The Clebsch-Mayer Theory of the Second Variation in the Calculus of Variations: A Case Study in the Influence of Dynamical Analysis on Pure Mathematics

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Abstract Carl Jacobi worked in the 1830s at the University of Königsberg on what became known as Hamilton-Jacobi theory, and also on the theory of the second variation in the calculus of variations. The first was a subject in dynamical analysis, while the second was a subject in pure mathematics. Insofar as the calculus of variations was concerned, Jacobi's contributions were seminal and original but presented in an incomplete and programmatic form. Together his writings stimulated active but independent traditions of research in both subjects. In the late 1850s and 1860s Alfred Clebsch and Adolph Mayer—mathematicians associated with the Königsberg school—established a new approach to the investigation of sufficient conditions in the calculus of variations by bringing methods from Hamilton-Jacobi theory to bear on the transformation of the second variation. In doing so they established the basis for research on the subject that was eventually codified in writings around 1900 of Camille Jordan, Adolf Kneser, Gustav von Escherich and Oskar Bolza.

1 Introduction

A central problem of the calculus of variations in the nineteenth century was identifying a set of conditions that are sufficient to ensure a maximum or minimum. While several necessary conditions had been found, the question of whether these conditions taken together were sufficient was an open subject of investigation.

In the second half of the 1830s Carl Gustav Jacobi carried out research in both dynamical analysis and the calculus of variations. In 1837 he published two seminal papers in Crelle's journal on these subjects. The paper on dynamics, "Ueber die Reduction der Integration der partielle Differentialgleichungen erster Ordnung,"

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(Jacobi 1837b) was stimulated by William Hamilton's discoveries as well as by some results of Joseph Lagrange and Siméon Poisson. The one on the calculus of variations, "Zur Theorie der Variationsrechnung und der Differentialgleichungen," (Jacobi 1837a) belonged to pure mathematics, and extended in an original way the work of Adrien Legendre and Lagrange on the problem of sufficiency. The two subjects were distinct and relatively independent of each other. Although dynamical analysis employed methods and results from the calculus of variations it was in no way connected to the concerns of the mathematical subject. The latter were made up of logical and theoretical elements or goals—the problem of sufficiency, questions of existence—that involved no reference to any physical domain or applied concerns.[1]

Jacobi's contributions to dynamics led to the creation of that branch of mathematical science known today as Hamilton-Jacobi theory. Jacobi continued to work on this subject and delivered lectures on it at the University of Königsberg in 1841–42 and the University of Berlin in 1848. The Königsberg lectures were edited by Carl Borchardt and Alfred Clebsch and published in 1866 in a volume titled *Vorlesungen über Dynamik* (Jacobi 1866a). The latter also contained a 167-page appendix on the more mathematical aspects of dynamics including the formulation of a theory of canonical transformations and a programmatic outline of the further mathematical development of this theory. The Berlin lectures were unpublished and less influential although they did circulate to some degree in German libraries in the second half of the century. They were finally published in 1996 (see Jacobi (1996) and Thiele (2000)).[2]

[1] The term "pure mathematics" is being used here in its conventional modern sense that was well established by the end of the nineteenth century. Pure mathematics is distinguished by its focus on conceptual, logical and formal aspects and its relative detachment from empiricism. By contrast, in the early nineteenth century pure mathematics referred to that which is not practical or pragmatic in character. The *Journal für die reine und angewandte Mathematik* was founded in 1826 under the editorship of Leopold Crelle, who served in this position until his death in 1856. The table of contents was divided into pure mathematics (with broad subject categories of analysis, geometry and mechanics) and applied mathematics. The latter included topics such as machines, hydraulics, crystallography and cartography. Work in mathematical dynamics was generally regarded as a pure subject. A memoir on the three-body problem that Jacobi published in 1843 was listed in the table of contents twice, first under the "pure" heading and then again under the "applied" heading. This subject schema was abandoned when Carl Borchardt became editor in 1856, 2 years before Clebsch's publications of 1858 on the second variation. (The journal was often referred to as "Crelle" even after Crelle's death.) See also Archibald (2001), Hacking (2014, Chapter 5) and Fraser (2018, 124).

[2] An account of the historical genesis of the original Hamilton-Jacobi theory is given by Nakane and Fraser (2002). Jacobi's results in dynamical analysis became the basis of an active thread of research in celestial mechanics. Mathematical results were valued primarily if they were applicable to the investigation of planetary motion. Notable researchers here were Adolphe Desboves, William Donkin, and in the 1890s, Henri Poincaré. Other researchers of note included Joseph Liouville, François Tisserand and Ludwig Charlier. For the development of Hamilton-Jacobi theory in the nineteenth century including its relations to celestial mechanics see Fraser and Nakane (2023).

Jacobi's ideas in the calculus of variations also spurred a vigorous program of research up into the 1880s. Investigators included Charles Delaunay, Victor A. Lebesgue, Joseph Bertrand (age 19!), Friedrich Eisenlohr, Gaspard Mainardi, Simon Spitzer, Otto Hesse, Alfred Clebsch, Rudolf Lipschitz and Adolph Mayer. For whatever reasons, Jacobi himself did not publish anything on the subject after 1840, and it is not clear to what extent he remained engaged with the work of others.

In addition to the contrast in subject matter (physics on the one hand and pure mathematics on the other) there were technical aspects that distinguished Jacobi's work in the two areas. In the calculus of variations, the primary problem of interest involved a single dependent variational function and derivatives of this function of arbitrary order. By contrast, in Hamilton-Jacobi dynamics one considered integrand functions of multiple dependent variables where only first derivatives of the variables occur. Furthermore, in dynamics questions about conditions that are sufficient to ensure the existence of maxima or minima did not arise, for two reasons: first, the physical instantiation of the formalism obviated any need to consider questions of existence; and second, logical concerns about sufficiency played at best a minor role in dynamical investigations.

