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LAGRANGE'S ANALYTICAL MATHEMATICS, ITS CARTESIAN ORIGINS AND RECEPTION IN COMTE'S POSITIVE PHILOSOPHY

I. Introduction

IN HIS *éloge* to Laplace, delivered in 1829, Joseph Fourier referred to the mathematical methods developed in the preceding century by Euler and Lagrange. In the course of his discussion he summarized Lagrange's mathematical style:

> The distinctive characteristic of his genius consists in the unity and grandeur of views. He was attracted above all to a simple thought, just and very elevated. . . . All his mathematical compositions are remarkable for a singular elegance, by the symmetry of forms and generality of methods, and if one may speak thus, by the perfection of the analytical style.[1]

Fourier's comments form an interesting contrast with an assessment presented two decades earlier by Jean Delambre, who was critical of the theoretical emphasis in Lagrange's astronomical work:

> [Lagrange] made of these problems, simple, common and already resolved, the same use that is made by other analysts of questions of pure curiosity, that they furnish examples of calculus and occasions to develop new analytical artifices, . . . More than once he expressed openly his wish to see purely analytical researches encouraged; and even when he proposes the greatest facility for ordinary calculations, it is principally Analysis that he brings to perfection.[2]

The comments of Fourier and Delambre highlight the central Lagrangian riddle, to characterize, and to interpret the meaning of, the distinctive and highly developed conception of analysis that is expressed in all of his work in mathematics. Lagrange never wrote a treatise on philosophy, and the occasional general remarks in his published memoirs reveal little except an unwavering commitment to analytical method. A discussion of his mathematics that is not simply technical must be based on inference and an examination of context, the background, character and later influence of his work.

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[1]J. Fourier, 'Éloge Historique à Laplace', delivered to l'Académie Royale des Sciences, 15 June, 1829; in sixth edition of Laplace's *Exposition du Système du Monde*, Vol. 1 (Paris: Bachelier, 1836), p. vi.

[2]J. Delambre, 'Notice sur la Vie et les Ouvrages de M. Le Comte J.-L. Lagrange', in: *Oeuvres de Lagrange*, Vol. 1 (Paris: Gauthier-Villars, 1867), pp. ix–li.

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Lagrange's analytical approach to pure and applied mathematics constituted the final step in a tradition of research that originated with Descartes and became consolidated in the 18th-century programme of analysis and mechanics. In the early 1800s his achievement was received and extended, sometimes in a substantially modified form, by researchers throughout Europe. Among this group was Auguste Comte, the founder of positive philosophy, who regarded Lagrange's scientific work as a model both of mathematics and of how mathematics is used in physics.

The present essay attempts to illuminate the Lagrangian riddle by examining the mathematical and philosophical origins of his conception of analysis and the quite specific interpretation it received in Comte's theory of scientific method.[3]

II. Cartesian Origins

During the 18th century, as inertial physics and Newtonian force became generally accepted, various Cartesian doctrines were discarded: the theory of vortices, the insistence on mechanical explanation and the doctrine of innate ideas. Already within mathematics Descartes' restriction of analysis to "algebraic" or non-transcendental curves had been decisively rejected by researchers of the preceding century.

Recent work in the history of science has clarified the character of 18th-century Cartesianism, showing that traditional proponents, the Cassinis and Fontenelle for example, were sceptical of theory and maintained a practical concern for observation in their work.[4] A researcher like d'Alembert, on the other hand, usually associated with the triumph of Newtonianism in France, has been shown to have harboured pronounced Cartesian tendencies in his approach to physical theory.[5]

[3]In the 19th century different foundational questions were investigated in mathematics and physics. Although physics increasingly used the methods of mathematics, its philosophical concerns — the adequacy of mechanism, the question of reduction and the validity of positivist and realist approaches — were unrelated to fundamental mathematical questions about the nature of the continuum, the need for logical rigour and the relation between logic, intuition and formalism.

This division is reflected in modern historiography of science. Although histories of physics emphasize mathematicization, questions of mathematical foundational import usually are not considered. However understandable this approach is for the post-1800 period, it must be qualified for the 18th century, when mathematical and physical conceptions were connected at a basic level in mature and technically advanced theories.

[4]See J. L. Greenberg, 'Geodesy in Paris in the 1730s and the Paduan Connection', *Historical Studies in the Physical Sciences* 13 (1983), 239–260, and R. Hahn, *The Anatomy of a Scientific Institution: The Paris Academy of Sciences 1666–1803* (Berkeley: University of California Press, 1971), pp. 32–33. Hahn refers to the position of the Paris Academy of Sciences in the early 18th century as "phenomenological positivism", a designation that is presumably not meant in a too formal philosophical sense.

