D'Alembert's Principle: The Original Formulation and Application in Jean d'Alembert's *Traité de Dynamique* (1743)

by

CRAIG FRASER*

Introduction

In 1743 the young French geometer Jean d'Alembert published his work *Treatise on Dynamics, in which the Laws of Equilibrium and Motion of Bodies are reduced to the smallest possible number and demonstrated in a new manner, and where a general Principle is given for finding the Motion of several Bodies which act on one another in any way*. D'Alembert's "general Principle" has since become the object of considerable celebration and misunderstanding in the history of mechanics. Although Truesdell [1960, 186–192] and Szabo [1979, 31–43] have done much to dispel this misunderstanding, their accounts re-

* Institute for the History and Philosophy of Science and Technology, University of Toronto, Room 36, Victoria College, Toronto, Canada M5S 1K7.

Centaurus 1985: vol. 28: pp. 31–61.

main unsatisfactory as a description of d'Alembert's original procedure. In Part One of the following study I shall describe how d'Alembert formulates and applies his principle in the *Traité de Dynamique*. In Part Two, to be published in the next issue of *Centaurus*, I shall examine several special uses which d'Alembert makes of his principle in the *Traité*. Part Two will also explain the origin of the standard interpretation of d'Alembert's principle.

The *Traité de Dynamique* consists of a preface and two parts.[1] In the Preface d'Alembert describes his philosophy of mechanics and outlines the plan of the *Traité*. In Part One he presents three laws of motion and provides arguments to justify their status as fundamental truths of mechanics. In addition, he discusses in detail three special topics: the proper measure of accelerative force and its role in central force problems; the motion of bodies changed by obstacles; the properties of the angular lever. Part One is written in an expository manner at a reasonably elementary level; it should, d'Alembert informs us, be accessible to "commençants" or beginners.

Part Two of the *Traité* opens with a statement of 'd'Alembert's principle' followed by three chapters in which it is applied. The most important of these, Chapter Three, consists of fourteen problem sets with detailed solutions. Here d'Alembert treats mechanical questions that had arisen in the earlier work of such geometers as James Hermann, Leonhard Euler and James, John and Daniel Bernoulli. The technical demands of this chapter are such as to restrict its audience to those at the forefront of research in rational mechanics. The *Traité* closes with a discussion of the celebrated principle of live forces.

In 1758 d'Alembert published a second edition of the *Traité* in which he incorporates some results of research completed after 1743 and expands his earlier treatment of selected topics. The second edition also contains sixty explanatory notes supplied by d'Alembert's young contemporary Étienne Bezout. Although the later edition clarifies in places points unclear in 1743, the overall structure of the *Traité* is not substantially altered.

a. D'Alembert's Principle

D'Alembert opens Part Two of the *Traité* with a statement of the principle he believed would provide a method for solving all problems of dynamics:

General Principle

Given a system of bodies arranged mutually in any manner whatever; let us suppose that a particular motion is impressed on each of the bodies, that it cannot follow because of the action of the others, to find that motion that each body should take.

Solution

Let A,B,C, etc. be the bodies composing the system, and let us suppose that the motions a,b,c, etc. be impressed on them, and which be forced because of the mutual action of the bodies to be changed into the motions $\bar{a},\bar{b},\bar{c}$, etc. It is clear that the motion a impressed on the body A can be regarded as composed of the motion $\bar{a}$ that it takes, and of another motion α; similarly, the motions b,c, etc. can be regarded as composed of the motions $\bar{b},\beta;\bar{c},x$; etc.; from which it follows that the motions of the bodies A,B,C, etc. would have been the same, if instead of giving the impulses a,b,c, one had given simultaneously the double impulses $\bar{a},\alpha;\bar{b},\beta;\bar{c},x$; etc. Now by supposition the bodies A,B,C, etc. took among themselves the motions $\bar{a},\bar{b},\bar{c}$, etc. Therefore the motions α,β,x, etc. must be such that they do not disturb the motions $\bar{a},\bar{b},\bar{c}$, etc., that is, that if the bodies had received only the motions α,β,x etc. these motions would have destroyed each other and the system would remain at rest.

From this results the following principle for finding the motion of several bodies which act on one another. Decompose the motions a,b,c etc. impressed on each body into two others $\bar{a},\alpha;\bar{b},\beta;c,x$; etc. which are such that if the motions $\bar{a},\bar{b},\bar{c}$, etc. were impressed alone on the bodies they would retain these motions without interfering with each other; and that if the motions α,β,x were impressed alone, the system would remain at rest; it is clear that $\bar{a},\bar{b},\bar{c}$ will be the motions that the bodies will take by virtue of their action. [1743, 50–51] [1758, 73–75]

D'Alembert appears to have derived the idea for his principle from the mechanics of impact, a subject which figures prominently in his discussion of the foundations of dynamics in Part One. In the chapter "On Motion Destroyed or Changed by Obstacles" he considers a "hard" particle which strikes obliquely a fixed impenetrable wall (Figure 1). (The concept of "hard" body is a central one in d'Alembert's mechanics. A hard body is impenetrable and non-deformable. Such bodies would today be treated analytically as perfectly *inelastic*.) Decompose the particle's pre-impact velocity u into two components v and w parallel and perpendicular to the wall. D'Alembert argues using a form of the principle of sufficient reason that w must be destroyed. (Assume the particle strikes the wall with perpendicular velocity w. Clearly no forward motion is possible. Thus the post-impact velocity is $-nw$, where n is a non-negative number. Since there is no reason why n should have any one positive value rather than another n must be zero.) Hence the post-impact velocity of the particle is the

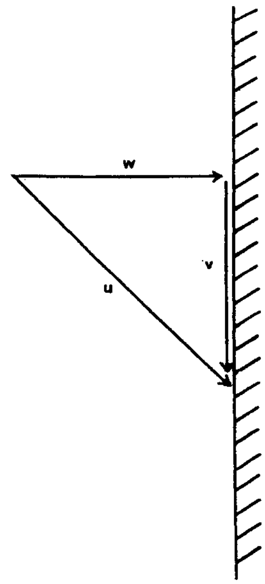

Figure 1. (Based on Figure 9, *Traité* (1743)).

component v of u parallel to the wall. The velocities u, v and w of the particle correspond to the motions $a, \bar{a}$ and α of A in the statement of d'Alembert's principle.

In Problem IX of Chapter Three of Part Two d'Alembert applies his principle to the collision of two hard bodies m and M. Assume m and M collide with velocities u and U directed along the line joining their centers. It is necessary to find the velocities after impact. D'Alembert writes $u = v + u-v$ and $U = V + U-V$, where v and V are the post-impact velocities of m and M. The quantities u, v, and $u-v$ correspond to the impressed, actual and "lost" motions of the body m; a similar interpretation holds for the body M. Because the actual motions v and V are followed unchanged v must equal V. In addition, the application of the "lost" velocities $u-v$ and $U-V$ to m and M must produce equilibrium. By the rule for equilibrium presented in Part One $m(u-v) + M(U-V) = 0$. (D'Alembert had "demonstrated" this rule using the properties of hard bodies and the laws of reason.[2] See Hankins [1970, 186–187].) Hence v or V is equal to $(mu+MU)/(M+m)$.

b. Some Mathematical Background

In those problems of Part Two of the *Traité* that involve continuously accelerated motion d'Alembert derives differential equations to describe the motion of the bodies of each system. He does so using his principle and the methods of the Leibnizian calculus. This calculus differs in important ways, both conceptual and technical, from today's subject. To understand his application of his principle in Part Two it will therefore be necessary to examine as background some technical features of the Leibnizian calculus. (A more detailed historical account is provided in Bos [1974].)