The focus of the present study is the research on the second variation that was stimulated by Jacobi's 1837 paper. The elaboration of Jacobi's results on the second variation may be divided into two streams: researches by Delaunay, Eisenlohr, Bertrand and Hesse up to 1857; and researches by Mainardi, Spitzer, Clebsch, Mayer and Husserl, from the 1850s to the 1880s and continuing from there into the new century. It is this second stream of research that is the primary focus of the present paper.[3]

2 The Problem in Its Elementary Form

The variational integral is given as

$$I = \int_{x_0}^{x_1} f\left(x, y, y'\right) dx. \tag{1}$$

[3] Isaac Todhunter's 1861 book provides an overview of work done on the calculus of variations up to that time and is notable for the range of writings examined. See in particular Chapter 10, "Commentators of Jacobi." Modern historical accounts of the developments in the calculus of variations discussed in the present paper are contained in Goldstine (1980) and Fraser (1996, 2003, 2018, 2019). Goldstine's book is a stimulating and informative study written by a specialist in the modern mathematical subject. His approach is "to select those papers and authors whose works have played key roles in the classical calculus of variations as we understand the subject today" (Goldstine 1980, vii). Historian of mathematics Helena Pycior (1983, 493) suggests that the book has some shortcomings: "Its style is not representative of the finest mathematical (or historical) exposition. It contains no adequate statement of purpose or audience, makes but few attempts to highlight major ideas through clear prose statements, and evidences throughout an extremely tight, technical style."

The δ-process is introduced to effect a comparison between the value of I along this curve and along a class of neighbouring curves. Set $\delta y = w(x)$, where $w(x_0) = w(x_1) = 0$. The first and second variations I_1 and I_2 are by definition

$$I_1 = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} w + \frac{\partial f}{\partial y'} w' \right) dx, \tag{2a}$$

$$I_2 = \int_{x_0}^{x_1} \left(\frac{\partial^2 f}{\partial y^2} w^2 + 2 \frac{\partial^2 f}{\partial y \partial y'} w w' + \frac{\partial^2 f}{\partial y'^2} w'^2 \right) dx. \tag{2b}$$

The difference ΔI in the value of I along the actual and comparison arc is

$$\Delta I = I_1 + \frac{1}{2} I_2 + \text{higher order terms}. \tag{3}$$

A solution curve $y = y(x)$ that makes I a maximum or minimum will satisfy the Euler-Lagrange equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$. If this equation holds it is easy to show that we have $I_1 = 0$. Because $w(x)$ is small, it is then clear that I_2 will dominate in this expansion. Hence the sign of ΔI in (3) is determined by the second variation I_2.

In what follows we introduce some standard abbreviations for the second partial derivatives:

$$P = \frac{\partial^2 f}{\partial y^2}, \; Q = \frac{\partial^2 f}{\partial y \partial y'}, \; R = \frac{\partial^2 f}{\partial y'^2}. \tag{4}$$

The expression (2b) for I_2 then is written

$$I_2 = \int_{x_0}^{x_1} \left(P w^2 + 2 Q w w' + R w'^2 \right) dx. \tag{5}$$

The central problem becomes one of transforming the second variation to a form that gives rise to a criterion to decide if a maximum or minimum holds.

In 1788 Legendre presented an important result. An auxiliary function $v = v(x)$ is introduced. Legendre showed that the second variation can be reduced to the form

$$I_2 = \int_{x_0}^{x_1} R \left(w' + \frac{Q + v}{R} w \right)^2 dx, \tag{6}$$

where the function v is a solution of the differential equation

$$R \left(P + v' \right) = (Q + v)^2. \tag{7}$$

Assume that such a solution v has been found. Consider the inequality

$$R = \frac{\partial^2 f}{\partial y'^2} \geq 0 \tag{8}$$

on $[x_0,x_1]$. For a given curve $y = y(x)$ that minimizes I, I_2 will be positive. It follows from (6) that for such a curve the inequality (8) holds. (Suppose R is negative on part of the interval $[x_0,x_1]$. Select a variation $w = w(x)$ that is positive on this subinterval and zero elsewhere. From (6) we see that for such a variation, $I_2 < 0$, which is a contradiction.) (8) would become known in the later subject as Legendre's necessary condition.

To obtain Legendre's condition one must show that a solution to (7) exists and remains finite on the given interval. In his *Théorie des fonctions analytiques*, Lagrange (1797, 206–210) called attention to this point and produced examples in which no finite solution exists. Suppose for example that $f(x,y,y') = y'^2 - y^2$. In this case $P = -2$, $Q = 0$ and $R = 2$ and (7) becomes $2(v' - 2) = v^2$. This equation integrates to $v = 2\tan(x + c)$, where c is a constant. It is clear that if $x_1 - x_0$ is greater than $\pi/2$ then no suitable solution of $2(v' - 2) = v^2$ will exist. Although Lagrange did not contribute any new results to the variational theory, he at least raised the question of conditions under which the Legendre transformation is valid.[4]

3 Jacobi's Discoveries

Jacobi investigated a more general problem than the example from Lagrange in the preceding section. The variational integrand f is taken to be a function of y, y' and higher-order derivatives of y:

$$I = \int_{x_0}^{x_1} f\left(x, y, y', \ldots, y^{(n)}\right) dx. \tag{9}$$

Jacobi concentrated in particular on the case $n = 2$:

$$I = \int_{x_0}^{x_1} f\left(x, y, y', y''\right) dx. \tag{10}$$

The investigation of the transformation of the second variation here requires a deeper level of analysis and is an order of magnitude more difficult than the case $f(x, y, y')$. This investigation would occupy the energies of researchers up into the 1870s and beyond.

