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Calculus of variations

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1 OPTIMIZATION PROBLEMS

A basic problem of the differential calculus is to find the point at which a given curve is a maximum or minimum. An immediate generalization is posed by problems in which it is required to find a curve from among a class of curves that renders a certain integral quantity an extremum. For example, it might be required to find the curve joining two points that makes the distance between them a minimum (a straight line), or to find the curve for which the solid formed by revolving that curve about an axis yields a surface of minimum area (a catenary). In the problem of the brachistochrone, one considers a smooth curve joining two points in a vertical plane and seeks the shape that minimizes the time it takes a body to slide down the curve (a cycloid). In each of these examples the quantity to be extremalized is expressed as a definite integral of the form $\int_a^b f(x,y,y')\,dx$, where $y=y(x)$ is a function of x, and $y'=dy/dx$ is its derivative. The fundamental problem of the calculus of variations is to investigate necessary and sufficient conditions for $y(x)$ to make this integral an extremum.

The fundamental problem may be modified by demanding that the extremalizing function satisfy an auxiliary condition or constraint. One thereby obtains the so-called 'isoperimetrical problems' which figured prominently in the early history of the subject. Thus it may be necessary to find the curve with a fixed perimeter that encloses a maximum area (the circle), or to find the shape of a hanging chain of given length whose centre of gravity is lowest (the catenary).

The theory may be further modified or generalized in a number of ways: by introducing into the integrand function f a variable s of the form $s=\int_a^x g(x,y,y')\,dx$; by permitting higher-order derivatives y'', y''', and so on in the integrand; by allowing the end-points of the curve to vary; by considering more than one independent variable, so that the quantity to be extremalized becomes a multiple integral; or by formulating the problem in parametric form with a parameter and several dependent variables in the integrand.

2 SURVEY OF THE HISTORY

The calculus of variations began as a disparate set of problems and techniques in the early eighteenth century. It was synthesized into a branch of mathematics by Leonhard Euler in the 1740s and was radically reformulated by Joseph Louis Lagrange using his δ-algorithm in the next decade. Both men also used variational ideas extensively in mechanics (§8.1). In his famous calculus textbooks published at the end of the century, Lagrange returned to the subject and developed it along formal analytical principles (§3.2).

In the nineteenth century several traditions of research emerged. Siméon-Denis Poisson, Mikhail Ostrogradsky, Charles Delaunay, Frédéric Sarrus and Augustin Louis Cauchy investigated variational problems involving multiple integrals, a subject that was closely connected to problems in potential theory and mathematical physics. William Rowan Hamilton, Carl Jacobi and Joseph Liouville concentrated on the formal study of analytical dynamics by means of variational principles. Jacobi continued Legendre's analysis of the second variation along more strictly mathematical lines, and achieved highly significant results concerning necessary conditions for an extremum. His students and successors at Königsberg, Ludwig Hesse, Alfred Clebsch and Christian Mayer, extended his theory in various directions.

Throughout the nineteenth century many writers understood variational mathematics in terms of the concepts and methods of operator and formal calculus. It was Karl Weierstrass's achievement in the 1870s to systematically ground the subject in the theory of a function of a real variable (§3.3). Weierstrass also investigated the basic problems of the calculus of variations in parametric form. His work was presented in his lectures at the University of Berlin and was disseminated by his students Paul Du Bois-Reymond, K. H. A. Schwarz and, especially, Oscar Bolza.

Significant research into the early twentieth century was carried out by David Hilbert, Adolf Kneser, Hans Hahn, Constantin Carathéodory, William Osgood and Gilbert Bliss. The modern 'classical' formulation of the calculus of variations embodying the results of the German 'school' was presented in textbooks by Kneser (*1900*) and Bolza (*1904*, *1909*). Significant contributions since then (which will not be explored here) include, in the 1930s, Marston Morse's calculus of variations in the large, and the development in recent decades of optimal control theory. Classical variational mathematics continues to play an important role in engineering mechanics and physics.

