basis. The algebraic viewpoint of Euler and Lagrange was rejected. To arrive at a proper historical appreciation of their work it is necessary to reflect on the meaning of analysis in the 18th century. Since Viète, analysis had referred generally to mathematical methods that employed equations, variables, and constants. With the extensive development of the calculus by Leibniz and his school, analysis became identified with all calculus-related subjects. In addition to this historical association, there was a deeper sense in which analytic methods were fundamental to the new mathematics. An analytic equation implied the existence of a relation that remained valid as the variables changed continuously in magnitude. Analytic algorithms and transformations presupposed a correspondence between local and global change, the basic concern of the calculus. It is this aspect of analysis that fascinated Euler and Lagrange and caused them to see in it the "true metaphysics" of the calculus.

OTHER DEVELOPMENTS

During the period 1600–1800 significant advances occurred in the theory of equations, foundations of Euclidean geometry, number theory, projective geometry, and probability theory. These subjects, which became mature branches of mathematics only in the 19th century, never rivaled analysis and mechanics as programs of research.

Theory of equations. After the dramatic successes of Niccolò Fontana Tartaglia and Lodovico Ferrari in the 16th century, the theory of equations developed slowly, as problems resisted solution by known techniques. In the later 18th century the subject experienced an infusion of new ideas. Interest was concentrated on two problems. The first was to establish the existence of a root of the general polynomial equation of degree n. The second was to express the roots as algebraic functions of the coefficients or to show why it was not, in general, possible to do so.

The proposition that the general polynomial with real coefficients has a root of the form $a + b\sqrt{-1}$ became known later as the fundamental theorem of algebra. By 1742 Euler recognized that roots appear in conjugate pairs; if $a + b\sqrt{-1}$ is a root, then so is $a - b\sqrt{-1}$. Thus, if $a + b\sqrt{-1}$ is a root of $f(x) = 0$, then $f(x) = (x^2 - 2ax - a^2 - b^2)g(x)$. The fundamental theorem was therefore equivalent to asserting that a polynomial may be decomposed into linear and quadratic factors. This result was of considerable importance for the theory of integration, since by the method of partial fractions it ensured that a rational function, the quotient of two polynomials, could always be integrated in terms of algebraic and elementary transcendental functions.

[The fun-]damental [theo]rem of [alge]bra

Although d'Alembert, Euler, and Lagrange worked on the fundamental theorem, the first successful proof was developed by Carl Friedrich Gauss in his doctoral dissertation of 1799. Earlier researchers had investigated special cases or had concentrated on showing that all possible roots were of the form $a \pm b\sqrt{-1}$. Gauss tackled the problem of existence directly. Expressing the unknown in terms of the polar coordinate variables r and θ, he showed that a solution of the polynomial would lie at the intersection of two curves of the form $T(r, \theta) = 0$ and $U(r, \theta) = 0$. By a careful and rigorous investigation he proved that the two curves intersect.

Gauss's demonstration of the fundamental theorem initiated a new approach to the question of mathematical existence. In the 18th century mathematicians were interested in the nature of particular analytic processes or the form that given solutions should take. Mathematical entities were regarded as things that were given, not as things whose existence needed to be established. Because analysis was applied in geometry and mechanics, the formalism seemed to possess a clear interpretation that obviated any need to consider questions of existence. Gauss's demonstration was the beginning of a change of attitude in mathematics, of a shift to the rigorous, internal development of the subject.

The problem of expressing the roots of a polynomial as functions of the coefficients was addressed by several mathematicians independently about 1770. The Cambridge mathematician Edward Waring published treatises in 1762 and 1770 on the theory of equations. In 1770 Lagrange presented a long expository memoir on the subject to the Berlin Academy, and in 1771 Alexandre Vandermonde submitted a paper to the French Academy of Sciences. Although the ideas of the three men were related, Lagrange's memoir was the most extensive and most influential historically.