Jacobi expressed in a particular way the relationship between the first and second variations in terms of the variational operation δ. He also introduced a new type

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[4] Lagrange's discussion of this point is limited to the case $n = 1$. What is missing in Lagrange's account is an analysis of the equation $I_2 = 0$ in terms of variations given as partial derivatives of the solutions to the Euler equation with respect to the arbitrary constants appearing in this solution. In this respect, he did not anticipate the crucial idea of a conjugate point. (Compare Goldstine (1980, 146) who suggests that Lagrange had "in an interesting way, a presage of Jacobi's condition on conjugate points.")

of variation that was fruitful in this investigation. The solution $y = y(x)$ to the variational problem must satisfy the Euler-Lagrange differential equation and will contain $2n$ arbitrary constants. Let a be one of these constants. We let the variation δy be obtained by varying a and define $\delta y = (\partial y/\partial a)da$.

The new variation led Jacobi to the concept of the conjugate point and the criterion known in modern textbooks as Jacobi's condition. Because $y = y(x)$ is a solution arc it follows that $I_1 = 0$. Now the third-order term can in general be made either positive or negative. Thus, in order for there to be a maximum or minimum it must be the case that there is no admissible variation for which $I_2 = 0$. (An admissible variation is one that is zero at the endpoints of the interval.) One is able to show that for the variation $\delta y = (\partial y/\partial a)da$ we have $I_2 = 0$. Suppose that for such a variation $\partial y/\partial a(x_0,a) = 0$ and thus $\delta y(x_0,a) = 0$. If it is also the case that $\delta y(x^*,a) = 0$ for some x^* with $x_0 < x^* \leq x_1$ then there is an admissible variation for which $I_2 = 0$ and there is no maximum or minimum. A value of x^* for which this is true corresponds to what is known as a conjugate point. Jacobi's condition asserts that in order for a minimum or minimum to exist there can be no conjugate point on the interval $[0, x_1]$. Expressed analytically, there is no x^* ($x_0 < x^* \leq x_1$) for which $\partial y/\partial a(x^*,a) = 0$.

Four years after the appearance of Jacobi (1837a) the young French mathematician Charles Delaunay published a substantial paper on the subject of Jacobi's memoir. Delaunay (1841) is a valuable aid in understanding Jacobi's results, with insightful explanations of key parts of the theory, that were by no means clear in the original article. Some years later Spitzer (1854, 1014) would write that "Delaunay's excellent work contributes not a little to the understanding of Jacobi."

The main part of Jacobi's memoir was devoted to a new transformation of the second variation that was based on a result about ordinary differential equations. This result was stated but not proved. Jacobi's transformation was different from Legendre's and seems, like many of his ideas, to have been a mysterious product of his mathematical mind. In the years following the appearance of Jacobi's paper researchers devoted a great deal of effort to the study of the transformation of the second variation along the lines opened up by him. Several proofs of the theorem on linear differential equations were published, most notably one by Lebesgue in 1841 in Liouville's journal. Other than Delaunay the most important figure in this tradition was Hesse. Hesse's 1857 memoir in *Journal für die reine und angewandte Mathematik* represented the high point in the development of the Jacobi transformation and also marked the end of this particular line of investigation. The Jacobi transformation is something of a relic in the history of nineteenth-century mathematics.[5] The idea at its base would be replaced by the theory of Spitzer, Clebsch and researchers that followed them.

[5] The Jacobi-Hesse transformation is of intrinsic mathematical interest and did not fall completely by the wayside. It was presented in some detail in Ernesto Pascal's 1899 textbook *Variationsrechnung*. Nevertheless, it would appear to have been an improbable development in the history of analysis.

4 Spitzer

Simon Spitzer was a Jewish Austrian mathematician known for his many contributions to analysis. A detailed historical study of his mathematical work remains to be done. Our interest here is an early two-part article on the calculus of variation that he published in 1854 and 1855 in the *Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften*.[6]

Following Jacobi, Spitzer developed the analysis in terms of variations given in terms of the partial derivatives of y (y being a solution of the Euler-Lagrange equation) with respect to the constants of integration that appear in y. However, he rejected Jacobi's new transformation of the second variation, writing (1854, 1025) "There has occurred to me another and much simpler way of determining the values of λ by means of which Jacobi's complicated and difficult transformation is avoided."

Spitzer retained Legendre's older method of transforming the second variation, in which the second variation is directly equated to an expression in positive or negative definite form. Altogether his writings constituted an impressive and extended treatise on the mathematics of the second variation. Hesse (1857, 414) stated that "Spitzer departs from the path trodden by Jacobi" and referred to Spitzer's writings as being "elaborated with as much intellectual acuity as diligence."

In part one Spitzer exhaustively analyzed the second variation for the integral $I = \int_{x_0}^{x_1} V\left(x, y, y', \ldots, y^{(n)}\right) dx$ for the three cases $n = 1, 2, 3$. (The variational integrand function is here denoted V by Spitzer.)

Following his Italian contemporary Gaspare Mainardi (1852) Spitzer in part two introduced a new element of generality into the variational problem. (Spitzer's account was more detailed and complete than Mainardi's and will be the subject of what follows.[7]) We assume that instead of one dependent variable y there is a second dependent variable z. The variational integrand function now takes the form

$$V = V\left(x, y, y', \ldots, y^{(n)}, z, z', \ldots, z^{(n)}\right). \tag{11}$$

Spitzer developed the theory in a limited way for the general integral (11). However, in order to go further and keep the treatment tractable he turned in §21 to the case where $n = 1$ in (11). Here the problem is to select functions y and z of x that maximize or minimize the variational integral, where the latter is now given in the form

[6] Simon Spitzer (1826–1887) and his wife Marie (née Bunzl) had a large family. Their descendants suffered from the persecution of Austrian Jews that intensified following the 1938 annexation of Austria by Germany. The Spitzers' son Alfons and his wife (then in their 70s) perished in 1942 in the Łódź Ghetto. Their grandson Fritz committed suicide late in 1938. Two of their granddaughters emigrated, Leoni Spitzer to England (where she died in 1940) and Helen Adolf to the United States.