In addition to the interest inherent in its development as a technical branch of mathematics, the calculus of variations is historically noteworthy for two reasons. The close relationship that existed between it and mechanics, at least until the late nineteenth century, illuminates the historical links that have connected mathematics and physics. Originating as a generalization of the ordinary calculus, variational mathematics is of foundational interest in illustrating changing conceptions of analysis. Its development by Lagrange serves as an example of the algebraic understanding of calculus that was dominant in the later eighteenth century; in the nineteenth century the different traditions that emerged indicate the diversity of research during the period and help to define the context, character and historical significance of the Weierstrassian programme of real analysis that has informed the modern organization of the subject.

3 THE EIGHTEENTH CENTURY

Early work in what later became known as the calculus of variations was part of the new infinitesimal calculus pioneered by Isaac Newton and Gottfried Wilhelm Leibniz. In the *Principia* (1687), Newton investigated the problem of the shape of a solid of revolution that yields the least resistance as it moves through a fluid in a direction parallel to its axis. A substantial British programme of variational research continued in the writings of Brook Taylor, Colin Maclaurin and Thomas Simpson into the middle of the eighteenth century. Nevertheless, it was Continental mathematicians following Leibnizian principles who contributed most decisively to the subsequent development of the subject, and it is this tradition that forms the subject of the following account.

The early Leibnizian calculus consisted of a sort of geometrical analysis in which differential algebra was employed in the study of 'fine' geometry (§3.4). The curve was analysed in the infinitesimal neighbourhood of a point, and related by means of an equation to its overall shape and behaviour. An important curve that was the solution of several variational problems was the cycloid, the path traced by a point on the perimeter of a circle as it rolls without slipping along a straight line. The cycloid could be simply described in terms of the infinitesimal calculus. Let the generating circle roll along the y-axis, and let the vertical distance be measured downward from the origin along the x-axis (Figure 1). An elementary geometrical

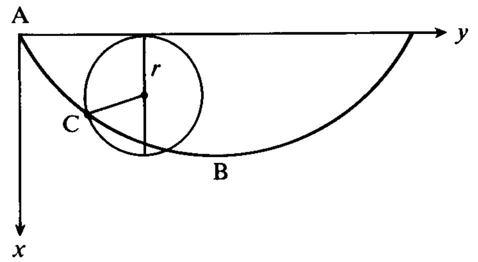

Figure 1 The cycloid

argument revealed that the equation of the cycloid is

$$\left(\frac{\mathrm{d}s}{\mathrm{d}y}\right)^2 = \frac{2r}{x}, \tag{1}$$

where $\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}y^2)^{1/2}$ is the differential element of path length.

The cycloid was most notably the solution to the brachistochrone problem. Following John Bernoulli's public challenge in 1696, solutions to this problem were proposed by his elder brother James, by John himself, and by Newton, Leibniz and the Marquis de l'Hôpital. All these people showed that the condition that the time of descent be least for the given curve leads to equation (1), and hence to the conclusion that the curve is a cycloid. John Bernoulli's solution was based on an optical analogy which, although interesting, did not lead anywhere within the subject. The analysis by his brother James was illustrative of the ideas that would develop into the calculus of variations. He considered three arbitrary, infinitesimally close points C, G and D on the given curve, and constructed a second neighbouring curve identical to the first except that the arc CGD was replaced by CLD (Figure 2). Because the curve minimizes the time of descent, it was clear that the time taken to traverse CGD is equal to the time taken to traverse CLD. Using this condition and the dynamical relation $\mathrm{d}s/\mathrm{d}t \propto \sqrt{x}$, Bernoulli was able to derive equation (1).

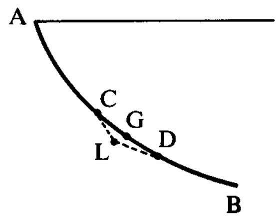

Figure 2 John Bernoulli's treatment of the brachistochrone problem

James Bernoulli also investigated isoperimetrical problems in which the extremalizing curve satisfies an auxiliary integral condition. His idea was to vary the curve at two successive ordinates, thereby obtaining an additional degree of freedom, and use the side constraint to obtain a differential equation. Following James's death in 1705, his brother John, then 38 years old, adopted his approach and developed it extensively in a paper that appeared in 1718 (Bernoulli *1991*).