[5]See R. Grimsley, *Jean D'Alembert (1717–1783)* (Oxford: Clarendon Press, 1963), and T. L. Hankins, *Jean d'Alembert: Science and the Enlightenment* (Oxford: Clarendon Press, 1970).

A relevant general characteristic of Cartesian physics has been noted in a recent study:

> The domain of investigation of mathematical physics is decided on the basis of a general metaphysical classification of what can be known. It is not decided by working within an already constituted physical theory; rather, what physical theory can and should deal with is determined prior to the constitution of the theory.[6]

Although Galileo had proclaimed that "the book of nature is written in mathematics", his contemporary Descartes assumed a stronger role for mathematics in scientific philosophy. The intrinsic mathematical character of Descartes' physics is summarized by Buchdahl:

> ... why should the concepts of the geometer (and their mathematical relations) have any significance for the relationships of extensions studied by mathematical physics? Descartes must be saying that what the geometer apprehends in body (viz. extension) is precisely what the geometer apprehends in the relationships between his figures. But how can we be sure that the space and matter of physics are thus tractable by the geometer's systems? Descartes must be saying that in seeing 'clearly and distinctly' what is involved in body, viz. extension, we *ipso facto* see also what is subject to the geometer's systematic constructions.[7]
> Descartes slides from treating mathematics as a necessary *tool* for physics to postulating it as its *object*. For, not only is the world to be studied by means of the setting up of relations between 'extensions'; the world (in its material aspect) *is* extension.[8]

In the 18th century, despite the presence of a generally accepted physical theory, an *a priori* belief in the intrinsic mathematical character of physical reality continued to underlie work in mechanics. This belief persisted when the original Cartesian requirement of mechanical explanations in terms of corporeal contact had been set aside as impracticable or unrealizable.[9]

[6] S. Gaukroger, 'Descartes' Project for a Mathematical Physics', in *Descartes, Philosophy, Mathematics and Physics*, S. Gaukroger (ed.) (Sussex: Harvester Press, 1980), pp. 97–140, p. 125.

[7] G. Buchdahl, *Metaphysics and the Philosophy of Science: The Classical Origins Descartes to Kant* (Oxford: Basil Blackwell, 1969), p. 115.

[8] *Ibid.*, pp. 116–117.

[9] C. G. Fraser in 'D'Alembert's Principle: The Original Formulation and Application in Jean D'Alembert's *Traité de Dynamique* (1743)', *Centaurus* **28** (1985), 31–61, 145–159, describes some striking examples that are presented entirely in terms of such concepts as impenetrable extension and linked geometrical systems. D'Alembert is able to avoid the concept of force because he derives the equations of motion from first principles of the calculus and mechanics. The subsequent employment by him and others of "Newtonian" analytical equations, in which force appears in an essential form, as well as the general failure by Lesage and others to develop a reasonable physical theory of gravitation, led to the abandonment of the ideal of a purely "mechanical" philosophy.

It is of course not entirely clear that the concept of force *necessarily* introduces a positivistic element into physics. A. Gabbey in 'Force and Inertia in the Seventeenth Century: Descartes and Newton', in *op. cit.*, note 6, pp. 230–262, argues that force is an integral part of Descartes' original physics. The concept is basic in modern realist scientific philosophies. The question of the logical status of force within a rational, mathematical mechanics of the sort current in the 18th century would seem to require more study.

Lagrange was a "mathematicist" (to use a term introduced by Brunschwicg) in his approach to physics; he substituted mathematical analysis for a deep absorption in the understanding of physical phenomena. He is best known for his variational formulation of mechanics and for having made mechanics, in his words, "a new branch of analysis". To understand his conception of analysis and its place in mathematical physics it is fruitful to review the thought of earlier Cartesians. Brunschwicg has noted a significant tension in Descartes' original programme:

> To extend geometry proper to problems of cosmology, and to reduce the problems of geometry to algebra; to generalize the science of Euclid in such a way as to reduce mechanics, physics and even biology to it, and to intellectualize the science of Euclid in such a way as to reduce it to algebra — these two tasks not only are not united, they appear the inverse of each other.[10]

The tension noted here becomes even more pronounced in the writings of such influential Cartesians as Nicolas Malebranche, who wished to extend Descartes' programme by a further reduction of algebra to arithmetic. There were powerful antecedents within the history of mathematics for such an emphasis. Viète had introduced the concepts of algebraic variable and parameter in his study of Diophantus' *Arithmetic.* The figures of geometry enjoyed an inherent generality that enabled theorems to be formulated; what was proven in reference to a given right-angled triangle was proved for all such triangles. In order to achieve the same generality in the theory of numbers the introduction of literal formalism was necessary. Thus Decartes spoke of a "kind of Arithmetic called Algebra, which designs to effect, when dealing with numbers, what the ancients achieved in the matter of figures." Newton's use of the term "universal arithmetic" indicated that the role of algebra was to assist investigation in the theory of numbers.