[7] Mainardi (1852, 169–171) devoted three pages of his memoir to the problem of multiple dependent variables whereas Spitzer (1855, 57–121) devoted 65 pages to this subject.

$$I = \int_{x_1}^{x_2} V\left(x, y, y', z, z'\right) dx. \tag{12}$$

The analysis of this problem parallels the earlier case involving a single variable y with $V = V(x, y, y', y'')$ and $I = \int_{x_1}^{x_2} V\left(x, y, y', y''\right) dx$. The various considerations and challenges that arose in the latter problem are also present in the optimization of I in (12).

Spitzer obtained the second variation I_2 corresponding to (12) and showed that it could be reduced to another form that allowed one to infer a general condition (corresponding to Legendre's necessary condition) for the variational problem involving two dependent variables. As in the first part of his investigation, he carried out in detail the quite substantial labours of calculation that were needed to do the analysis in full. (Partial derivatives of V (denoted by letters) and of y (with respect to the constants of integration) and determinate methods appear throughout the extensive computations. Figure 1 is typical of some of the pages from Spitzer's memoir.) Included was a consideration of the auxiliary differential equations that were required for the transformation. The constants arising in the solution to these equations must satisfy certain conditions in order for the transformation to be possible. (The constants are given in terms of the constants of integration that appear in the solution to the Euler-Lagrange equations for the problem.) His major result was stated in §23 (Spitzer 1855, 93).

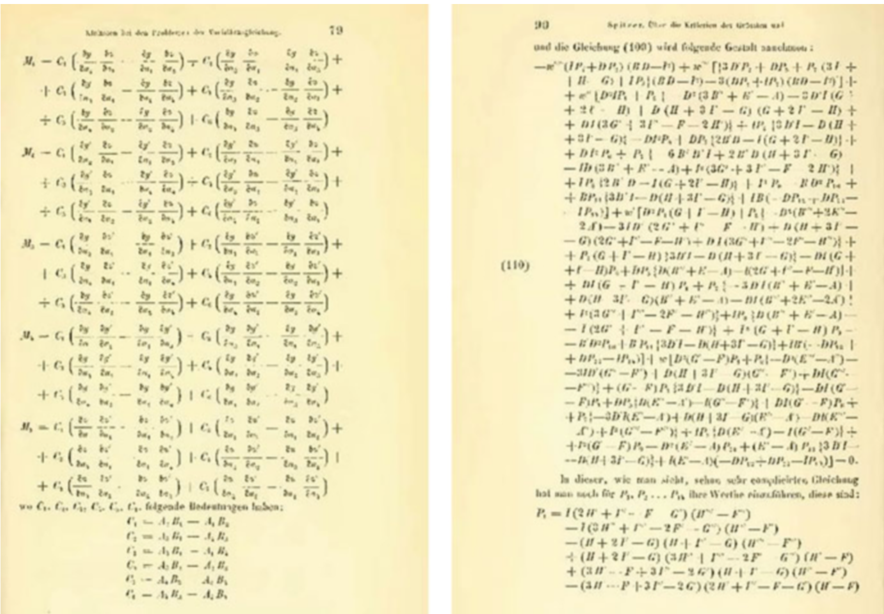

Fig. 1 Pages from Spitzer (1855)

For a maximum or minimum to occur it must be the case that $\frac{\partial^2 V}{\partial y'^2}$ and $\frac{\partial^2 V}{\partial z'^2}$ have the same sign on $[x_1, x_2]$ (positive for a minimum and negative for a maximum) and the term $\frac{\partial^2 V}{\partial y'^2}\frac{\partial^2 V}{\partial z'^2} - \left(\frac{\partial^2 V}{\partial y' \partial z'}\right)^2$ must be positive on this interval. (These conditions ensure that the second variation I_2 has the same sign on the interval.)

5 Clebsch and Mayer

Alfred Clebsch's (Fig. 2) contributions to the calculus of variations are contained in three papers he published in 1858a, b and 1859 in *Journal für die reine und angewandte Mathematik*. Clebsch was a product of the Königsberg school, having studied at the University of Königsberg in the early 1850s under Friedrich Richelot, Franz Neumann and Hesse. Hesse and Richelot had been students of Jacobi, while Neumann was Jacobi's colleague.[8] Clebsch's investigation of the second variation was done early in his career, when he was in his middle twenties, at a time when he was also working on problems in hydrodynamics and mechanics. He was also the editor of Jacobi's *Vorlesungen über Dynamik*; although Jacobi died in 1851 this work was only published in 1866. By this date Clebsch's interests had shifted and

Fig. 2 Alfred Clebsch (1833–1872). (Image from Wikimedia Commons, Public domain)

[8] On the Königsberg school, Richelot and Mayer see Fraser (2018, 126–127).

Fig. 3 Adolph Mayer (1839–1908). (This portrait is reproduced from the 1908 obituary of Mayer in volume 17 of *Jahresbericht der Deutschen Mathematiker-Vereinigung*. In the obituary his first name is given throughout as Adolf)

he would go on to achieve distinction for his research in algebraic geometry and the theory of invariants.