In two memoirs published in the St Petersburg Academy of Sciences in 1738 and 1741, Euler extracted from the various solutions of John and Daniel Bernoulli, as well as the researches of Taylor, a unified approach to integral variational problems. These investigations were further developed and made the subject of his classic treatise *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* ('Method of Finding Curved Lines that Show some Property of Maximum or Minimum') (*1744*). Published when he was 37 years old, this book was a remarkable synthesis in which he virtually created the calculus of variations as a branch of mathematics. He realized that the different integrals in the earlier problems were all instances of the single form $\int_a^b f(x, y, y')\,\mathrm{d}x$. He derived a canonical differential equation, known today as the Euler or Euler–Lagrange equation, as the fundamental necessary condition of the variational problem:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{2}$$

This equation was obtained by investigating the expression

$$\mathrm{d}f = M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}y', \tag{3}$$

introduced in a slightly different form by Taylor in 1715, when the curve was altered at a single ordinate.

Of general interest was Chapter 4 of Euler's treatise. He noted that, although the derivation of the variational equations had been carried out with respect to an orthogonal coordinate system, the derivation and the equations themselves remained valid when alternate coordinate variables were employed. He proceeded to investigate problems using various coordinate descriptions. Euler's observation here is of considerable historical significance: it is the explicit recognition in a definite mathematical context that the essential nature of calculus is defined by its analytical character, and not by any geometrical interpretation conferred upon the formalism.

This theoretical insight notwithstanding, the analytical scope of Euler's variational calculus was restricted in practice by the detailed process involved in each individual derivation. Euler himself called for further research on algorithms that would facilitate the procedures of the subject. The call was answered in 1760 when the 24-year-old Lagrange published his

δ-algorithm in the memoirs of the Turin Society. Formalistic in spirit, his method was based on an entirely new idea involving the simultaneous variation of the range of values of the extremalizing curve. In this formulation the variational problem

$$\delta \int_a^b f(x, y, y')\,\mathrm{d}x = 0 \tag{4}$$

was reduced by means of an integration by parts to

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,\mathrm{d}x = 0. \tag{5}$$

From this equation he inferred the basic equation (2).

Lagrange showed that Euler's theory could be efficiently recovered using his new method, and demonstrated its superiority in handling problems involving variable end-points. Euler, in turn, adopted the δ-algorithm and coined the name 'calculus of variations' for the new mathematics. In the years that followed, both men published memoirs exploring the analytical possibilities of the subject. Considerably later, in the second edition of his *Leçons sur le calcul des fonctions* (1806), the elderly Lagrange attempted to develop the calculus of variations algebraically as part of his larger project of basing analysis on Taylor's theorem.

In a series of researches that began in 1760 and culminated with his *Méchanique analitique* (1788), Lagrange employed variational techniques in mechanics. He provided variational formulations for the principle of least action, the principle of virtual work and d'Alembert's principle, and derived the 'Lagrangian' differential equations of motion (§8.1). His emphasis throughout was on the formal mathematicization of mechanics as opposed to conceptual development or experimental verification.

With Lagrange's theory, the calculus of variations had completed the formative phase of its historical development (Fraser *1985*). Although later researchers would introduce new concepts, deepen the theory and extend its geometrical applications, the basic structural character of the subject was in place.

4 THE NINETEENTH CENTURY

Throughout the nineteenth century a growing number of researchers worked on the calculus of variations. In Lecat's bibliography *1913*, the number of papers increases from 65 in the period 1800–24 to 130 in 1825–50. This trend continued in subsequent decades as the subject became established as a major branch of real analysis. Although research was carried out in France, Italy, Britain and the United States, German-speaking Europe came to dominate the field, particularly in the closing years of the century.

In the early decades considerable effort was devoted to extending the established theory for single-variable integration to multiple integrals. The integration by parts in Lagrange's original derivation of equation (2) required in this case results that correspond to the later Green's theorem and the divergence theorem (§3.17). In the simplest case, where $z = z(x, y)$ and the boundary R is unvaried in the variational process, the condition

$$\delta \iint_R f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \mathrm{d}x \mathrm{d}y = 0 \tag{6}$$

leads to the partial differential equation

$$\frac{\partial f}{\partial z} - \frac{\partial f_p}{\partial x} - \frac{\partial f_q}{\partial y} = 0, \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}. \tag{7}$$

In the derivation of equation (7), a line integral appears that is zero because the variation is taken to be zero on the boundary. The problem was to find expressions for this integral when the boundary is itself varied. This question, as well as the corresponding one for triple integrals, occupied the attention of Poisson and Ostrogradsky in long memoirs published in the 1830s. The most successful assault on the problem was by Frédéric Sarrus, in a paper published in 1846. His approach was adopted and extended by Cauchy and by François Moigno.