Malebranche's conception of an arithmetically-based algebra would have heightened the tension already present in Cartesian science between a non-spatial mathematical analysis on the one hand and a geometrized physics on the other. It is therefore interesting to note that Malebranche later substantially modified this conception and that a significantly altered understanding of mathematical foundation, one in which geometry occupied the significant implicit role, developed in 18th-century analysis. To the details of this story we now turn.

Malebranche's understanding of mathematics was presented in successive editions of his *Recherche de la Vérité* (1674–1712) and in his disciple Jean

[10]L. Brunschwicg, *Les Étapes de la Philosophie Mathématique* (Paris, 1912), p. 124; reprinted in 1972 with preface by J. T. Desanti.

Prestet's *Élémens des Mathématiques*, first published in 1675.[11] His early mathematical philosophy was grounded in the well known 17th-century distinction between "intellect" ("esprit") and "imagination". The imagination constituted a kind of internal sense organ that presented to the intellect sensory images retrieved from memory.[12] Geometry necessarily involved the exercise of imagination and was therefore less fundamental than arithmetic and algebra, which were "the true Logic which serves to discover the truth and to give to the intellect all the extent of which it is capable". In the *Élémens des Mathématiques* Prestet wrote

> There is nothing in this Book that appeals to the senses or to the imagination. All that is contained in it tends only to clarify the intellect. . .[13]
> . . .[the elements of arithmetic and algebra] are properly the general science or the foundation and principle of all mathematics, and not Geometry which depends in several places on a knowledge of these Elements.[14]

Malebranche and Prestet were presenting a more articulate and definite mathematical philosophy than one finds in Descartes.[15] Their emphasis on numerical algebra as the basis of mathematics in turn raised difficult technical questions. It was by no means clear what an arithmetical resolution of the problem of incommensurability and continuity would consist of. One could using geometry rigorously construct irrational quantities, but the notion of existence required for an arithmetical approach was beyond the reach of current mathematics and would not develop until the 19th century.[16]

[11]N. Malebranche, *Recherche de la Vérité ou l'on traite de la nature de l'esprit de l'homme et de l'usage qu'il en doit faire pour éviter l'erreur dans les sciences*, published in six editions beginning in 1674–1675 and ending in 1712. The standard critical edition with variant readings appears as Vols 1 and 2 of Malebranche's *Oeuvres complètes*, G. Rodis-Lewis (ed.) 1962 (V.1) and 1962 (V.2). The part on the philosophy of mathematics appears in V.2 in the sixth book. See especially pages 282–294, which contain passages that also appear in J. Prestet, *Élémens des Mathématiques, ou, principes généraux de toutes les sciences, qui ont les grandeurs pour object* (Paris, 1675).

[12]K. Park, L. Daston and P. L. Galison, 'Bacon, Galileo and Descartes on Imagination and Analogy', *Isis* **75** (1984), 327–342, p. 291.

[13]*Op. cit.*, note 11, dedication.

[14]*Ibid.*, preface, p. vi.

[15]Malebranche's arithmetical mathematics is the subject of A. Robinet, *Malebranche de l'Académie des Sciences. L'Oeuvre Scientifique, 1674–1715* (Paris: Librairie Vrin, 1970), pp. 17–45.

[16]In defence of an arithmetical treatment of irrationals Prestet *op. cit.*, note 11, preface v, writes: "Il est me semble evident que ce nombre $\sqrt{20}$ est beaucoup plus connu que la soûtendante d'un angle droit dont les côtez sont 2 & 4. Car on sçait ou moins que $\sqrt{20}$ est environ $4\frac{1}{2}$, & si on le veut sçavoir plus au juste, on le sçaura par les règles de l'approximation des racines. Mais on ne sçait pas la grandeur de la ligne qui soutient un angle droit, quoy qu'elle soit presente aux yeux on a l'imagination." This argument is made less convincing by the fact that he is unable to provide an arithmetical definition of incommensurable magnitudes, or even to show in what sense they exist. Later in the treatise (p. 101) he admits that "l'on ne puisse en avoir une connoissance exacte & parfait, cependant elles sont tres-réelles, & la Géométrie nous fournit les moyens de les déterminer exactement par lignes." Thus geometry establishes the existence of a magnitude by actually constructing it, and then provides the means (Pythagoras's theorem, theory of odd and even numbers) to show that it is incommensurable relative to some line segment taken as unit.

A more general problem concerned the direction of contemporary mathematical research. The deeper problems of mathematics involved the curve, an object whose properties were of increasing interest in geometry, mechanics, astronomy and optics. The Cartesian physicist Christian Huygens was developing a geometrical mechanics based on the detailed investigation of such special problems as pendulum motion and the collision of bodies.[17] A programmatic emphasis on arithmetic seemed either at odds with or irrelevant to such a project.