Adolph Mayer's (Fig. 3) interest in the calculus of variations was stimulated by the lectures at Königsberg in 1864–65 of Richelot, who suggested the topic of his *Habilitationsschrift*. In December of 1866 Mayer, then 27, defended the *Habilitationsschrift* before the philosophical faculty of the University of Leipzig. This work was devoted to an investigation in the general setting of the Lagrange problem connecting the transformation of the second variation with the theory of the conjugate point. It was published in 1866 as *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale*. Mayer published a much abbreviated and refined version of the work 2 years later in *Journal für die reine und angewandte Mathematik*. In addition to doing fundamental original work, Mayer also provided clear and expository accounts of Clebsch's achievements. (von Escherich (1898, 1191) stated that Clebsch's investigation of the second variation only became of interest after Mayer "substantiated Clebsch's result in a simpler, more transparent manner."[9]) In addition to Clebsch, Mayer also cited in his *Beiträge* works of Spitzer (1854–1855), Hesse (1857) and a memoir of Rudolf Lipschitz that had just appeared in 1866. In the introduction Mayer (1866, vii) stated that the true nature of the problem was not as clear in Lipschitz's account as it was in Clebsch's and so he

[9] Jacobi's (1837a) was interpreted and explained more clearly by Delaunay (1841), and Clebsch's (1858a, b) were interpreted and explained more clearly by Mayer (1868). Jacobi and Clebsch were the creators while Delaunay and Mayer provided new ideas, lucidity and detail.

would follow Clebsch in his development.[10] Neither Lipschitz nor Clebsch explored the conditions needed to ensure that the functional entities in the transformation exist, a point that was at least touched on by Spitzer and would become the primary focus of Mayer's investigation.

One aspect of Clebsch's mathematical style is noteworthy. All mathematicians that preceded him had provided worked-out accounts of the theory for simpler cases involving first, second and third derivatives of one or two dependent variables. Clebsch by contrast presented the theory once and for all in complete generality and eschewed any exposition of elementary or more accessible cases. The mathematical development was to be experienced at a level of high generality and presumably only at that level. This elevation of generality of presentation came to characterize mathematical work in the calculus of variations. It reflected the emergence of a new mathematical mentality, one that anticipated to some degree the celebrated "modernism" of twentieth-century mathematics.

Clebsch seems to have been stimulated by Hesse's 1857 article. Hesse's contribution was both substantial and stylistic, indicating the sophisticated formal and notational procedures that would be necessary for the very general statement and treatment of the problem preferred by Clebsch. Hesse is the only researcher other than Jacobi cited by Clebsch. Nevertheless, the actual ideas at the base of Clebsch's approach are similar to those, not of Hesse, but of Spitzer.[11] Clebsch followed Spitzer in considering the problem where there are multiple dependent variables and their first derivatives in the variational integrand function. It should be noted that his notation was slightly different from Spitzer's in Eq. (12) above. The variational integrand f is given by Clebsch as a function of $x, y_1, y_2, \ldots, y_n$ and $y_1', y_2', \ldots y_n{}'$. That is,

$$f = f\left(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots y_n{}'\right), \tag{13}$$

in which the variables $y_1, y_2, \ldots$ appear first and then the first derivatives $y_1', y_2', \ldots$ follow.[12] This was of course the standard everywhere in mathematical dynamical

[10] Lipschitz (1866) was not widely discussed by later researchers and seems to have had limited influence. It made no use of Hamilton-Jacobi methods which were an integral part of the later development of the subject. At the turn of the century the theory of the second variation was known simply as the "Clebsch-Mayer theory."

[11] Spitzer had rejected Jacobi's new transformation, which as we noted above, he found complicated and apparently somewhat artificial. Hesse's 1857 paper was motivated by a desire to replicate Spitzer's results while adhering to a general and rigorous formulation of Jacobi's transformation. (At the beginning of his article, Hesse (1857, 227) wrote: "In the following treatise I have tried first to uncover the actual source from which Jacobi drew his results, and second to trace the Jacobian transformation of the second variation back to Spitzer's form.") Clebsch refers to Hesse's article but makes no mention of Spitzer, although his point of departure was anticipated by Spitzer. See also footnote 5 above.

[12] Clebsch does not actually give (13) but states (1858a, 336) verbally that f is a function of $y_1, y_2, \ldots, y_n$ and their first derivatives. In (1858b, 336) he states that f is a function of $x, y_1, y_2, \ldots, y_n$

writing, and Clebsch's formulation indicated a familiarity with dynamical conventions that was absent in Spitzer.

Clebsch broke new ground by modifying the statement of the variational problem to include auxiliary constraints in the form of differential equations. For purposes of exposition, we depart from his general presentation and show how it works in the case of two dependent variables. We have the dependent variables y_1, y_2 and one side constraint. The problem is to maximize or minimize

$$I = \int_{x_0}^{x_1} f\left(x, y_1, y_2, y_1', y_2'\right) dx, \tag{14}$$

where y_1, y_2 are assumed to satisfy the differential equation

$$\varphi\left(x, y_1, y_2, y_1', y_2'\right) = 0, \tag{15}$$

in which φ is a given function of x, y_1, y_2, y_1', y_2'. Let the function $\Omega\left(x, y_1, y_2, y_1', y_2'\right)$ be defined as

$$\Omega = f + \lambda\varphi. \tag{16}$$

Here $\lambda = \lambda(x)$ is a multiplier. Consider the problem of maximizing or minimizing the integral

$$I = \int_{x_0}^{x_1} \Omega\left(x, y_1, y_2, y_1', y_2'\right) dx. \tag{17}$$

The desired maximum or minimum will be a solution that satisfies the constraint equation (15) and is also a solution of the Euler-Lagrange equations for the optimization of I:

$$\frac{\partial\Omega}{\partial y_1} - \frac{d}{dx}\frac{\partial\Omega}{\partial y_1{}'} = 0, \; \frac{\partial\Omega}{\partial y_2} - \frac{d}{dx}\frac{\partial\Omega}{\partial y_2{}'} = 0. \tag{18}$$

In the modern subject a problem with constraints in the form of differential equations is known as a "problem of Lagrange." Lagrange had originally developed the multiplier rule to handle examples in which an auxiliary variable z appears in the integrand, a variable connected to x and y by means of a differential equation. This class of problems formed the subject of Chapter 3 of Euler's *Methodus inveniendi* (1744). Clebsch (1858a, 267–268) by contrast seems to have been the first to have observed that the rule also yields the variational equations in the standard free problem of maximizing or minimizing the integral $\int_{x_0}^{x_1} f\left(x, y, y', \ldots, y^{(n)}\right) dx$.

and dy_1dx,dy_2dx, . . . ,dy_ndx. (Clebsch writes out the derivatives of all orders as quotients and does not use the more standard prime notation employed by Spitzer, Hesse and others.)