The Leibnizian calculus in the first half of the 18th century consisted of an algebraical theory that was interpreted geometrically. The algebra comprised a set of rules and algorithms that governed the use of the symbol d and was based on two postulates: $d(x+y) = dx+dy$ and $d(xy) = ydx+xdy$. The differential algebra was used to analyze the properties of a curve, the primary object of study in the calculus. The differential dx was set equal to the difference of the value of x at two consecutive "infinitely" close points in the geometrical configuration. Higher order differentials were set equal to the difference of successive lower order differentials. Euclidian geometry and the algebraic procedures of the calculus were used to derive a differential equation to describe some property of the curve.

The dual algebraical and geometrical character of the Leibnizian calculus was reflected in mathematical dynamics in the alternate ways accelerative force was measured. The effect of a force acting on a freely moving particle could be measured analytically by a relation of the form $dde = \varphi dt^2$, where e is the distance travelled by the particle, t is the time and φ is an algebraic expression composed from the several variables of the problem. Alternatively, the effect of the force might be given directly in geometry by a small line representing the motion imparted to the particle during an instant by the force.

Some of the issues involved in the geometric interpretation of the differential algebra arise in the section "On the Comparison of Accelerative Forces" of Part One of the *Traité* ([1743, 20–22], [1758, 22–34]). (The background to this section concerns a dispute that had occurred in earlier 18th century discussions of central force problems. Details of this controversy and its relation to d'Alembert's work are

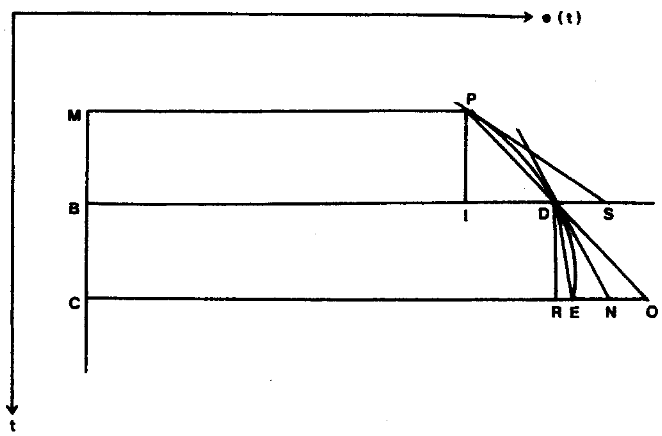

Figure 2. (Based on d'Alembert's supplementary Figure 3, *Traité* (1758)).

provided in Hankins [1970, 222–232].) Assume a force acts on a freely moving particle. Let e be the distance travelled by the particle in time t. D'Alembert plots e as a function of time to obtain a curve PDE (Figure 2). The letters M, B, C represent equally separated infinitesimally close times. The points P, D, E are the corresponding points on the curve. It is possible to treat this curve in two ways: rigorously, as the curve is actually given; or polygonally, made up of infinitesimal chords joining the points P to D and D to E. D'Alembert uses the terms "Courbe rigoureuse" and "Courbe polygone" to describe this distinction. The distinction has implications for two questions: the calculation of the effect of the force; the calculation of the second differential.

Consider first the question of the effect of the force. The two methods of treating the curve lead to different measures of this force. Let N be the intersection of the tangent at D with the line CR extended. The effect of the force when the curve is treated rigorously is defined to be the distance NE. If O is the intersection of the extension of the chord PD with CR extended then the quantity OE is the measure of the force's effect when the curve is treated polygonally. D'Alembert supposes the curve is suitably approximated at the point D by its circle of curvature; he then uses properties of the circle to establish the identity $OE = 2(NE)$. He remarks that either method of estimating the effect

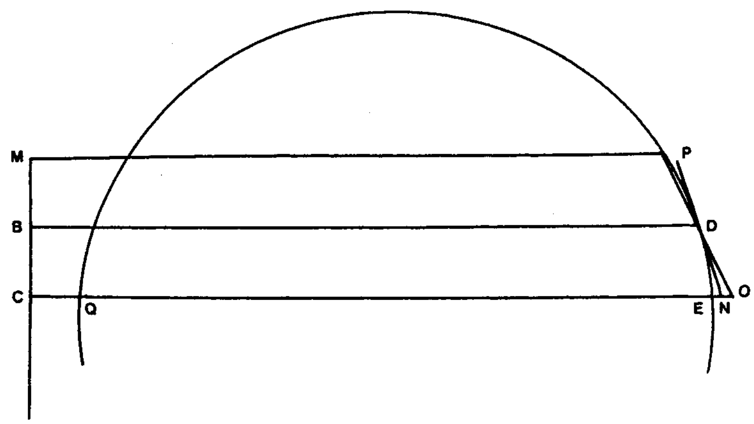

Figure 3. (Based on d'Alembert's Figure 5, *Traité* (1743)).

of the force is valid as long as one is consistent. In comparing the effects of several forces a decision must be made to treat the curve rigorously or polygonally; the two resulting sets of values will differ by a factor of two.

To obtain $OE = 2(NE)$ d'Alembert supposes the circle of curvature at the point D contains the three points P, D and E. Let Q be the second intersection of this circle and the line CO (Figure 3). By the familiar property of the circle (Euclid III (39)) $(DN)^2 = (NE)(NQ)$ and $2(DO)^2 = (OD)(OP) = (OE)(OQ)$. In the following calculations third and higher order infinitesimals are neglected. Included in the implicit axioms of the Leibnizian calculus was the assumption that the difference of successive first order infinitesimals is second order (see our discussion of d'Alembert's order analysis in section c(i)). Thus $RE-ID$ is second order, from which it can be established that the difference between DN and DO is second order. Therefore $2(NE)(NQ) = (OE)(OQ) = OE(NQ+NO)$. But $OE \cdot NO$ is third order, so that $2NE = OE$.

We can understand this result if we designate the times at M, B and C t, $t+dt$ and $t+2dt$ and set $MP = e(t)$, $BD = e(t+dt)$ and $CE = e(t+2dt)$, and calculate NE and OE to the second order – using the Taylor expansion – in the following way:

$$NE = NR - ER \quad OE = OR - ER,$$
$$NR = e'(t+dt) \cdot dt = e'(t)dt + e''(t)dt^2,$$
$$ER = e(t+2dt) - e(t+dt) = e'(t)dt + \tfrac{3}{2}\, e''(t)dt^2,$$
$$OR = e(t+dt) - e(t) = e'(t)dt + \tfrac{1}{2}\, e''(t)dt^2,$$

hence

$$NE = -\tfrac{1}{2}\, e''(t)dt \quad \text{and} \quad OE = -e''(t)dt^2.$$

Let us turn now to d'Alembert's calculation of the second differential of e. To obtain dde d'Alembert supposes ID and RE are equal to the first differential of e, de, at times M and B. The value of dde is therefore the second difference $RE-ID = RE-RO = -(OE)$. To calculate dde in this way is to treat the curve polygonally. In the second edition of the *Traité* (1758) d'Alembert comments on this calculation:

> It is not useless to remark that when we have first the equation between e and t in finite terms, and we derive from it by ordinary differentiation the equation dde = φdt², the value of dde that we find by this calculation is precisely OE, the true second difference of BD; we might first question, given the very nature of the differential calculus, if the value of dde found by this differentiation truly represents OE, or some other line, for example NE. But we may convince ourselves by the calculus itself that the quantity φdt² is equal to OE.
>
> [1758, 27–28]

What d'Alembert is saying here is that the value for dde given by the differential algorithm is the same as the value for dde given earlier ([1758, 21]) by the 'polygonal' calculation $RE-ID = -(OE)$. Since we deduced that $OE = -e''(t)(dt)^2$ we can observe that the 'polygonal' calculation indeed leads to $dde = e''(t)dt^2$.