The curve was also the primary object of study in the Leibnizian calculus, a branch of mathematics that employed the methods and symbolism of analytic geometry. Malebranche played an enlightened role in promoting the new calculus in Paris, defending research against a conservative and obtuse Cartesian opposition.[18] In the later editions of *La Recherche*, and especially in the edition of 1712, he revised the text in order to modify his earlier emphasis on arithmetic. Gone were the passages that asserted the primacy of arithmetical algebra over geometry; arithmetic now referred only to calculation on whole numbers. The term analysis was distinguished from algebra and the theory of equations, referring more generally to the solution of mathematical problems, particularly ones that required the infinitesimal calculus.

The effect of the development and consolidation of the infinitesimal calculus was to shift the underlying focus of analysis away from arithmetical algebra to a geometrical theory of magnitude based in the geometry of curves.[19] This paradigm was reinforced by the close association in which Continental analysis and mechanics developed in the 18th century.

Jean d'Alembert was a major 18th-century researcher in analysis and mechanics whose early career was directly linked to the Malebranche circle.[20] In the Preliminary Discourse to the *Encyclopédie* (1751), in his philosophical discussion of the conceptual foundations of mathematics, he accepted (against Descartes and Malebranche) the Lockean doctrine that ideas originate in our sense impressions of the external world.[21] The objects of arithmetic, geometry

[17]See H. J. M. Bos, introduction to Christian Huygens' *The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks*, transl. with notes by Richard J. Blackwell (Ames: Iowa State University Press, 1986), p. xvi, and E. J. Dijksterhuis, *Mechanization of the World Picture* (Oxford: Clarendon Press, 1961), pp. 368–379, 457–462.

[18]See Robinet, *op. cit.*, note 15, pp. 47–65.

[19]L. Nový, in *Origins of Modern Algebra* (Leyden: Noordhoff International Publishing, 1973), p. 16, distinguishes two concepts of algebra in the 18th century: "One of these considered algebra to be a science of equations and of their solutions, and the other a science of quantities in general." Nový is not concerned in his study with calculus-related applications of symbolic formalism ("analysis") characteristic of advanced research during the period. Thus the different senses of algebra that he identifies must be distinguished from the concept of analysis associated with the calculus.

[20]Hankins, *op. cit.*, note 5, chapter 2.

[21]J. D'Alembert, *Discours Préliminaire de l'Encyclopédie* (1751). Critical edition with variant readings, F. Picavet (ed.) (Paris, 1894).

and mechanics, he explained, are obtained by abstraction from space and the material bodies about us. This conception referred only to the origins of mathematical ideas and in no way implied that the foundations of the subject were subject to empirical verification. One did not need to make observations and perform measurements in order to validate geometrical theorems. Mathematics — including geometry and mechanics — was for d'Alembert a deductive, rational science, based on *a priori* truths about number, extension and time. Its method, moreover, was decidedly Cartesian, involving the generation of the whole subject from a small number of clear and distinct propositions.[22]

Lagrange's conception of analysis developed against this general mathematical and philosophical background.[23] His contribution consisted of a highly distinctive emphasis on analysis, understood in a rather formal, uninterpreted sense, as a means to unify the different branches of mathematics, both pure and applied. Throughout his writings he notes its successful application to problems that had traditionally demanded synthetic solutions.[24] Analytic procedures based on abstract relational operations and involving no intervention of visual imagery appealed to his sense of mathematical taste.

In addition to its methodological usefulness, Lagrange's viewpoint enjoyed strong support at the very foundation of the calculus. An analytic equation implied the existence of a relation that remained valid as the variables changed continuously in magnitude. Analytic algorithms and transformations presupposed a correspondence between local and global change, the basic consideration in the application of the calculus to the curve.[25]

[22]The account here disagrees with the interpretation presented by M. Förster, *Beiträge zur Kenntnis des Characters und der Philosophie d'Alemberts* (Hamburg, 1892), and G. Misch *Zur Enstehung des französischen Positivismus. I. Die philosophische Begründung des Positivismus in den Schriften von d'Alembert und Turgot* (Berlin, 1900). Förster (p. 90) writes "Wir finden bei d'Alembert nicht nur Andeutungen, die auf den Positivismus hinweisen, sondern wir sehen die hauptsächlichsten Züge desselben im Keime vorhanden." The supporting argument which follows this assertion is I believe based on an insufficiently detailed characterization of d'Alembert's mathematical philosophy.