For example, consider an integrand function f of the form $f = f\left(x, y_1, y_2, y_2'\right)$. We introduce the auxiliary function $\varphi = y_1' - y_2$ and impose the side constraint $y_1' - y_2 = 0$. The function Ω is then $\Omega = f + \lambda(y_1' - y_2)$. The Euler equations (18) become

$$\frac{\partial f}{\partial y_1} - \frac{d\lambda}{dx} = 0, \frac{\partial f}{\partial y_2} - \lambda - \frac{d}{dx}\frac{\partial f}{\partial y_2{}'} = 0, \tag{19}$$

which reduce to the known Euler-Lagrange equational form

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1{}'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y_1''} = 0. \tag{20}$$

From the time of Legendre until the 1850s the theory of the second variation had been formulated primarily for $I = \int_{x_0}^{x_1} f\left(x, y, y', \ldots, y^{(n)}\right) dx$. Clebsch's reformulation of this problem as one of constrained extrema had the important consequence of eliminating higher-order derivatives in the variational integrand. In order to do this one had to introduce multiple dependent variables. It was also necessary to apply a powerful mathematical tool, the multiplier rule. Justification of this rule is by no means simple, and a study of the first rigorous proofs reveals something of the new and substantial ideas involved in its application. (It was not until almost 30 years later that Mayer (1886) gave a proof of the multiplier rule in the calculus of variations.)

Within this framework, Clebsch (1858a) was able to develop a general transformation of the second variation beginning with functions of the form

$$\Omega = \Omega\left(x, y_1, y_2, \ldots, y_m, y_1', y_2', \ldots, y_m'\right). \tag{21}$$

The starting point is to take the integrand **F** in the second variation and express it in the form

$$F = F + \frac{dB}{dx}. \tag{22}$$

F is in positive or negative definite form and B is an expression involving variations that vanish at the endpoints. When integrated (22) gives the desired transformation of the second variation.

Clebsch produced a formidable piece of analysis that led finally to a generalization of Legendre's necessary condition that is known in some modern textbooks as Clebsch's condition (Bolza 1909, 609). Consider a free system with no constraints. If a given $y_1, y_2, \ldots, y_n$ minimizes the variational integral, then the following inequality must hold:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\Omega}{\partial y_i{}'\partial y_k{}'}\zeta_i\zeta_k \geq 0, \tag{23}$$

where ζ_i and ζ_k are magnitudes given as determinates related to the variations of y_i and y_k. If constraints are present, then ζ_i and ζ_k must also satisfy these constraints.

For the case of two variables y_1, y_2, (23) becomes

$$\frac{\partial^2 \Omega}{\partial {y_1}'^2}{\zeta_1}^2 + 2\frac{\partial^2 \Omega}{\partial {y_1}'\partial {y_2}'}\zeta_1\zeta_2 + \frac{\partial^2 \Omega}{\partial {y_2}'^2}{\zeta_2}^2 \geq 0. \tag{24}$$

This condition is equivalent to stipulating that $\frac{\partial^2 \Omega}{\partial {y_1}'^2}$ and $\frac{\partial^2 \Omega}{\partial {y_2}'^2}$ are positive and that

$$\left(\frac{\partial^2 \Omega}{\partial {y_1}'\partial {y_2}'}\right)^2 - \frac{\partial^2 \Omega}{\partial {y_1}'^2}\frac{\partial^2 \Omega}{\partial {y_2}'^2} \leq 0. \tag{25}$$

As we saw above, this result had already been obtained by Spitzer (1855, 93).

In a second memoir Clebsch (1858b) further developed his transformation of the second variation using methods from Hamilton-Jacobi theory in dynamical analysis.[13] (Lipschitz (1866, 27) noted Clebsch's use of methods drawn from "the investigations of Hamilton and Jacobi on mechanical problems.") This was a critical moment in history because it brought into the subject ideas from a previously unrelated branch of analysis. The possibility of doing this at all was of course opened up the introduction of multiple dependent variables and the use of the multiplier rule to reduce the general variational integrand to the form $\Omega\left(x, y_1, y_2, \ldots, y_m, y_1', y_2', \ldots, y_m'\right)$, the same form as the integrand of variational integrals in dynamics.

Clebsch's memoirs are difficult to follow.[14] (Recall von Escherich's (1898, 1191) remark given above.) Mayer presented the relevant results from Hamilton-Jacobi theory in a more readable form in the opening part of his treatise. He stated that the method originated in Hamilton's work on mechanical problems and was perfected by Jacobi. For simplicity of exposition, we will present the analysis for the free case where there are no side constraints. We begin with the variational equations (what are called the Euler-Lagrange equations although Clebsch and Mayer did not call them that):

[13] Clebsch was not the first to employ Hamilton-Jacobi methods in the calculus of variations. Mikhail Ostrogradsky had already done so in a memoir published in 1850, although Clebsch did not seem to have been aware of Ostrogradsky's research. Ostrogradsky was concerned with showing how Hamilton-Jacobi methods could be extended to more general variational integrands containing higher derivatives of the optimizing curve-function $y = y(x)$. However, the second variation and its transformation were not taken up as a subject of investigation. Nevertheless, Ostrogradsky developed his analysis for multiple dependent variables and in this respect anticipated the work of Spitzer and Clebsch.