Lagrange presented a detailed analysis of the solution by radicals of second-, third-, and fourth-degree equations and investigated why these solutions failed when the degree was greater than or equal to five. He introduced the novel idea of considering functions of the roots and examining the values they assumed as the roots were permuted. He was able to show that the solution of an equation depends on the construction of a second resolvent equation, but he was unable to provide a general procedure for solving the resolvent when the degree of the original equation was greater than four. Although his theory left the subject in an unfinished condition, it provided a solid basis for future work. The search for a general solution to the polynomial equation would provide the greatest single impetus for the transformation of algebra in the 19th century.

[Para]llel [postu]late

Foundations of geometry. Although the emphasis of mathematics after 1650 was increasingly on analysis, foundational questions in classical geometry continued to arouse interest. Attention centred on the fifth postulate of Book I of the *Elements*, which Euclid had used to prove the existence of a unique parallel through a point to a given line. Since antiquity, Greek, Islāmic, and European geometers had attempted unsuccessfully to show that the parallel postulate need not be a postulate but could instead be deduced from the other postulates of Euclidean geometry. During the period 1600–1800 mathematicians continued these efforts by trying to show that the postulate was equivalent to some result that was considered self-evident. Although the decisive breakthrough to non-Euclidean geometry would not occur until the 19th century, researchers did achieve a deeper and more systematic understanding of the classical properties of space.

Interest in the parallel postulate developed in the 16th century after the recovery and Latin translation of Proclus' commentary on Euclid's *Elements*. The Italian researchers Christopher Clavius in 1574 and Giordano Vitale in 1680 showed that the postulate is equivalent to asserting that the line equidistant from a straight line is a straight line. In 1693 John Wallis, Savilian Professor of Geometry at Oxford, attempted a different demonstration, proving that the axiom follows from the assumption that to every figure there exists a similar figure of arbitrary magnitude.

In 1733 the Italian Girolamo Saccheri published his *Euclides ab Omni Naevo Vindicatus* ("Euclid Cleared of Every Flaw"). This was an important work of synthesis in which he provided a complete analysis of the problem of parallels in terms of Omar Khayyam's quadrilateral. Using the Euclidean assumption that straight lines do not enclose an area, he was able to exclude geometries that contain no parallels. It remained to prove the existence of a unique parallel through a point to a given line. To do this, Saccheri adopted the procedure of reductio ad absurdum; he assumed the existence of more than one parallel and attempted to derive a contradiction. After a long and detailed investigation, he was able to convince himself (mistakenly) that he had found the desired contradiction.

In 1766 Johann Heinrich Lambert of the Berlin Academy composed *Die Theorie der Parallellinien* ("The Theory of Parallel Lines"; published 1786), a penetrating study of the fifth postulate in Euclidean geometry. Among other theorems Lambert proved is that the parallel axiom is equivalent to the assertion that the sum of the angles of a triangle is equal to two right angles. He combined this fact with Wallis' result to arrive at an unexpected characterization of classical space. According to Lambert, if the parallel postulate is rejected, it follows that for every angle θ less than $2R/3$ (R is a right angle) an equilateral triangle can be constructed with corner angle θ. By Wallis' result any triangle similar to this triangle must be congruent to it. It is therefore possible to associate with every angle a definite length, the side of the corresponding equilateral triangle. Since the measurement of angles is absolute, independent of any convention concerning the selection of units, it follows that an absolute unit of length exists. Hence, to accept the parallel postulate is to deny the possibility of an absolute concept of length.

The final 18th-century contribution to the theory of parallels was Adrien-Marie Legendre's textbook *Éléments de géométrie* (*Elements of Geometry and Trigonometry*), the first edition of which appeared in 1794. Legendre presented an elegant demonstration that purported to show that the sum of the angles of a triangle is equal to two right angles. He believed that he had conclusively established the validity of the parallel postulate. His work attracted a large audience and was influential in informing readers of the new ideas in geometry.

The influence of Kant

The 18th-century failure to develop a non-Euclidean geometry was rooted in deeply held philosophical beliefs. In his *Critique of Pure Reason* (1781), Immanuel Kant had emphasized the synthetic a priori character of mathematical judgments. From this standpoint, statements of geometry and arithmetic were necessarily true propositions with definite empirical content. The existence of similar figures of different size, or the conventional character of units of length, appeared self-evident to mathematicians of the period.