[23]Lagrange's *Oeuvres* were published in 14 volumes from 1867 to 1892. Didactic works of note here are the *Méchanique Analitique* (1788), the second edition of which appeared in two volumes as the *Mécanique Analytique (1811, 1815)*, reprinted as *Oeuvres* 11 and 12; *Théorie des Fonctions Analytiques* (1797), second edition 1813, reprinted as *Oeuvres* 9; *Leçons sur le Calcul des Fonctions* (1801), reissued in 1804 in *Journal de l'École Polytechnique*, 12 cahier, tome 5, second edition of 1806 includes 'un traité complet du calcul des variations', reprinted as *Oeuvres* 10.

The early development of Lagrange's theory is described by C. G. Fraser, 'J. L. Lagrange's Early Contributions to the Principles and Methods of Mechanics', *Archive for History of Exact Sciences* **28** (1983), 197–241, and 'J. L. Lagrange's Changing Approach to the Foundations of the Calculus of Variations', *Archive for History of Exact Sciences* **32** (1985), 151–191.

[24]This theme is especially prominent in his writings of the early 1770s. See 'Sur le Problème de Kepler', *Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, t.xxv, 1771, *Oeuvres* 3, pp. 113–138, 'Nouvelle Solution du Problème du Mouvement de Rotation d'un Corps de Figure Quelconque qui n'est animé par Aucune Force Accélératrice', *Nouveaux Mémoires de l'Académie ... de Berlin*, année 1773, *Oeuvres* 3, pp. 579–616, 'Sur l'Attraction des Sphéroides Elliptiques', *ibid.*, pp. 619–658.

[25]C. G. Fraser, 'Joseph Louis Lagrange's Algebraic Vision of the Calculus', *Historia Mathematica* **14** (1987), 38–53.

The emphasis on analysis was also connected to the 18th-century conception of how generality is achieved in mathematics. A part of mathematics, the calculus for example, was regarded as a kind of instrument that possessed a given range of application. The existence of a relationship among variables implied an extended domain of validity; an analytical method or algorithm implied a uniform and general mode of operation.[26] The analytical character of mathematics conferred upon it its unity and generality:

> The abstract ideas ... [of] mathematics correspond, in fact, to a sort of "nature" which is manifested in their cohesion and order. The internal coherence of mathematical notions extends beyond the simple logical verifications that our demonstrations furnish. In the order that is assigned to them by their function, they manifest a unity characteristic of intrinsic "existence" ... the spirit of analysis consists not in demonstrating the validity of the transformations implied, but in defining correlatively their functioning and their domain ... [the principles of mathematics announce] the unity of mathematical existence which our theorems indicate by certain properties but cannot limit the autonomous development of.[27]

An essential logical feature of analysis consisted of its abstract or uninterpreted character. Lagrange himself drew attention to the non-visual style of his mathematics in the preface to the *Mécanique Analytique*. If one looks to the broader mathematical context, however, it is clear that geometrical objects — curves and surfaces — constituted the substratum for 18th-century work in analysis. Lagrange, and Euler before him, in separating the calculus from geometry, accepted as a matter of faith, as a sort of philosophical principle, the continued applicability of analysis to the geometry of curves and surfaces.

Malebranche's early notion of an arithmetical foundation had fallen by the wayside and would not be revived until the 19th century. Although one can discern arithmetical tendencies in Lagrange's late work, the fundamental orientation of his mathematics remained toward the concept of algebraic analysis, understood in its 18th-century geometrical context.[28]

Mechanics was regarded either as a development of geometry obtained by the introduction of the concepts of impenetrability and time, or as something that, like geometry, was reducible to the study of relations among analytical

[26]C. G. Fraser, 'The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century', *Archive for History of Exact Sciences* **39** (1989), 317–335.

[27]P. Ducassé, *Essai sur les Origines Intuitives du Positivisme* (Paris: Félix Alcan, 1939), p. 71. Ducassé's comments appear in a distinctively philosophical discussion of the mathematical background to Comte's positivism. Although this author does not address points of technical mathematical detail, he succeeds in conveying a reasonable sense for the Lagrangian notion of analysis.

[28]J. V. Grabiner, in *The Origins of Cauchy's Rigorous Calculus* (Cambridge, MA: M.I.T. Press, 1981), draws attention to sections of Lagrange, *op. cit.*, note 23 (1797, 1801 and 1806), that quite probably provided some of the background for Cauchy's formulation in the 1820s of an arithmetical foundation for the calculus. Fraser, *op. cit.*, note 25, emphasizes the more fundamental formal, algebraic character of Lagrange's theory and contrasts the latter with Cauchy's arithmetical approach.

variables. There is nothing in Lagrange's emphasis on analysis to suggest that he questioned the received philosophical understanding of geometry and mechanics. The logical status of these subjects was clear; their basic propositions were necessary truths, derivable *a priori* from the principles of reason.