[14] Goldstine (1980, Chapter 6) provides detailed descriptions of Clebsch (1858a, b) (as well as of Mayer (1868)). His account adheres to the original papers closely with little explanation of their content. For one part of Clebsch (1858b) Goldstine finds "it reasonable to replace his discussion by a correctly formulated one" (p. 258).

$$\frac{\partial f}{\partial y_i} = \frac{d}{dx}\frac{\partial f}{\partial y_i{}'}\ (i = 1, 2, \ldots, n)\,. \tag{26}$$

Consider a function V of the variables $x, y_1, y_2, \ldots, y_n$ defined by the equations

$$\frac{\partial f}{\partial y_i{}'} = \frac{\partial V}{\partial y_i}, \frac{\partial V}{\partial x} = f - \sum\nolimits_{h=1}^{n} y_h{}' \frac{\partial V}{\partial y_h}. \tag{27}$$

(V is simply Hamilton's principal function and $\sum_{h=1}^{n} y_h{}' \frac{\partial V}{\partial y_h} - f$ is the Hamiltonian H, although this terminology is not employed.) Using (27) one obtains the derivatives $y_i{}'$ as functions of $x, y_1, y_2, \ldots, y_n$. Equation (27) then becomes a partial differential equation for V in terms of $x, y_1, y_2, \ldots, y_n$. Mayer called this the Hamiltonian partial differential equation for the variational problem $\delta I = 0$.[15] He proceeded to give something known in the modern subject as Jacobi's theorem, for which Mayer cited Clebsch (1858b, 337–340). Let $V = V(x, y_1, y_2, \ldots, y_n, a_1, a_2, \ldots, a_n)$ be a complete solution of (27), where $a_1, a_2, \ldots, a_n$ are arbitrary constants. Consider a second set of n arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$. (Mayer stated that these may be called canonical constants corresponding to $a_1, a_2, \ldots, a_n$.) Then the solution of (26) is given by the equations

$$\frac{\partial V}{\partial y_i} = \frac{\partial f}{\partial y_i{}'}, \frac{\partial V}{\partial a_i} = \alpha_i.\ (i = 1, 2, \ldots, n) \tag{28}$$

Having introduced these results, Mayer embarked on a derivation of the transformation of the second variation along the lines of Clebsch (1858a), which involved no reference to canonical methods. At the conclusion of this derivation, he noted that difficulties will arise if certain functional terms in the transformation become infinite at isolated points. It is here that Mayer turned to Hamilton-Jacobi methods to refine the analysis. From this point on these methods are basic to his development of the theory. The investigation is recast in terms of the results contained in Clebsch (1858b). Hamilton-Jacobi methods—including the standard canonical equations—are also the foundation of Mayer's 1868 article in *Journal für die reine und angewandte Mathematik*.

In the Clebsch-Mayer theory the mathematics of the second variation is complicated, involving multiple summations, arrays of variables and constants and detailed computation of determinates. Everything is presented in complete generality with a full set of side constraints in the form of differential equations. It is not possible here to go into detail on this subject. However, we will try to provide an indication of the advantages conferred by the use of Hamilton-Jacobi methods (Mayer 1866,

[15] Nowhere does anything corresponding to the mass appear (it is taken to be one). The conjugate variables p_i also do not appear. Hamilton's canonical equations are not given; reference is made only to the Lagrangian equations (26). In Mayer's article in *Journal für die reine und angewandte Mathematik* 2 years later the conventional formulation is given in terms of conjugate variables and canonical equations.

35). Following Jacobi the variational process is defined by taking a solution of the Euler-Lagrange equations and defining the variation as the partial derivatives of the dependent variables $y_1, y_2, \ldots, y_n$ with respect to any one of the constants of integration a_i appearing in the y_i. The various expressions and summations that arise in the derivation of the transformation involve these variations. Suppose the solution of the Euler-Lagrange equations is given in the form (28) above using a complete solution V of the Hamilton-Jacobi equation. Then we have

$$(\alpha_k) = \frac{\partial V}{\partial a_k}. \tag{29}$$

Here the parenthetical notion (α_k) is used to indicated that α_k is regarded as a function of $x, y_1, y_2, \ldots, y_n, a_1, a_2, \ldots, a_n$. We have from (29)

$$\frac{\partial\,(\alpha_k)}{\partial a_h} = \frac{\partial^2 V}{\partial a_h \partial a_k} = \frac{\partial^2 V}{\partial a_k \partial a_h} = \frac{\partial\,(\alpha_h)}{\partial a_k}. \tag{30}$$

Equation (30) is used to simplify expressions appearing in the transformation. Mayer (1866, 35) observed that a fundamental theorem that had been obtained in the first part of his *Beiträge* became simpler and more elegant if canonical constants of integration are employed. As the treatise progresses Hamilton-Jacobi methods become integral to the development of the theory and predominate in Mayer's 1868 article in *Journal für die reine und angewandte Mathematik*. Equations (29) and (30) are used to obtain identities involving summations of mixed partial derivatives. Using these identities and various substitutions one is led in a rather complex and lengthy procedure to a new derivation of the transformation of the second variation that is preferable to the one presented in Clebsch (1858a).

Mayer's account of Clebsch's transformation was only a prelude to his main goal, which was to show that if Jacobi's condition holds then one is able to obtain suitable functions in the transformation that remain finite on the interval in question. While the ideas here were foreshadowed in the work of Spitzer and Hesse the main problem had been stated in a treatise on the calculus of variations published in 1861 by François Moigno. These authors drew attention to the need to find a system of functions for which the determinant in the denominator of the integrand of the transformed second variation is non-zero throughout the given interval. The arbitrary constants contained in these functions must satisfy certain auxiliary relations. Moigno (1861, 192) observed: "The question of recognizing if such a system of constants does or does not exist is in reality the most delicate part of Jacobi's theory, and it awaits yet a general solution."

