Abraham Robinson

ONE OF THE MOST remarkable developments of recent mathematics has been the rehabilitation of the infinitesimal in the calculus. Abraham Robinson, a mathematical logician who worked in the 1950s and the 1960s at the University of Toronto and Yale University, devised an approach to the calculus known as non-standard analysis that permits calculation with infinitesimals, elements that are non-zero but smaller than any finite quantity.

The idea of the differential dx as an infinitely small increment in the variable x was basic to Leibnizian analysis in the eighteenth century. In later mathematics the continuing heuristic and pedagogical value of infinitesimals was recognized. Nevertheless, it was believed that the use of infinitesimals was not quite rigorous, and that in any sound presentation of the calculus they would need to be eliminated. This belief was apparently confirmed in the nineteenth century with the emergence and acceptance of Cauchy's classical foundation based on the concept of limits.

Robinson expressed the differential and integral calculus as a consistent formal system in which infinitesimal elements of various orders appear. A formal system consists of a clearly specified mathematical language with a definite syntax and rules of inference. Robinson developed the resulting language and model into a comprehensive presentation of the calculus known as non-standard analysis. At the time of his death in 1974 many mathematicians were engaged in the research and development of this new branch of mathematics.

Abraham Robinson was a German Jew who left Germany in the 1930s to escape Fascist persecution. After years in Palestine and England he arrived in 1951 at the University of Toronto where he stayed until 1957 working on the theory of models. The department of mathematics at Toronto was the most prominent one in Canada and included such renowned figures as the applied mathematician J.L. Synge and the geometer H.M.S. Coxeter. The quiet atmosphere of the city in those days is described in the memoirs of Leopold Infeld, another distinguished visitor to the University: "It must be good to die in Toronto. The transition between life and death would be continuous, painless and scarcely noticeable in this silent town."

Augustin-Louis Cauchy

ALTHOUGH THE CALCULUS was invented in the seventeenth century, a hundred and fifty years passed before mathematicians developed the modern foundation based on the concept of limit. In three textbooks published in the 1820s the French mathematician Augustin-Louis Cauchy presented research that began a new era in the history of analysis.

At the end of the eighteenth century, mathematicians had become increasingly concerned with finding a logically sound foundation for the calculus. At that time the subject was justified either as a set of operations on formulas that yielded acceptable results or as an analysis of the geometry of curves that was intuitively satisfying. In 1784 Berlin Academy of Sciences offered a prize for the best exposition of the role of the infinite in the differential and integral calculus. Although several memoirs were submitted and the prize was awarded, none of the proposals was entirely satisfactory and none achieved widespread acceptance.

Cauchy came to the problem of the foundation of the calculus as a young lecturer at the Ecole polytechnique. He realized that the right approach involved the concepts of limit and continuity, which he formulated clearly for the first time. Beginning with these notions he systematically developed a theory of differentiable and integrable functions of a real variable. In attending to logical rigour and basing the calculus on the continuum of real numbers he began a new period in mathematical analysis. Cauchy is deservedly known as the first "modern" mathematician.

Cauchy and his German contemporary Carl Friedrich Gauss were the two leading mathematicians of the early nineteenth century. Although an innovator in mathematics, Cauchy remained in religion and politics a devout Catholic and absolutist. He refused to pledge allegiance to the Orleanist monarch Louis-Phillipe, installed during the July Revolution of 1830, and followed the deposed Charles X into exile to Prague. He eventually returned to Paris.

Cauchy continued to impress his contemporaries with his odd behaviour. He would sometimes begin research on a subject, forget that he had worked on it years before, and completely duplicate his earlier results. Unlike Gauss, who made public only his deepest and most mature work, Cauchy during his long career published all research in which he became engaged. In addition to his work in the calculus he made fundamental contributions to complex analysis, the theory of elasticity, and algebra. More concepts and theorems have been named for Cauchy than for any other mathematician.

Gottfried Wilhelm Leibniz

THE FIRST PUBLICATION of the differential calculus took place in 1684 in the *Acta Eruditorum*, a European scientific journal that had been founded in 1682. The author of the article, Gottfried Wilhelm Leibniz, was a thirty-eight-year-old German philosopher, mathematician, and diplomat. Leibniz had duplicated the achievement of his English contemporary, Isaac Newton, who several years earlier had communicated privately to friends his results in the calculus. Although controversy over priority developed later, it is clear that both men worked independently and pursued different courses in their development of the new analysis.