III. Reception in Comte's Positive Philosophy

The larger part of volume one of Auguste Comte's *Cours de Philosophie Positive* (1830) is devoted to mathematics.[29] Comte wished to present an expository, descriptive account of the subject that would convey its main character to a general reader. His work, unusual for the period in its philosophical concerns, differs from later similar investigations in attempting to provide an overview of the entire subject, including its most advanced parts.[30]

Comte's outlook reflected the general feeling, expressed in France at the end of the 18th and in the early 19th century, that mathematics had reached a more-or-less completed state. He writes

> Mathematics is now sufficiently developed, both in itself and as to its most essential application, to have arrived at that state of consistency in which we ought to strive to arrange its different parts in a single system, in order to prepare for further advances. We may even observe that the last important improvements of the science have directly paved the way for this important philosophical operation, by impressing on its principal parts a character of unity which did not previously exist; such is eminently and beyond all comparison the spirit of the works of the immortal author of *Théorie des Fonctions* and *Mécanique Analytique*.[31]

Comte regarded himself as developing methodological implications inherent in Lagrange's mathematics. His treatise shares family resemblances with works of several European researchers of the time, George Peacock in England, Martin Ohm in Prussia and Bernhard Bolzano in Bohemia.[32] All of these people on philosophical grounds separated geometry from arithmetic and

[29]The fifth edition, identical to the first, appeared in 1907 (Paris: Reinwald). Page references are to this edition. An English translation by W. M. Gillispie of large parts of volume one appeared in 1851 (New York: Harper & Brothers) under the title *The Philosophy of Mathematics*. Omitted are the initial chapters on method and the chapter on mechanics. The translation is slightly abridged in places. Unless otherwise indicated I have used this translation. All other translations from the French in the present essay are my own.

[30]In addition to elementary mathematics, Comte discusses such topics as Lagrange's theory of singular solutions to differential equations and Fourier's results on the representation of "discontinuous" functions by trigonometric series. He is also familiar with advanced work of Abel and Jacobi in the theory of elliptic functions.

[31]*Op. cit.*, note 29, p. 65.

[32]H. Pycior, 'George Peacock and the British Origins of Symbolic Algebra', *Historia Mathematica* **8** (1981), 23–45, B. Bekemeier, *Martin Ohm (1792–1872): Universitäts- und Schulmathematik in der neuhumanistischen Bildungsreform* (Göttingen: Vandenhoek & Ruprecht, 1987), and V. Jarník, *Bolzano and the Foundations of Mathematical Analysis*, translation of Bolzano's papers with introduction by J. Folta (Prague: Society of Czechoslovak Mathematicians and Physicists, 1981).

algebra. With the partial exception of Bolzano, all wished to base the calculus on formal, algebraic principles. All considered Lagrange the leading mathematical thinker of the preceding period and all viewed themselves as developing conceptions contained in his work.

What distinguished Comte among this group was his desire to integrate mathematics into a general system of positive knowledge. His conception of mathematics was strongly shaped by his empirical philosophy of science, which rejected "metaphysics" and posited prediction and verification as criteria to separate science from descriptive, fact-finding enterprises.[33] Mathematics, he asserts in *La Philosophie Positive*, is concerned with "the *indirect* measurement of magnitudes" and seeks "*to determine certain magnitudes from others by means of relations between them*"; it links unknown quantities and "those which admit of direct measurement".[34] He continues:

> . . . the spirit of mathematics consists in always regarding all the quantities which any phenomena can present, as connected and interwoven with one another, with the view of deducing them from one another.[35]

Comte distinguishes between "abstract" mathematics, or "calculation" ("*calcul*"), and "concrete" mathematics, whose character is "experimental, physical, phenomenal". Abstract mathematics is comprised of arithmetic, algebra and the calculus, the latter being regarded as a sort of algebra.[36] Concrete mathematics consists of geometry and mechanics, although this demarcation is not final, and could in time be extended to "thermology" or any other discipline that has become mathematicized.

Geometry and mechanics are for Comte empirical subjects that derive relations among physical, measured magnitudes. (The examples he gives as illustration are the law of falling bodies and the relationship among the sides and angles of a triangle.) Given these relations, the role of abstract mathematics is to obtain expressions for the magnitudes, and to evaluate these expressions numerically. Comte summarizes the character of abstract and concrete mathematics:

[33]L. Laudan, in 'Towards a Reassessment of Comte's 'Méthode Positive' ', *Science and Hypothesis: Historical Essays on Scientific Methodology* (Dordrecht: Reidel Publishing Company, 1981), pp. 141–162, notes (p. 141) that "remarkably little has been written about the details of Comte's theory of scientific method and his philosophy of science . . . it seems to me that such an inquiry is worth undertaking . . . his views on certain questions are both original and perceptive."

[34]*Op. cit.*, note 29, p. 71, Comte's emphasis.