In a footnote to the second edition of the *Traité* [1758, 27–28] Étienne Bezout added calculations corresponding to those just carried through. Thus he finds – although in a slightly different terminology – that in the polygonal approach $dde = RE-ID = e''(t)dt^2$; in the rigorous approach he introduces the intersection, S, of the tangent at P with the line BD and sets $dde = RN-IS = e'(t+dt)dt - e'(t)dt = e''(t)dt^2$.

D'Alembert himself illustrates the point of obtaining the same second differential in the article "Différentiel" in Diderot's *Encyclopédie* [1754, 988]. He considers the parabola $y = x^2$. Applying the differential algorithm twice (assuming dx is constant) he obtains $dy = 2xdx$ and $ddy = 2dx^2$. On the other hand, 'polygonal' reasoning leads to the three successive ordinates x^2, $x^2+2dxdx+dx$, $x^2+4xdx+4dx^2$ corresponding to the three abscissae x, $x+dx$, $x+2dx$. Take the difference of the third and second ordinate and the difference of the second and first ordinate: $2xdx+3dx^2$ and $2xdx+dx^2$. Take now the difference of the differences: $2dx^2$. The value $2dx^2$ is the same as that yielded directly by the differential algorithm.

Since he has chosen a function $y = f(x)$ for which $f'''(x)$, $f^{(4)}(x)$ etc. is equal to 0 it is no wonder that he gets exactly the same second differential: When dx is assumed constant the differential approach leads to $ddy = f''(x)dx^2$, whereas the polygonal approach gives

$$ddy = f(x+2dx) - f(x+dx) - (f(x+dx) - f(x)) = \\ f''(x)dx^2 + dx^3(f'''(x) + \ldots).$$

Thus in general the two are the same when third order infinitesimals are neglected; for the parabola they are exactly the same.

In examining the distance-time graph d'Alembert has restricted himself to an analysis of the tangential component of the particle's acceleration. (This is a limitation on his analysis of which he does not appear to be completely aware.) In a subsequent section of Part One he turns to the study of a particle moving under the action of a central force ([1743, 27–30], [1758, 40–44]). In this case, he also takes up the question of the rigorous and polygonal curve. However, here it is the actual physical trajectory of the particle in space that is being analyzed. The points P, D, E on the curve PDE represent three infinitesimally close points in space occupied by the particle. The lines OE and NE are now directed line segments representing (vector) accelerations. As in the earlier analysis of the distance-time graph d'Alembert obtains $OE = 2(NE)$. Once again he cautions on the need for consistency in measuring the force's effect. If the curve is treated rigorously NE is this measure; if treated polygonally the measure is OE.

In many of the problems of Chapter Three of Part Two that involve continuously accelerated motion d'Alembert proceeds either polygonally or in a way that is analogous to such a procedure. He does so with no explanation and with no reference to the issues raised and discussed in Part One. A knowledge of these issues nevertheless illuminates the relationship in his analysis between the differential algebra and the geometrical polygonal procedure. Consider again the example of a particle moving in space. Assume we analyze the trajectory of the particle using a Cartesian x-y-z co-ordinate system. Let x_p, x_D, x_E be the x- values of the particle at P, D, E; suppose these values are occupied at times t, $t+dt$, $t+2dt$. D'Alembert typically sets $dx = x_D - x_p$ and calculates the second differential as follows:

$$ddx = (x_E - x_D) - (x_D - x_p),$$

or, in modernized notation,

$$ddx = (x(t+2dt) - x(t+dt)) - (x(t+dt) - x(t)).$$

The value ddx obtained in this way is the same, as we saw above, as the value $\ddot{x}\, dt^2$ given directly by the differential algorithm.

The measure of the accelerative force in the polygonal approach also coincides with the value that would be provided by an analytical relation of the form $dde = \varphi dt^2$. D'Alembert noted that the measure of the force in the polygonal approach is the line OE (Figure 2). OE is equal to the polygonal second difference of e, and, as we saw above, this second difference equals the quantity dde in the relation $dde = \varphi dt^2$.

One respect in which the polygonal procedure differs from the differential algebra is in the calculation of the first differential. Let $(dx)_p$ denote the value of dx in the polygonal approach. We have the following relation:

$$(dx)_p = x(t+dt) - x(t) = \dot{x}(t)dt + \tfrac{1}{2}\ddot{x}(t)dt^2.$$

$(dx)_p$ therefore differs from the analytical or 'rigorous' differential $dx = \dot{x}(t)dt$ by the second order term $\frac{1}{2}\ddot{x}(t)dt^2$. This technical difference in the calculation of the first differential plays no role in the derivation of the equations of mechanics. These equations may be expressed in a form composed of second-order terms. Any first-order differential appearing in the equations thus expressed must be multiplied by another first-order differential. Because $(dx)_p$ differs from $\dot{x}(t)dt$ by a second-order term the error introduced by employing one differential rather than the other would be third-order and would therefore be negligible.

Note finally that d'Alembert presents an interpretation of the speed of the particle in the polygonal curve. It is unnecessary to know this interpretation in order to follow his application of his principle in Part Two of the *Traité*. The interpretation nevertheless possesses some historical interest as an indication of how d'Alembert visualized the speed in the polygonal approach. It is described in Appendix Two.

c. Two Problems with Solutions

D'Alembert's solutions to Problems II and X of Chapter Three of Part Two illustrate how he applies his principle to problems of continuously accelerated motion. These problems are representative of the simple mechanical examples studied by geometers of the period. Versions of them appear in a memoir composed by Clairaut in 1742, a treatise introduced with a note explaining that the problems presented "have nearly all been proposed by the savants Messr. Bernoulli and Euler" ([1742, 213]).

From a modern view (and possibly also to his contemporaries) d'Alembert's solutions seem complicated. The reader may wish to compare the account which follows to Appendix One, where the modern solutions to Problems II and X are presented.

i. Problem II

In Problem II d'Alembert examines a system consisting of a massless rigid rod situated in a plane. The rod is free to rotate about one end G which is fixed in the plane. A body A is attached to the other end of the rod; a second body D is free to slide along its length. No external forces act on the system. The problem is to determine the speeds of A and D at any instant and the path of D.