Although there is little mention of Sarrus's theory in the modern literature, it was highly regarded by his contemporaries. It was awarded a prize by the Paris Academy in 1842, when Sarrus was 44 years old, and was judged by Isaac Todhunter *1861* to be 'the most important original contribution which has been made to the Calculus of Variations in this century'. The theory's historical context is the French interest in differential geometry of surfaces, and it forms part of the background to Gaston Darboux's own researches in this subject later in the century.

In his seminal paper of 1836, Carl Jacobi, then 32 years old and professor of mathematics at Königsberg, investigated the question of when solutions to the single-integral variational problem lead to an actual minimum. He began with a result proved by Legendre in 1786 concerning the second variation. Legendre had shown that, in addition to the Euler–Lagrange equation (2), a necessary condition that a given function $y = y(x)$ on $[a, b]$ be a minimum is that $\partial^2 f / \partial y'^2 > 0$ on $[a, b]$. He assumed in his demonstration that a certain differential equation has a finite integral on $[a, b]$. Jacobi investigated this equation further and showed that its integrals were given by differentiating the family of solutions $y = y(x, \alpha, \beta)$ of equation (2) with respect to α and β. He introduced a certain function $\Delta(x, a)$, defined in terms of these solutions, and considered values k of x for which $\Delta(k, a) = 0$. Such values are known as conjugate values with respect to the one-parameter family of solutions to equation (2) passing through $(a, y(a))$. He showed that, if there is a conjugate value k such that $k \leqslant b$, then the given arc is not a minimum; if $k > b$, he inferred, the arc is a minimum.

Jacobi provided a simple geometric interpretation for his theory in terms of the envelope to the family of extremals of the variational problem. Over the next twenty years a series of papers appeared that elaborated and extended his results.

Mathematical writers early in this century regarded Weierstrass's Berlin lectures of 1872–81 as a turning-point in the history of the subject. Bliss (*1925*: 177) wrote that 'The memoirs and treatises on the calculus of variations up to the latter part of the nineteenth century frequently leave one in doubt as to the validity and the precise character of the results which they contain.' He credited Weierstrass with bringing to the field a sense of rigour and an awareness of the need for sharp formulation of fundamental concepts (§3.3). This critical spirit was evident in the writings of Weierstrass's student Du Bois-Reymond, who in 1879 provided a rigorous demonstration of the fundamental lemma, the result that allows one to infer the validity of the Euler–Lagrange equations from the vanishing of the first variation.

Among Weierstrass's many contributions was the parametric theory, in which x and y are regarded as functions of a parameter t and the extremalizing integral is written as

$$\int_{t_0}^{t_1} f(x, y, x', y')\,\mathrm{d}t, \tag{8}$$

where $x' = \mathrm{d}x/\mathrm{d}t$ and $y' = \mathrm{d}y/\mathrm{d}t$. The class of comparison arcs here includes curves whose derivatives differ by finite amounts from the given curve. Kneser later introduced the term 'strong' variation to distinguish this case from the usual one where the variations are 'weak'. Weierstrass developed the parametric theory of the first and second variations, discovered a new necessary condition and completed important researches on sufficiency.

The strength of the parametric approach is its ability to handle geometric applications. The problem of finding a geodesic or shortest line on a surface was explored by Darboux, Ernst Zermelo and Kneser in papers published in the 1880s and 1890s. Then, in a related series of analytical researches, David Hilbert began a very general investigation of existence results and sufficiency conditions. His work brought the classical phase of the subject to a close. The modern research that led to Morse's theory, carried out to a considerable extent by Hilbert's American students, evolved by bringing his methods and 'the tools of topology to bear on the classical calculus of variations in order to develop a macro-analysis' (Goldstine *1981*: 371).

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