The general problem here was also taken up in the lectures of Friedrich Richelot in the early 1860s. It was solved definitively by Mayer in his 1866 *Beiträge*. The solution is reached by means of equations and identities given by Hamilton-Jacobi theory. Mayer's result resolved the basic question of sufficiency for single-integral variational problems. Mayer announced this result in the opening paragraph of the forward to his *Beiträge*:

> As a result of a suggestion of my esteemed teacher, Professor Richelot, I have attempted in the past years to determine the criteria for the minimum of a few particular integrals. I found in these integrals a remarkable relationship between the denominator of the reduction and another determinant which plays a major role in the Jacobi criterion. The reduction of the second variation contains a certain number of arbitrary constants. It is shown that by a suitable determination of these constants both determinants become identical. This theorem, which is not restricted to the treatment of special problems but is generally valid, if combined with an important observation of Professor Richelot's concerning the impossibility of the vanishing of the second variation under certain conditions, leads finally by a completely different route to the proof and extension of the Jacobi criterion to the general case of the calculus of variations involving a single independent variable. (Mayer 1866, v)

6 Consolidation

After Clebsch published his papers on the second variation he turned to other subjects and died at age 39 in 1872. Mayer continued to work in the calculus of variations but did not make substantial further contributions to the theory of the second variation. However, he remained engaged with this subject. Adolf Kneser's notable article on the second variation (Kneser 1898) took as its starting point the "Clebsch-Mayer theory." He acknowledged (p. 321) personal correspondence with Mayer in the writing of the article.

In 1882 Edmund Husserl completed his dissertation under the supervision of Leo Königsberger at the University of Vienna on the calculus of variations. The main subject of this work was Mayer's theory of the second variation, concerning which Husserl arrived at fundamental and novel insights (see Fraser (2019)). He also made perceptive comments on the foundations of the calculus of variations. Husserl's dissertation was not published until 1983 (in French translation) and appears to have had limited influence. Husserl himself turned away from mathematics to philosophy. Nevertheless, his dissertation remains of considerable intrinsic interest and also indicates that there was active interest in the 1880s among mathematicians in the Clebsch-Mayer theory of the second variation.

Later accounts of the Clebsch-Mayer theory, by Camille Jordan (1887), Kneser (1898), von Escherich (1898) and Bolza (1909), developed the subject along the lines laid down by its original authors.[16] These works show a concern with formulating the theory at a high level of generality; the exposition is characterized by formal complexity and elaborate notation. Although Clebsch and Mayer placed primary emphasis on the transformation of the second variation by means of Hamilton-Jacobi methods, they nonetheless also provided a transformation that did

[16] In the preface to Jordan (1887, p. v) mention is made of "emprunts" from the work of several authors, including "MM. Clebsch and Mayer sur la seconde variation des intégrales." This preface is not reproduced in the second edition of 1896 and subsequent editions and there is no indication of where the relevant results originated.

not refer to these methods. (The latter evidently was the case for Clebsch (1858a).) By contrast, Jordan and von Escherich introduced Hamilton-Jacobi methods from the outset and the analysis was developed from this standpoint. Jordan followed the second method employed by Mayer (1866) in his Beiträge to prove the main sufficiency result. Using the canonical constants of integration, Jordan derived after some effort a certain equality involving determinate functions and used it to establish the desired result.

The researches of Clebsch and Mayer were at a higher level than contemporary work on the calculus of variations in textbooks and journal literature. The subsequent development of the theory by such prominent figures as von Escherich and Kneser reflected its status at the turn of the century. The Clebsch-Mayer theory receded somewhat into the background with the advent of Karl Weierstrass's field-theoretic approach to the problem of sufficiency, a development that took place from 1890 into the 1920s (see Thiele (2007).) A focus on the second variation was nonetheless a significant part of the history of the calculus of variations and the Clebsch-Mayer synthesis remains today an impressive and substantial achievement.

7 Conclusion

In his writings on dynamics Jacobi emphasized that it was possible to develop the subject on a more purely mathematical level. Concerning the theory of partial differential equations he wrote,

> Hamilton's theorems themselves contribute to the perfection of this theory in a significant and unexpected way, although the author has not emphasized this purely analytical interest. (1866b, 304)

In this same work Jacobi (1866b, 468–470) concluded the final section on canonical transformations with a programmatic outline of further mathematical work to be done. Transformations that were introduced to facilitate the integration of the dynamical equations of motion were also viewed on a mathematical level as a subject of interest. Jacobi's proposal was not developed by him, probably because of other interests, poor health and early death, but it was taken up in a more geometrical setting by Sophus Lie in the 1870s. (For details on these developments see Fraser and Nakane (2023, 273).)

The mathematical possibilities Jacobi saw in Hamilton-Jacobi theory involved partial differential equations and transformations that were intrinsic to the subject matter under consideration. It was a natural step to go from dynamical transformations to a more purely mathematical study of these objects, particularly for someone like Jacobi who was primarily a mathematician to begin with. There was a consistency and natural affinity between dynamical analysis and its development into cognate areas of mathematics.

By contrast, Clebsch's application of Hamilton-Jacobi theory to the study of the second variation involved an insertion of dynamical analysis into an area of pure mathematics with no connection to the physical domain or with applied mathematics. Dynamical analysis and the theory of the second variation were not cognate subject areas. Physicists were not interested in the investigation of necessary and sufficient conditions. Prior to Clebsch (1858b) no one had thought to apply Hamilton-Jacobi methods to the study of the second variation. It was an indication of Clebsch's remarkable originality and mathematical perspicacity that he made this connection.

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