To solve Problem II d'Alembert analyzes the system during an infinitesimal time period. The motions of A and D are represented geometrically by line segments. In all calculations he neglects infinitesimals of the third order or higher. That is, two quantities are taken to be equal if their difference is an infinitesimal of the third order or higher. I shall use the phrase "up to second order" to refer to this level

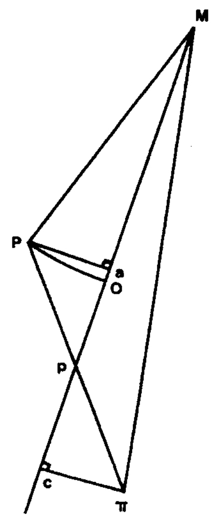

Figure 4. (D'Alembert's Figure 24, *Traité* (1743)).

of approximation. As an example, consider the sector PMO (Figure 4), where the angle PMO is infinitesimal. The arc PO, the chord PO and the perpendicular Pa are all equal up to second order. The line Oa is an infinitesimal of the second order.

D'Alembert had preceded Problem II with the demonstration of a geometrical lemma needed in its solution. Assume in the triangle $MP\pi$ that the angle $PM\pi$ is infinitesimal, $Pp=p\pi$ and $MO=MP$ (Figure 4). D'Alembert establishes the following results:

$$M\pi - PM = 2pO + \frac{PO^2}{PM} \tag{1}$$

$$\measuredangle pM\pi - \measuredangle PMp = \left(\frac{-2pO}{PM}\right)(\measuredangle PMp). \tag{2}$$

(The quantity PO may be taken to be the chord or the arc; (1) is valid in either case up to second order.) (1) and (2) are presented as a corollary to the geometrical lemma; the latter is itself established for a configuration more general than the triangle $PM\pi$. I shall for simplicity describe d'Alembert's demonstration as it would apply directly to this triangle. From P and π drop the perpendiculars Pa and πc to pM. Then

$$PM-pM = (PM-Ma) + (Ma-pM) = \frac{(Pa)^2}{2PM} - ap. \qquad (3)$$

The equality $PM-Ma = (Pa)^2/2(PM)$ can be derived from the two relations

$$PM-Ma = PM(1\text{-}\cos\theta) = (PM)\frac{\theta^2}{2}$$

$$Pa = (PM)\sin\theta = (PM)\theta,$$

where $\theta = \measuredangle PMp$. Equation (4) follows in a similar manner:

$$pM-\pi M = \frac{-(\pi c)^2}{2(\pi M)} - cp. \qquad (4)$$

By adding (3) and (4) we obtain, up to second order,

$$PM-M\pi = -2(ap). \qquad (5)$$

Substituting the value for ap given by (3) into (5) yields

$$PM-M\pi = 2(PM-pM) - \frac{(Pa)^2}{PM}. \qquad (6)$$

Because Pa and PO are equal up to second order, (1) follows from (6). To obtain (2) notice first that

$$\measuredangle pM\pi = \frac{c\pi}{cM} = \frac{Pa}{PM+2(ap)} = \frac{Pa}{Ma} - \frac{2(ap)}{Ma} \cdot \frac{Pa}{Ma}, \qquad (7)$$

up to second order. But $Pa/Ma = \measuredangle PMp$ and $(\measuredangle PMp)(ap/Ma) = \measuredangle PMp(pO/PM)$ up to second order. (2) therefore follows from (7).

Equations (1) and (2) are used by d'Alembert to analyze the motion of the body D. These equations express the following fact: If $Pp\pi$ is the path of a particle moving freely under the action of no force and M is the origin of a polar co-ordinate system, then the radial and transverse accelerations of the particle are zero. (The following modern interpretation may be useful in recognizing this fact. Assume the particle as it moves from P to π with constant velocity occupies the positions P, p and π at times t, $t+dt$ and $t+2dt$. In polar co-ordinates equation (1) becomes

$$r(t+2dt)-r(t) = 2[r(t+dt)-r(t)] + \frac{r^2 d\theta^2}{r}$$

which leads to

$$\ddot{r}-r\dot{\theta}^2 = 0. \tag{1'}$$

Similarly equation (2) becomes

$$[\theta(t+2dt)-\theta(t+dt)]-[\theta(t+dt)-\theta(t)] = -\frac{2[r(t+dt)-r(t)]}{r}(\theta(t+dt)-\theta(t))$$

which leads to

$$r\ddot{\theta}+2\dot{r}\dot{\theta} = 0. \tag{2'}$$

The left sides of (1′) and (2′) are the well known expressions for the acceleration in the radial and perpendicular direction.)

D'Alembert begins his solution to Problem II by assuming that the bodies A and D travel during a given instant the arc AB and the line DE (Figure 5). In a second instant equal to the first A would if free travel by its circular motion the arc BC = arc AB; D would travel the line $Ei=DE$. Because of the constraint resulting from the presence of D the body A actually travels BC in an instant larger than the first. Let BQ be the arc that would be travelled uniformly by A with the circular speed it possesses at B during this larger instant. Let Eo be the line that would be travelled by D with its speed at E during the same in-

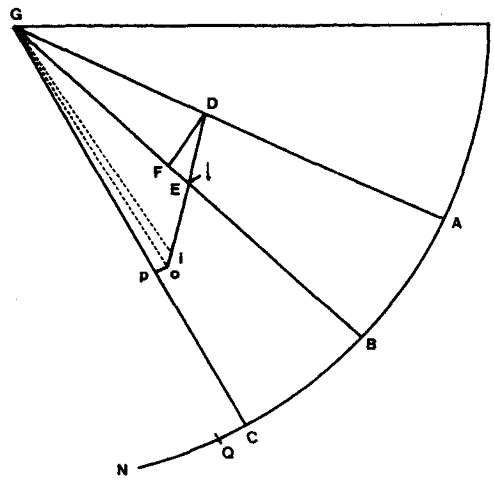

Figure 5. (D'Alembert's Figure 25, *Traité* (1743)).

stant. Because D is constrained to move on the rod it ends up at p. D'Alembert invokes his principle to obtain the following decompositions:

$$BQ : \text{composed of } BC \text{ and } CQ, \tag{8}$$
$$Eo : \text{composed of } Ep \text{ and po}. \tag{9}$$

BQ and Eo represent the impressed or free motions of A and D; BC and Ep represent the actual motions and CQ and po represent the lost motions. (Note that CQ, po and io are second order quantities; this fact will be used later in calculations where third order infinitesimals are neglected.) By his principle equilibrium would subsist if CQ and po were the motions of A and D. Since D is free to slide along the rod, $po=El$ must be perpendicular to GB. D'Alembert uses as a condition for equilibrium the fact that the total moment of the lost motions about G is zero:

$$A(CQ)(GA) = D(El)(GE). \tag{10}$$

D'Alembert proceeds to derive a differential equation for the path of D. Let $GA=a$, $GD=y$, $AB=dx$, and $CQ=\alpha$. The line FD is then

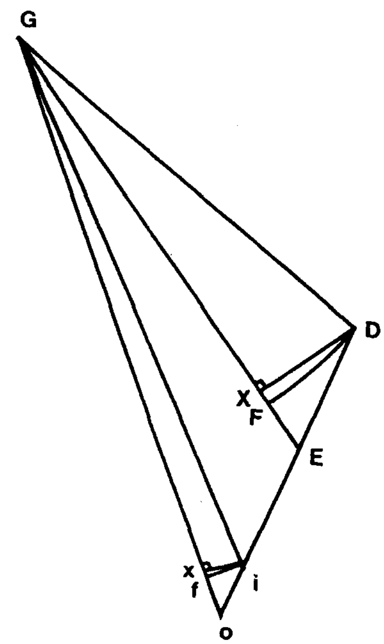

Figure 6.