[35]*Ibid.*, p. 71.

[36]A common feature in the work of Comte, Ohm and Bolzano (note 32) was their explicit separation on philosophical grounds of geometry from arithmetic and algebra, and their grouping of the differential and integral calculus with the latter. Given that the three worked independently and followed different approaches, this commonality is rather striking. Only Bolzano, by providing an analysis of arithmetical continuity, was able to develop a theory that would become a permanent part of mathematics.

The abstract part [of mathematics] alone is then purely instrumental, and is only an immense and admirable extension of natural logic to a certain class of deductions. On the other hand, geometry and mechanics must be viewed as real natural sciences, founded on observation, like all the rest. . . .[37]

Comte stresses that the difficult and primary problem of abstract mathematics is to obtain suitable analytical expressions from the relations furnished by the phenomena; their actual numerical evaluation occupies a secondary role. These expressions contain functions that are built up from elementary algebraic and transcendental functions using composition. In the course of the derivation it may be necessary to introduce "magnitudes", such as imaginaries, that have no direct numerical interpretation:

. . . functions may refer not only to the magnitudes which the problem presents of itself, but also to all the other auxiliary magnitudes which are connected with it, and which we will often be able to introduce, simply as a mathematical artifice, with the sole object of facilitating the discovery of the equations of the phenomena.[38]

He later elaborates on the place of imaginaries:

. . . since the spirit of mathematical analysis consists in considering magnitudes in reference to their relations only, and without any regard to their determinate value, analysts are obliged to admit indifferently every kind of expression which can be engendered by algebraic combinations. The common embarrassment on this subject seems to me to proceed essentially from an unconscious confusion between the idea of *function* and the idea of *value*, or, what comes to the same thing, between the *algebraic* and the *arithmetical* point of view.[39]

The view that mathematics introduces formal expressions whose only purpose is to facilitate calculation occurs again in Comte's account of the calculus. He regarded infinitesimals and derivatives as auxiliary quantities

uniformly associated with those which are the proper object of the investigation, substituted expressly to facilitate the analytical expression of the mathematical laws of the phenomena, although they must be finally eliminated by means of a special calculation.[40]

The introduction of derivatives was simply a "general logical artifice", whose necessity led Comte to call transcendental analysis "the calculus of indirect functions".

Comte's account of the calculus was accompanied by a historical summary of the major stages in its development. He regarded Lagrange's *Théorie des Fonctions Analytiques* (1797), a work that used Taylor's theorem to base the

[37] *Op. cit.*, note 29, p. 62. This passage is not included in the English translation of 1851.
[38] *Ibid.*, p. 118.
[39] *Ibid.*, p. 118.
[40] *Ibid.*, p. 142.

calculus on algebraic analysis, as being the most satisfactory presentation of the subject. His own contribution was the observation that the calculus involves "indirect" procedures, an idea of questionable intrinsic value and little influence on later mathematics.

In Comte's mathematical philosophy the concrete part of the subject occupies a logically prior place, in the sense that it derives from the phenomena the initial magnitudes and relations that are subjected to further analysis and computation. Comte seemed to believe that the conceptual and deductive character of geometry and mechanics was adequately expressed by their role in this process. He emphatically rejects the older rational understanding of geometry:

> ... most minds at present conceive [geometry] to be a purely logical science quite independent of observation. It is nevertheless evident, to any one who examines with attention the character of geometrical reasoning, even in the present state of abstract geometry, that ... there always exists with respect to every body studied by geometers, a certain number of primitive phenomena, which, since they are not established by any reasoning, must be founded on observation alone, and which form the necessary basis of all the deductions.[41]

In contrast to d'Alembert, for whom geometrical ideas were abstractions originating in our most basic experience of the world outside the self, Comte emphasizes that geometry is a science of *observation*, of the *measurement* of extension. He dismisses those "fantastic discussions of metaphysicians" in which geometrical entities are presented as extra-sensory abstract objects.

When he turns to mechanics Comte vigorously denounces the widespread use of "ontological considerations". Mechanics, more so even than geometry, is a science of observation, of "general facts", and the attempt by such earlier researchers as Daniel Bernoulli, d'Alembert and Laplace to provide analytical demonstrations of dynamical propositions (composition of forces, law of inertia) is entirely misguided.

Comte presents a non-technical, rather conventional survey of mechanics, basing it on Lagrangian variational principles. He emphasizes that this subject does not concern itself with the causes or origins of forces, but only with the motions they produce, a point of view that would become an important tenet of positivist physical science.