Throughout the seventeenth century, mathematicians had accumulated, in their study of the geometrical curve, methods to solve two kinds of problems—the determination of tangents and the calculation of areas and path-lengths. In the 1670s Leibniz studied the algebraic techniques René Descartes had introduced to analyze curves and invented a calculus that provided a general algorithm for the solution of tangent and area problems. In 1693 he published the fundamental theorem of the calculus, demonstrating the inverse character of differentiation and integration. The fundamental theorem resolved the basic problem of the calculus, to describe mathematically change along a curve, and to connect change at a point (slope) to change over the entire curve (area, path-length).

In arriving at the idea for the calculus Leibniz was guided by his study of numerical sequences. Beginning with a given sequence, one forms the sum and difference sequences, whose general terms are given respectively as the sum of the first n members and the difference of the nth and $(n-1)$st members. The particular examples that Leibniz investigated were generated from the arithmetic and harmonic sequences, 1, 2, 3, . . ., and 1/1, 1/2, 1/3, . . . He examined arrays of sequences and used their sum and difference properties to determine relations among them. He connected this interest in combinatorial analysis to his investigation of the geometry of curves to arrive at his fundamental insight concerning the inverse character of differentiation and integration.

Although Leibniz shared credit for the invention of the calculus with Newton, his own work was historically much more influential. Newton was slow to publish and preferred to work alone. Leibniz, by contrast, established a vigorous school of mathematicians to promote his methods, among whom were l'Hôpital in Paris and the Bernoulli brothers in Basel. The notation and algorithms of the Leibnizian calculus were more effective and easier to use than Newton's awkward fluxional calculus. Throughout his life Leibniz was concerned with the philosophical problem of language, and he consciously emphasized the importance of notation in his new calculus. The decline of English mathematics following Newton's death may be explained in part by the failure of English researchers to appreciate the power and superiority of Leibniz's methods.

Isaac Newton

FROM ITS BEGINNINGS until the nineteenth century the calculus was closely associated with problems in mechanics and theoretical astronomy. The great mathematicians of the eighteenth century—Euler, d'Alembert, Clairaut and Lagrange—were also leading mathematical physicists. The physical tradition in analysis originated with Isaac Newton, the greatest scientist of the seventeenth century and the co-inventor with Leibniz of the differential and integral calculus. The son of an English farmer, Newton became in 1669 the Lucasian professor of mathematics at Cambridge University. According to the statutes of the University, the Lucasian professor was to read and expound "some part of Geometry, Astronomy, Geography, Optics, Statics, or some other mathematical discipline."

The celebrated Newtonian synthesis was presented in 1687 in Newton's *Principia mathematica philosophiae naturalis* (Mathematical Principles of Natural Philosophy). Essentially a treatise on the mathematical dynamics of particle motion, the *Principia* combined the concept of universal gravitation with Galileo's laws of terrestial motion and Kepler's laws of planetary motion. At the beginning of the treatise Newton introduced in the form of eleven lemmas his calculus or method of first and last ratios, a geometrical theory of limits that provided the mathematical basis of his dynamics.

Newton's use of the calculus in the *Principia* is illustrated by Proposition 11 of Book One: If the trajectory of a particle moving under the action of a central force is an ellipse with the centre of force at one focus, then the force is inversely proportional to the square of the distance of the particle from the centre. Because the planets were known by Kepler's laws to move in ellipses with the Sun at one focus, this proposition provided support for his inverse-square law of gravitation. To establish the proposition Newton derived an approximate measure for the force using small lines defined in terms of the radius (the line from the force centre to the particle) and the tangent to the curve at a point. This result expressed geometrically the proportionality of force to acceleration. Using properties of the ellipse known from classical geometry Newton calculated the limit of this measure and showed that it was equal to a constant times one over the square of the radius.

The *Principia mathematica* was Newton's most important contribution to science. Although he devoted several other treatises to the calculus, delays in publication meant that his results were duplicated and published independently by other researchers. The *Principia* was written in an austere style that discouraged most readers. Indeed, his mathematical dynamics became influential only after it was translated into the idiom of the Leibnizian calculus in the early 1700s. The great achievements of classical mechanics in the eighteenth century owed their physical basis to Newton but were developed mathematically using the calculus of Leibniz.