$(ydx)/a$; d'Alembert sets FE equal to dy. (As will become clear in the subsequent analysis, d'Alembert is taking GF to be equal to GD.) In all calculations he neglects third and higher order differentials. We have first the relation expressing the equality of the times with which A and D traverse AB, DE and BQ, Eo:

$$\frac{BC}{CQ} = \frac{DE}{io}. \tag{11}$$

By (2) $\measuredangle iGE = \measuredangle EGD(1-2(FE)/GD)$. Since $\measuredangle EGD = dx/a$ we obtain

$$\measuredangle iGE = \measuredangle EGD - \frac{2dydx}{ay}. \tag{12}$$

Let us now draw a chord *if* and a perpendicular *ix* from *i* to *Go* (Figure 6). In addition, draw a perpendicular *DX* from *D* to *GE*. Then

$$\measuredangle iGo = \frac{if}{Gi} = \text{(to second order) } \frac{ix}{Gi}.$$

But *ix*/*io* differs from *DX*/*DE* by a first order quantity, a fact that may be ascertained after some calculation. Also, $DX/DE = DF/DE$ up to second order. Hence *ix*/*io* differs from *DF*/*DE* by a first-order quantity. Thus, because *io* is second order, $ix = (DF/DE)io$ up to second order. Consequently – by (11) –

$$\measuredangle iGo = \frac{io}{Gi} \times \frac{DF}{DE} = \frac{(CQ)(DF)}{(Gi)(BC)}.$$

Now $DF/BC = GD/GA$. Also, *CQ* is second order and *GD*/*Gi* differs from unity by a first order quantity. Letting $CQ=\alpha$ and $GA=a$ we therefore obtain the equation:

$$\measuredangle iGo = \frac{\alpha}{a}. \tag{13}$$

The angle *oGp* is equal to *po*/*Gp*, which, up to second order, equals *po*/*y*. Expressing (10) in terms of α, y and *po* gives $po = (A\alpha a)/(Dy)$. Hence

$$\measuredangle oGp = \frac{A\alpha a}{Dy^2}. \tag{14}$$

We now add equations (12), (13) and (14):

$$\measuredangle EGp = \measuredangle EGD - \frac{2dydx}{ay} + \frac{\alpha}{a} + \frac{A\alpha a}{Dy^2}. \tag{15}$$

Since $\measuredangle EGp = \measuredangle EGD$ we obtain

$$-\frac{2dydx}{ay}+\frac{\alpha}{a}+\frac{A\alpha a}{Dy^2}=0,$$

which, expressed in terms of α, becomes

$$\alpha=\frac{2Dydydx}{Aa^2+Dy^2}. \tag{16}$$

D'Alembert continues by calculating ddy. Applying (1) to the triangle DGi he derives the relation

$$Gi-GD = 2FE+\frac{(DF)^2}{GD}=2dy+\frac{ydx^2}{a^2}. \tag{17}$$

Now $Gp{=}Go$ up to second order. Consider again Figure 6. By comparing the triangles DFE and ifo we conclude, after some calculation, that fo/io and FE/DE differ by a first order quantity. Hence, because io is second order, $Gp-Gi = Go-Gi = fo = (io)(FE/DE)$ up to second order. Thus from (11) we obtain

$$Gp-Gi=\frac{CQ}{AB}dy=\frac{\alpha dy}{dx} \tag{18}$$

A value for $Gp-GE = (Gp-Gi)+(Gi-GD)+(GD-GE)$ is then given by

$$Gp-GE = dy+\frac{\alpha dy}{dx}+\frac{ydx^2}{a^2}. \tag{19}$$

Since $ddy = (Gp-GE)-dy$, equation (19) becomes

$$ddy=\frac{\alpha dy}{dx}+\frac{ydx^2}{a^2}. \tag{20}$$

By substituting the value for α given by (16) into (20) d'Alembert obtains a differential equation describing the path DEp of D:

$$ddy = \frac{ydx^2}{a^2} + \frac{2Dydy^2}{Aaa+Dyy}. \tag{21}$$

He proceeds to integrate (21). Let $dx=pdy/a$. Because $dx=AB=BC$ is constant we have $0 = dpdy/a + pddy/a$ or $ddy = (-dpdy)/p$. Substituting this value of ddy into (21) and adjusting terms yields

$$-\frac{a^4dp}{p^3} - \frac{2ydya^4D}{p^2(Aaa+Dyy)} = ydy. \tag{22}$$

D'Alembert multiplies (22) by the factor $1/(Aaa+Dyy)^2$ and integrates:

$$\frac{a^4}{2p^2(Aaa+Dyy)^2} = G - \frac{1}{2\mathrm{D}(Aaa+Dyy)} \tag{23}$$

where G is a constant. By substituting $(adx)/dy=p$ into (23) we obtain the result

$$\frac{ady\sqrt{D}}{\sqrt{(Aaa+Dyy)[2GD(Aaa+Dyy)-1]}} = dx, \tag{24}$$

an equation which, when integrated, furnishes an integral connecting x and y. (This integral is what in later mathematics would be called an *elliptic* integral.)

Having derived an expression for the path of D d'Alembert calculates the speeds u and v of A and D. He first presents the relation

$$-\frac{du}{u} = \frac{CQ}{BC}, \tag{25}$$

a result which is explained by Étienne Bezout in a footnote to the edition of 1758 ([1758, 107]). Since $udt = BC$ and BC is constant we have $d(udt) = dudt + uddt = 0$ or $-du/u = ddt/dt$. Clearly, however, $ddt/dt = CQ/BC$, so that $-du/u = CQ/BC$, as desired. Because $CQ=\alpha$ and $BC=dx$ we obtain from (16)

$$-\frac{du}{u}=\frac{2Dydy}{Aa^2+Dy^2}. \tag{26}$$

D'Alembert proceeds to integrate (26):

$$\frac{u}{g}=\frac{Aa^2+Db^2}{Aa^2+Dy^2}, \tag{27}$$

where g and b are the initial values of u and y. He continues by calculating an expression similar to (25) for the speed v of D. He points out, however, that v is given "more elegantly" by the principle of conservation of live forces, a principle he says he will demonstrate later. By this principle Dv^2+Au^2 is equal to a constant, so that

$$v^2=\frac{Dh^2+Ag^2-Au^2}{D}, \tag{28}$$

h being the initial speed of D. With (28) the solution to Problem II is complete.

(In a series of remarks following Problem II d'Alembert extends his analysis to the case in which external forces act on the system. These remarks are mainly of interest in illustrating how he applies his principle when external forces are present. Since I deal with this matter in my presentation of Problem X I omit discussion of them.)

Let us turn now to a critical examination of d'Alembert's solution. The first point to be noticed concerning this solution is that d'Alembert is making the quantity x the independent variable in the problem. (To say x is the independent variable was understood by geometers of the period to mean dx is held constant in all calculations. See Bos [1974].) The solution is based on the following two postulates: the total moment about G of the constraint forces acting on A and D is zero; the radial acceleration of D is zero. Equations (16) and (20) are the analytical statement of these postulates, expressed in terms of x instead of t as the independent variable. The quantity α is equal to $(dx/dt)ddt$.