IV. Conclusion

Comte's account of mathematics derived from his interpretation of its role in empirical science. Based on a study of mathematical method within this context rather than on conceptual analysis, his account could not provide a

[41] *Ibid.*, p. 193.

serious philosophy of mathematics, and perhaps was not intended to do so. It was hardly satisfactory as philosophical doctrine to suggest that concrete mathematics derived from the phenomena relations that were then analyzed by arithmetic and algebra. The concepts and results of the latter were presumably involved in the original derivation and it would be necessary to provide an account of this. More generally, his instrumental conception of "abstract" mathematics and his characterization of geometry and mechanics as "observational" sciences prevented a clear statement of foundational questions. The problem of arithmetical continuity alone was entirely intractable within such an approach.

During the 1820s, at the time positive philosophy was being created, Cauchy in a series of treatises began to put the calculus on a rigorous arithmetical basis.[42] With the acceptance and consolidation of his theory, which replaced Lagrange's earlier foundation, Comte's account of mathematics was superseded as an advanced presentation of the subject.

Although Lagrange's notion of analysis was rejected by Cauchy and later researchers, it functioned coherently within his mathematics and in the broader context of 18th-century exact science. His consistent style and impressive technical achievements imparted to the notion a substantiality and led to a unity of mathematical and physical conception not evident in later physics. The ontology implicit in his analysis, involving a theory of magnitude based in the geometry of curves, provided a reasonable technical foundation and answered to contemporary rational views about the nature of mathematics.

Lagrange inherited a philosophical tradition that viewed nature as an intrinsically mathematical construction. In the 1830s Siméon Poisson criticized his analytical mechanics for being too abstract and too idealized, proposing in its place a programme of molecular physics, exemplified by his own work on elasticity.[43] It is in the context of Poisson's research that Comte's achievement should be evaluated. He provided a philosophical reinterpretation of Lagrange's mathematics that adapted it to the needs of phenomenal empirical science, thereby establishing the basic orientation of 19th-century positivist physics.

While Comte was informed of contemporary mathematical research, he underestimated the possibilities of the subject. His personal situation put him at odds with Poisson and Cauchy, whose work he was unable or unwilling to

[42] See Grabiner, *op. cit.*, note 28, and I. Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (Cambridge, MA: M.I.T. Press, 1970).

[43] See D. H. Arnold, *The Mécanique Physique of Siméon Denis Poisson: The Evolution and Isolation in France of his Approach to Physical Theory* (University of Toronto Ph.D. dissertation, 1978). A revised version of this work appeared as a series of articles in the early 1980s in the *Archive for History of Exact Sciences*.

understand.[44] Foundational questions, increasingly important in the 19th century, hardly exist in his instrumental conception of mathematics. Finally, he did not properly appreciate the resources that mathematics offered to physical theory, a failing characteristic of later positivist physics.[45]

[44]One of the most striking things about Comte's mathematical formation was that although he attended Cauchy's lectures at the École Polytechnique and later was associated with this institution as a teacher of mathematics, he never had any inkling of the significance of Cauchy's analysis. H. Gouhier in *La Vie d'Auguste Comte* (Paris: Vrin, 1965) and P. Tannery, 'Auguste Comte et l'Histoire des Sciences', *Revue Générale des Sciences Pures et Appliquées* **16** (1905), 410–417, p. 411, comment on Comte's isolation from contemporary mathematics and physics. Comte expressed his bitterness in a letter of 1 May, 1841 to his friend Valat: "Du reste, je suis, au fonde, de ton avis sur le caractère peu rationnel de notre prétentieuse et superficielle époque, ou les vaines gambades de nos étroites spécialités usurpent, même en mathématique (et surtout en mathématique), la juste considération due aux conceptions réelement scientifiques, dont le temps est en effet passé, comme tu l'as fort judicieusement senti. Nous avons maintenant la monnaie des Lagrange, des Monge, des Fourier etc., dans les Poisson, les Cauchy etc.! Cette dégéneration graduelle ne peut que persister provisoirement, jusqu'à ce que la culture des sciences soit enfin régénérée d'après l'ascendant rationnel de la philosophie finale: jusque'là, les marchands en détail pourront continuer à faire aisément de grosses fortunes avec de bien futiles matériaux". From *Correspondance Générale et Confessions, Tome 2, 1841–1845*, P. E. de Berrêdo Carneiro and P. Arnauld (eds) (Paris, 1975), p. 8.

[45]Although the last generalization would require further documentation, the views of a leading positivist such as Mach, as presented in his *Die Mechanik in Ihrer Entwicklung historisch-kritisch Dargestellt* (Prague, 1883) (transl. Thomas J. McCormack, *The Science of Mechanics: A Critical and Historical Account of Its Development* [1893] (Lasalle: Open Court, 1960)), are suggestive. His understanding of mathematics and the role of infinitesimal calculus in physics is largely that of Comte; Lagrange is also for him the model mathematical physicist. In his discussion of the calculus of variations Mach seems to lack any sense for the well-established contemporary conception of the subject.