Sofia Kovalevskaia

UNTIL THE NINETEENTH CENTURY few women had the opportunity to study science or to pursue a career as a professional scientist. It was in mathematics that the first woman was admitted as a regular member of the international research community. Sofia Kovalevskaia, the daughter of a Russian general, was the first woman to be awarded a doctorate and the first woman outside of Italy to hold a chair at a European university. At the time of her death in 1890 she participated as an equal in the scientific community and stood as a peer among the outstanding mathematicians of Europe.

Coming from a well-to-do background in Russia, Kovalevskaia travelled to Germany in 1869 to study mathematics at Heidelberg and Berlin. Barred from lectures at the University of Berlin because she was a woman, Kovalevskaia studied privately for three years with the distinguished mathematician Karl Weierstrass. In 1874 she was awarded a doctorate *in absentia* from the University of Göttingen. After several years of study and travel she was appointed lecturer in mathematics at the University of Stockholm, a position that was arranged by her friend the Swedish mathematician Gösta Mittag-Leffler. In 1889 Mittag-Leffler secured a life professorship at Stockholm for her.

Kovalevskaia wrote her doctoral dissertation on a problem in partial differential equations. These equations occur frequently in nature, whenever a changing quantity depends on more than one independent variable. The initial and most basic mathematical question in the theory of partial differential equations is to determine conditions under which a solution exists. Using techniques that involved the expansion of functions in power series, Kovalevskaia derived conditions that assure the existence of one and only one solution of the given equation. The theorem she proved remains a basic result in the modern subject.

Apart from her research in mathematics, Sofia Kovalevskaia was committed throughout her career to the support of progressive socialist movements in Europe. In several literary works she presented her views on public education, feminism, and socialism. In her beautifully written *Memories of Childhood* Kovalevskaia traced her political development and that of her sister Aniuta in a provincial gentry family. Kovalevskaia's political sympathies contributed to the opposition she encountered during her life. After her death a tsarist government minister is said to have opposed the translation of a memoir dealing with her life with the words: "People have already concerned themselves too much with a woman who, in the last analysis, was a nihilist."

Maria Gaetana Agnesi

IN THE FIRST PART of the eighteenth century advanced mathematics on the Continent was devoted to the organization and extension of calculus to problems in the geometry of curves. Maria Gaetana Agnesi, the first woman in modern times who can be accurately called a mathematician, published in 1748 a two-volume treatise presenting a systematic exposition of work in the subject. The *Istituzioni analitiche ad uso della gioventù italiana* (Analytical Lectures for the Use of Italian Youth) covered the range from elementary algebra to coordinate geometry, and then on to differential calculus, integral calculus, infinite series, and the solution of differential equations. The work won immediate acclaim from the European mathematical community and was translated into both French and English.

Calculus as it was then practised was quite intricate in its application to the study of curves, requiring the careful calculation of infinitesimals of different orders. Its complexity was evident in a problem as basic as the determination of the radius of curvature of a curve at a point. In many memoirs published during the period the mathematical exposition was very informal; it was assumed that the necessary details could be supplied by the reader. In contrast, Agnesi's book provided a thorough and sophisticated treatment of difficult technical problems that avoided the weaknesses of other works.

A committee of the Paris Academy of Sciences, which authorized the French translation of Agnesi's book, reported in 1749: "This work is characterized by its careful organization, its clarity, and its precision. There is no other book, in any language, which would enable a reader to penetrate as deeply, or as rapidly, into the fundamental concepts of analysis. We consider this treatise the most complete and best written work of its kind." The editor of the English edition wrote "He [John Colson, the English translator] found her work to be so excellent that he was at the pains of learning the Italian language at an advanced age for the sole purpose of translating her book into English, that the British Youth might have the benefit of it as well as the Youth of Italy."

Maria Agnesi was the daughter of a wealthy professor of mathematics at the University of Bologna. Her father, encouraged by her interest in scientific matters, secured a series of distinguished tutors and arranged soirées attended by scholars and local celebrities. At the age of eleven she was fluent in or thoroughly familiar with French, Latin, Greek, German, Spanish, and Hebrew. In adulthood she became retiring, preferring to work alone in mathematics or devoting herself to religious studies and social work. After the publication of the *Istituzioni analitiche*, however, she gradually withdrew from mathematics. In 1762, she declared that she was no longer concerned with the subject of calculus.