In his first remark following the solution to Problem II d'Alembert states he has avoided making dt constant in order to obtain an expression for the curve that does not require a knowledge of the speeds. It is unclear to me what advantage he sees in such a procedure. This procedure, which he follows in several other problems (III, IV and VIII), rather complicates his solutions and is an aspect of his analysis that Lagrange ([1811, 256]) would later criticize.[3] (In d'Alembert's solution the motion is analyzed from the outset using the independent variable x. His analysis should be contrasted to the modern approach to central force problems, where a variable other than the time is the independent variable in certain formulas. The modern procedure is a mathematical one; the formulas are themselves derived from time-differential equations of motion.)

The principle of conservation of live forces, used by d'Alembert to obtain (28), is the assertion of the constancy of the live force or *vis viva* (later (twice) the kinetic energy) in the absence of external forces. In Part II(c) we examine d'Alembert's presentation of this principle in the last chapter of the *Traité*.

Note that d'Alembert is treating the motion of D "polygonally", in the sense explained in Section (b). Thus the path of D consists of the two polygonal segments DE and Ep. The differential dy is equal to $EG-GD$, the difference of the value of y at x and $x+dx$. The second differential ddy equals $(Gp-GE)-dy$, the difference of two successive values of dy. A similar interpretation holds for the second differential ddt of the time.

A final noteworthy feature of d'Alembert's solution concerns the assumptions he makes about the order of infinitesimal quantities. The order of the product of two infinitesimals is the sum of their orders. Also, the difference of two successive values of a given first order infinitesimal is second order. Thus d'Alembert assumes that CQ and io are second order. (He treats this fact as self-evident. He was probably guided by examples such as the acceleration: since dde/dt^2 is finite, the second difference dde of the distance travelled is proportional to dt^2 and is therefore second order.)

ii. Problem X

In Problem X d'Alembert considers an irregularly shaped object $KARQ$ of mass m which is free to slide along a frictionless plane QR (Figure 7). A body of mass M is situated on the curve KQ which forms

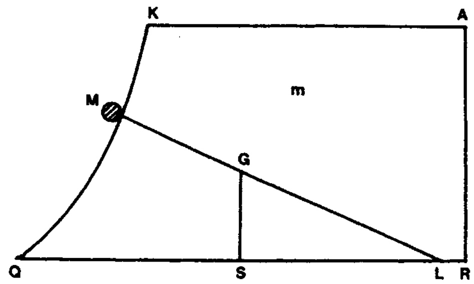

Figure 7. (D'Alembert's Figure 43, *Traité* (1743)).

the left edge of $KARQ$. A force acts on M in a direction perpendicular to QR. The two bodies possess given initial velocities; the problem is to determine the motion of the system as M slides down KQ.

D'Alembert preceded the presentation of Problem X with a lemma used in its solution. Consider the system described above with the vertical force acting on M removed. If m has velocity v_1 directed along RQ and M has velocity v_2 such that the system is in equilibrium then:

i) v_2 is directed along the perpendicular ML to the curve KQ,
ii) $(mv_1)/(Mv_2) = (SL)/(GL)$.

This result, stated by d'Alembert without proof, is apparent: for equilibrium to subsist the component of v_2 tangent to KQ must be zero; also, since the vertical component of v_2 is destroyed, and since the horizontal component equals $v_2(SL)/(GL)$, we must have $(Mv_2)(SL)/(GL) = mv_1$.

In Problem X d'Alembert analyzes the section of the curve KQ along which M moves during an infinitesimal time period (Figure 8). The section consists of two infinitesimal polygonal segments AB and BC; d'Alembert assumes the vertical projections of these segments are equal. At the beginning of the time period M is located at A. In the next instant it travels the line AB'; ABC travels the line AA' to assume the position $A'B'C'$. In the following instant m if free would travel the line $A'A'' = AA'$. The body M if free would travel $B'j = AB'$ as well as an additional distance $j\delta$ due to the action of the vertical force; its free motion would therefore consist of the line $B'\delta$. Because of the constraint of impenetrability m and M actually travel the

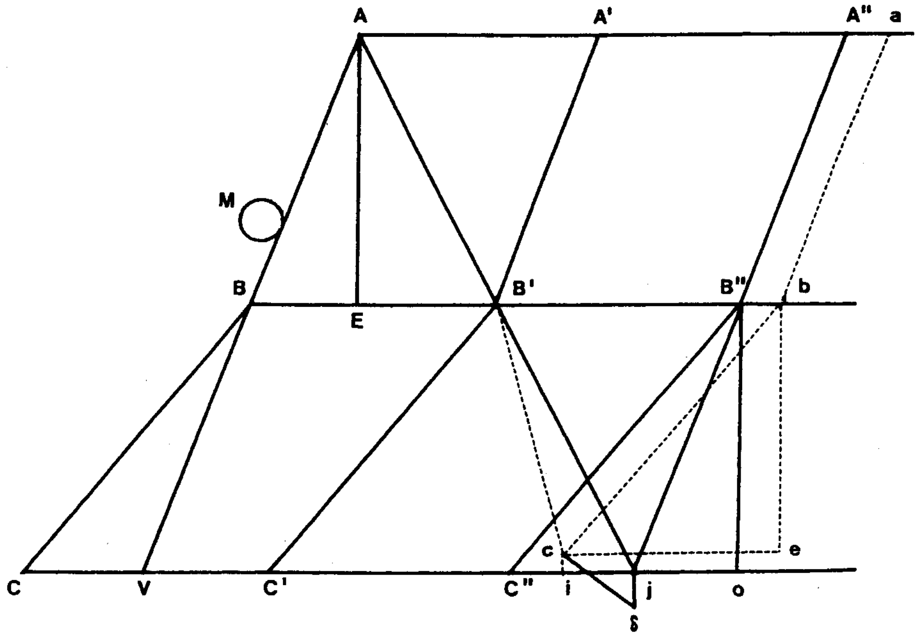

Figure 8. (D'Alembert's Figure 44, *Traité* (1743)).

lines $A'a$ and $B'c$ in the next instant. D'Alembert's principle gives the decompositions:

$$B'B'' : \text{composed of } B'b \text{ and } -(B''b), \tag{29}$$
$$B'\delta : \text{composed of } B'c \text{ and } c\delta. \tag{30}$$

The lines $-B''b$ and $c\delta$ represent the lost motions of m and M. Equilibrium would by this principle subsist if the bodies possessed the lost motions alone. The previous lemma is now invoked to conclude that

$$\frac{m(A''a)}{M(c\delta)} = \frac{ij}{c\delta}. \tag{31}$$

Thus $m(A''a) = M(ij)$. But $ij = ec - jo - A''a$. Hence

$$m(\text{A}''\text{a}) = M(ec - jo) - M(A''a). \tag{32}$$

D'Alembert sets $AA'=du$ and $BE=dy$. (The variable u designates the horizontal distance from some fixed line to the point K (Figure 7) in the body m. The variable y represents the horizontal distance travelled by M, measured from K in the moving system.) Clearly $A''a = A'a-AA' = ddu$. Also, since $jo=BE$ it is clear that $ec-jo = ddy$. Substituting these values into (32) we obtain the final equation for Problem X:

$$mddu = Mddy-Mddu,$$

or

$$(M+m)ddu = Mddy. \tag{33}$$

D'Alembert states that (33) is an "equation general and very simple for finding the motion of the two bodies, whatever be the force which acts on M, provided that this force be always perpendicular to QR".