(C.G.F.)

Mathematics in the 19th and 20th centuries

Most of the powerful abstract mathematical theories in use today originated in the 19th century, so any historical account of the period should be supplemented by reference to detailed treatments of these topics. Yet, mathematics a field (dividing 1 by 2 does not yield an integer). But Weil showed that simplified versions (posed over a field) of any question about integer solutions to polynomials could be profitably asked. This transferred the questions to the domain of algebraic geometry. To count the number of solutions, Weil proposed that, since the questions were now geometric, they should be amenable to the techniques of algebraic topology. This was an audacious move, since there was no suitable theory of algebraic topology available, but Weil conjectured what results it should yield. The difficulty of Weil's conjectures may be judged by the fact that the last of them was a generalization to this setting of the famous Riemann hypothesis about the zeta function, and they rapidly became the focus of international attention.

Weil, along with Claude Chevalley, Henri Cartan, Jean Dieudonné, and others, created a group of young French mathematicians who began to publish virtually an encyclopaedia of mathematics under the name Nicolas Bourbaki, taken by Weil from an obscure general of the Franco-German War. Bourbaki became a self-selecting group of young mathematicians who were strong on algebra, and the individual Bourbaki members were interested in the Weil conjectures. In the end, they succeeded completely. A new kind of algebraic topology was developed, and the Weil conjectures were proved. The generalized Riemann hypothesis was the last to surrender, being established by the Belgian Pierre Deligne in the early 1970s. Strangely, its resolution still leaves the original Riemann hypothesis unsolved.

Bourbaki was a key figure in the rethinking of structural mathematics. Algebraic topology was axiomatized by Samuel Eilenberg, a Polish-born American mathematician and Bourbaki member, and the American mathematician Norman Steenrod. Saunders Mac Lane, also of the United States, and Eilenberg extended this axiomatic approach until many types of mathematical structures were presented in families, called categories. Hence there was a category consisting of all groups and all maps between them that preserve multiplication, and there was another category of all topological spaces and all continuous maps between them. To do algebraic topology was to transfer a problem posed in one category (that of topological spaces) to another (usually that of commutative groups or rings). When he created the right algebraic topology for the Weil conjectures, the German-born French mathematician Alexandre Grothendieck, a Bourbaki of enormous energy, produced a new description of algebraic geometry. In his hands it became infused with the language of category theory. The route to algebraic geometry became steeper than ever, but the views from the summit have a naturalness and a profundity that have brought many experts to prefer it to the earlier formulations, including Weil's.

Grothendieck's formulation makes algebraic geometry the study of equations defined over rings rather than fields. Accordingly, it raises the possibility that questions about the integers can be answered directly. Building on the work of like-minded mathematicians in the United States, France, and Russia, the German Gerd Faltings triumphantly vindicated this approach when he solved the Englishman Louis Mordell's conjecture in 1983. This conjecture states that almost all polynomial equations that define curves have at most finitely many rational solutions; the cases excluded from the conjecture are the simple ones that are much better understood.

Meanwhile, Gerhard Frey of Germany had pointed out that, if Fermat's last theorem is false, so that there are integers u, v, w such that $u^p + v^p = w^p$ (p greater than 5), then for these values of u, v, and p the curve $y^2 = x(x - u^p)(x + v^p)$ has properties that contradict major conjectures of the Japanese mathematicians Taniyama Yutaka and Shimura Goro about elliptic curves. Frey's observation, refined by Jean-Pierre Serre of France and proved by the American Ken Ribet, meant that by 1990 Taniyama's unproved conjectures were known to imply Fermat's last theorem.

In 1993 the English mathematician Andrew Wiles established the Shimura-Taniyama conjectures in a large range of cases that included Frey's curve and therefore Fermat's last theorem—a major feat even without the connection to Fermat. It soon became clear that the argument had a serious flaw; but in May 1995 Wiles, assisted by another English mathematician, Richard Taylor, published a different and valid approach. In so doing, Wiles not only solved the most famous outstanding conjecture in mathematics but also triumphantly vindicated the sophisticated and difficult methods of modern number theory. (J.J.G.)

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