The differentials in equation (33) are differentials with respect to time. (33) asserts that the horizontal projection of the center-of-gravity moves with constant velocity. Earlier in the *Traité* ([1743, 52–68], [1758, 75–96]) d'Alembert had provided a discussion of the dynamical properties of the center-of-gravity of a constrained system. He recognized that in connected systems where the external forces are perpendicular, the horizontal projection of the center-of-gravity travels equal distances in equal times. He apparently wished to derive (33) directly from his principle rather than present it as a consequence of this fact. (He recognized the existence of other solutions, but preferred his own, "because it is extremely simple".) His procedure here indicates that at this point in the history of mathematical mechanics the relationship between general dynamical theorems and the analytical differential equations of motion was not yet fully assimilated.

Although d'Alembert provides no details, it is clear how one would proceed from (33) to a complete description of the motion of the system. By integrating (33) twice we would obtain a relation among u,y and the time t. A knowledge of the shape of the curve KQ would allow us to express y in terms of the height z of M above the base QR. The final equation needed to solve the problem would then be provided by equating the total force perpendicular to QR acting on M to $M\ddot{z}$. Alternatively, and more characteristic of d'Alembert's own approach, we could use the equation of live force (i.e., energy):

from the pivot G. The ordered pairs $(a,x/a)$ and $(y,x/a)$ are the polar co-ordinates of A and D. Using the known expression for the acceleration in polar co-ordinates we obtain

$$(-\dot{x}^2/a,\ddot{x}) \text{ and } (\ddot{y}-y\dot{x}^2/a^2, y\ddot{x}/a+2\dot{y}\dot{x}/a)$$

as the values for the radial and transverse acceleration of A and D. We therefore arrive at the two equations:

$$Aa\ddot{x} + Dy\left(\frac{y\ddot{x}}{a} + \frac{2\dot{y}\dot{x}}{a}\right) = 0, \tag{1}$$

$$\ddot{y} - \frac{y\dot{x}^2}{a^2} = 0. \tag{2}$$

(1) asserts that the angular momentum is constant; (2) asserts that the force acting on D along the rod is zero. (1) and (2) are simply d'Alembert's equations (16) and (20) when the parameter time is the independent variable in the problem.

We now rewrite (1) in the form $(Aa^2+Dy^2)\ddot{x}+ 2Dy\dot{y}\dot{x} = 0$ and integrate:

$$\dot{x} = \frac{c_1}{Aa^2+Dy^2}, \tag{3}$$

where c_1 is a constant of integration. Substituting this value for $\dot{x}$ into (2) and integrating yields

$$\dot{y} = \sqrt{2c_2 - \frac{c_1^2}{Da^2(Aa^2+Dy^2)}}, \tag{4}$$

where c_2 is a second constant of integration. By dividing (3) by (4) and adjusting constants $(G=c_2a^2/c_1^2)$ we obtain the final equation:

$$dx = \frac{\sqrt{D}\, ady}{\sqrt{(Aa^2+Dy^2)[2GD(Aa^2+Dy^2)-1]}}. \tag{5}$$

The determination of the speeds is straightforward. (3) furnishes a

$$M((\dot{u}-\dot{y})^2+\dot{z}^2) + m\dot{u}^2 = W(u,z), \tag{34}$$

where W is an integral to be computed from a knowledge of the external force acting on M.

Problem X is of interest as illustration of how d'Alembert applies his principle when external forces are present. Consider again the decomposition furnished by this principle for the body M:

$$B'\delta : \text{composed of } B'c \text{ and } c\delta. \tag{35}$$

The quantity $B'\delta$ is the 'sum' of $B'j$ and $j\delta$. $B'j=AB'$ represents the velocity of M when it is at B' (d'Alembert sometimes refers to this quantity as the "primitive impressed velocity"); $j\delta$ represents the impressed increment of velocity imparted to m by the vertical force. Because no external forces act on m its impressed motion is represented solely by the quantity $A'A''$. The resulting interaction of M and m is treated by d'Alembert as one of collision arising from the impenetrability of the bodies. He is therefore able to invoke the previous lemma on collision to obtain a condition on the lost motions; this condition in turn leads to the final solution to the problem.

Note finally that d'Alembert is treating the motion of M and m 'polygonally'. The path of M consists of the two polygonal segments AB' and $B'c$. The differential du of the horizontal distance u travelled by m is equal to AA', the difference of u at two successive instants. The second differential ddu in turn equals the difference of two successive values of du.

Appendix One. Problems II and X: The Modern Solutions

i) Problem II

The body A is attached to the end of a massless rod GA of length a. The rod is free to rotate about G (Figure 5). The body D is free to slide along the rod. No external forces act on the system. A and D possess given initial velocities and the problem is to determine the subsequent motion.

At a given instant let x be the distance of A measured along its circular path from a fixed initial reference line. Let y be the distance of D

value for the speed $\dot{x}$ of A. The speed v of D is then given by the equation of energy:

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}v^2 = \text{constant}. \tag{6}$$

ii. Problem X

The object $KARQ$ moves on a frictionless plane QR. The body M is situated on the curve KQ which forms the left edge of $KARQ$. A force acts on M in a direction perpendicular to the base QR. The two bodies possess given initial velocities; the problem is to determine the motion of the system as M slides down KQ.

Assume the base QR of $KARQ$ lies on the u-axis of a $u-z$ co-ordinate system. Suppose at a given instant that M is at the point C on the curve KQ. Consider the following designations (Figure 9):

(u_M, z) = co-ordinates of M
(u_m, b) = co-ordinates of K in m
F = vertical force acting on M
θ = angle between the normal to the curve KQ at C and the vertical
R_M = magnitude of force of reaction exerted on M by m
R_m = magnitude of net force acting on m as a result of the force exerted by M and the reaction at the base.

R_M and R_m are the constraint forces which act on M and m. R_M acts on M along the normal at C and is directed away from KQ. R_m acts on m horizontally to the right. The accelerations corresponding to these forces, if reversed, constitute the "lost motions". By d'Alembert's principle, equilibrium would subsist if M and m possessed the reversed accelerations alone. Hence by his introductory lemma on equilibrium:

$$\frac{R_m}{R_M} = \sin\theta. \tag{6}$$

The relation (6) would be obtained today using the equality of action-

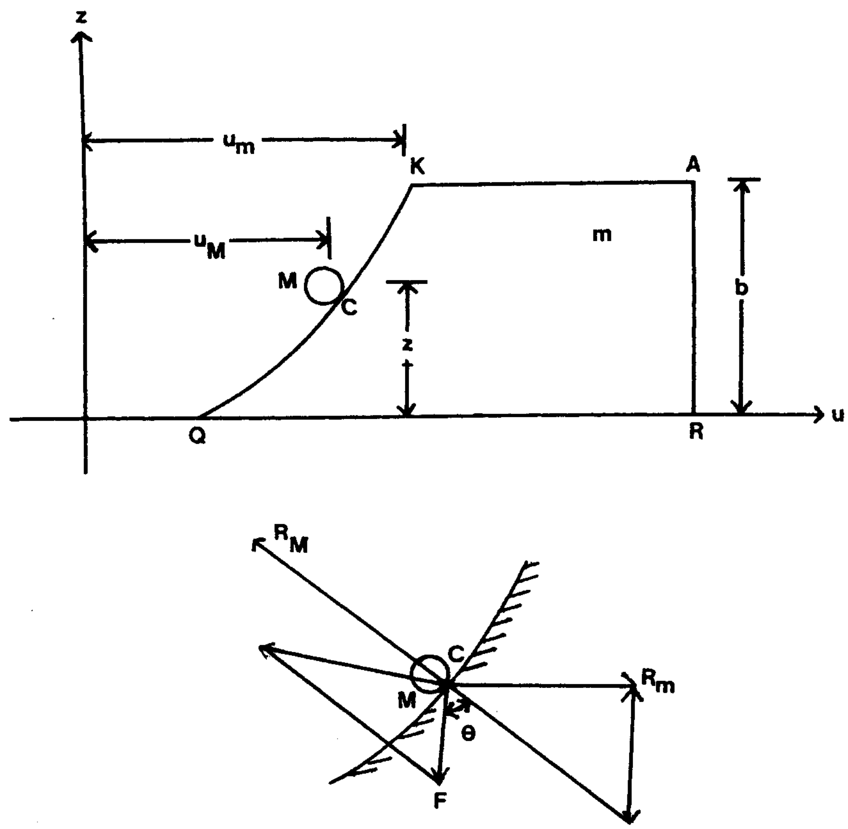

Figure 9.

reaction (R_M = force exerted on m by M) and taking horizontal projections. Now

$$R_m = m\ddot{u}_m. \tag{7}$$

In addition, because $-R_M \sin\theta$ is the u-component of the total force acting on M,

$$-R_M\sin\theta = M\ddot{u}_M. \tag{8}$$

(6), (7) and (8) correspond to two steps in d'Alembert's solution: the step where he sets $c\delta$ and $-(A''a)$ equal to the lost motions of M and m; the assumption involved in the lemma on equilibrium, namely, that the horizontal projection M(ij) of M($c\delta$) equals m(A″a). (6), (7) and (8) yield the desired equation:

$$0 = m\ddot{u}_m + M\ddot{u}_M. \tag{9}$$

Note that the total force acting on M is composed of F and R_M. The forces F and $-R_M$ correspond to the impressed motion $j\delta$ and the lost motion $c\delta$ of M in d'Alembert's solution. Note also that

$$R_M = F\cos\theta + Mv^2/\varrho,$$

where v is the speed of M and ϱ is the radius of curvature of its trajectory when it is at C. The equation obtained by equating the z-component of the total force acting on M to $M\ddot{z}$ is therefore:

$$F\cos^2\theta + (Mv^2/\varrho)\cos\theta - F = M\ddot{z}. \tag{10}$$

Either (10) or the equation of energy could be used to find the time-motion of the system.

Appendix Two. The Polygonal Curve

In his discussion of the polygonal and rigorous curve in Part One of the *Traité* ([1758, 29–31]) d'Alembert includes an interpretation of the speed of the particle in the two approaches. Consider again the distance-time graph of the particle (Figure 2). The infinitesimal distance OE is the measure of the effect of the force when the curve is treated polygonally. D'Alembert says that OE "should be regarded as travelled uniformly with a uniform motion equal to the infinitely small speed that the body has acquired at the end of the instant BC". He continues

> since in the polygonal curve the effect of the accelerative power is represented by a uniform motion, one should not suppose in this hypothesis that the speed of the body accelerates continuously ("by degrees") during the instant BC, but that at the beginning of

> this instant BC, when the body has travelled the space BD, the speed receives suddenly and by a single blow all the increase or decrease that it actually acquires only at the end of the instant BC.
>
> [1758, 30]

The polygonal interpretation should be contrasted with the rigorous treatment of the curve, where, as a result of the action of the force, the distance $NE=\frac{1}{2}(OE)$ is supposed to be travelled with a (uniformly) accelerated motion during BC. To treat the curve rigorously is to suppose the accelerative power

> imparts to the mobile during this instant [BC] a sequence of small equal and reiterated blows; and the sum of these small blows is equal to the single blow that the same power is assumed to impart to the body at the beginning of the instant BC in the hypothesis of the polygonal curve.
>
> [1758, 30–31]

In d'Alembert's polygonal curve the particle is assumed to move with constant speed during the time interval MB; at B this speed is suddenly, by a "single blow", increased or decreased by the total speed the particle actually gains or loses during the next time interval BC; the particle then moves uniformly with this new speed during BC. The changing speed of the particle therefore consists of a succession of discrete constant speeds.

Acknowledgments

I would like to thank Jed Buchwald for his help and encouragement. I am grateful to Philip Lervig for valuable conversations and correspondence. I have benefited from the comments of Henk Bos, Ole Knudsen, Kirsti Andersen and Yves Gingras.

NOTES

1. For a discussion of the historical background to the *Traité de Dynamique* see T. L. Hankins' scientific biography [1970] of d'Alembert.
2. For an interesting historical connection between this rule and the virtual work principle presented by d'Alembert in the last chapter of the *Traité* see Szabo [1979, 441–442]. I discuss this chapter in Part (II)(c).

3. Lagrange's criticism does not appear in the first edition (1788) of the *Mécanique Analytique*. It is interesting to compare Section Two of Part Two "Formule générale de la dynamique pour le mouvement d'un système de corps animés par des forces quelconques" in the two (1788, 1811) editions. In 1788 Lagrange follows d'Alembert's original presentation of his principle much more closely than he would later in the second edition. (Lagrange finished the *Mécanique Analytique* in 1782, a year before d'Alembert's death. At the time he was d'Alembert's closest professional correspondent.)

BIBLIOGRAPHY

Bos, H. J. M. [1974] "Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus". *Archive for History of Exact Science* 14 (1974), pp. 1–90.

Clairaut, Alexis. [1742] "Sur quelques principes qui donnent la solution d'un grand nombre de problèmes de dynamique". *Mémoires de Mathématique et de Physique de l'Académie Royal des Sciences de Paris de l'Anneé 1742* (1745).

d'Alembert, Jean. [1743] *Traité de Dynamique, dans lequel les Loix de l'Equilibre et du Mouvement des Corps sont réduites au plus petit nombre possible, et démontrées d'une maniére nouvelle, et où l'on donne un Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une maniére quelconque.* (Paris, 1743).

d'Alembert, Jean. [1754] "Différentiel", in *Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers,* vol. 4, Paris, 1754: 985–989.

d'Alembert, Jean. [1758] *Traité de dynamique* (Second Edition) (Paris, 1758). The second edition was reprinted in 1968 (Johnson Reprint Corporation) with an introduction and bibliography by Thomas L. Hankins.

Hankins, Thomas L. [1970] *Jean d'Alembert Science and the Enlightenment.* (Oxford, 1970).

Lagrange, Joseph Louis. [1788] *Méchanique Analitique.* (Paris, 1788).

Lagrange, Joseph Louis. [1811] *Méchanique Analytique.* (Paris, 1811). The second edition is reprinted as Lagrange's *Oeuvres* 11 (Paris, 1888). My page references are to *Oeuvres* 11.

Szabo, Istvan. [1979] *Geschichte der mechanischen Prinzipien und ihrer wichtigsten Anwendungen.* (Second Edition) (Birkhäuser, 1979).

Truesdell, C. [1960] "The rational mechanics of flexible or elastic bodies, 1638–1788". L. Euler, *Opera Omnia* S.2 11^2 (1960).