8

Euler and analysis: case studies and historiographical perspectives

Craig Fraser
University of Toronto

Andrew Schroter
Branksome Hall, Toronto

But [the historian's] particular business lies, not with this bare and general similarity, but with the detailed dissimilarity of past and present. He is concerned with the past as past, and with each moment of the past in so far as it is unlike any other moment (Oakeshott, 1933, p. 106).

Abstract: Two parts of analysis to which Leonhard Euler contributed in the 1740s and 1750s are the calculus of variations and the theory of infinite series. Certain concepts from these subjects occupy a fundamental place in modern analysis, but do not appear in the work of either Euler or his contemporaries. In the case of variational calculus there is the concept of the invariance of the variational equations; in the case of infinite series there is the concept of summability. However, some modern mathematicians have suggested that early forms of these concepts are *implicitly* present in Euler's writings. We examine Euler's work in calculus of variations and infinite series and reflect on this work in relation to modern theories.

8.1 Introduction

The present study explores the notion of anachronism in the history of mathematics in relation to some mathematical work of Leonhard Euler. The focus is on the 1740s and early 1750s, during Euler's Berlin period, when he was approaching the height of his mathematical powers and productivity. We consider aspects of two subjects that he investigated in an original and ground-breaking way: calculus of variations and the

[a] From *Anachronisms in the History of Mathematics: Essays on the Historical Interpretation of Mathematical Texts*, edited by Niccolò Guicciardini © 2021 Cambridge University Press.

theory of infinite series. Certain concepts occupy a fundamental place in the modern subject, but do not appear in the work of either Euler or his contemporaries. In the case of variational calculus there is the concept of the invariance of the variational equations; in the case of infinite series there is the concept of summability. While both concepts are a product of research since the later part of the nineteenth century, modern historical commentators have discerned the presence of intuitions or embryonic ideas of invariance and summability in Euler's writings. Our goal will be to look more closely at what Euler did and to evaluate the historical claims being made for his mathematical prescience.

8.2 Euler and the invariance of the variational equations

8.2.1 Euler's variational equation

In 1744 Euler published a major book on what would later be called the calculus of variations, his *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes* (hereafter referred to as *Methodus Inveniendi*).[1] In Chapter 2 of this work we are given the general problem of finding the curve that maximizes or minimizes a definite integral $\int_a^b Zdx$, where Z is a function of x, y, and $p = \frac{dy}{dx}$. The curve is given in an orthogonal Cartesian system, depicted in Figure 8.1. Suppose that $dZ = Mdx + Ndy + Pdp$. Through a process of reasoning that involved disturbing a single ordinate of the curve, Euler was able to show that the optimizing curve satisfies the differential equation $N - \frac{dP}{dx} = 0$. In modern notation this equation is written as $\frac{\partial Z}{\partial y} - \frac{d}{dx}\frac{\partial Z}{\partial y'} = 0$, and is known as the Euler equation or the Euler–Lagrange equation for the variational problem.

In §33 of Chapter 2 Euler considers the problem of the shortest distance between two points in the plane. In an orthogonal coordinate system the differential element of path length is $\sqrt{(dx^2 + dy^2)} = \sqrt{(1 + pp)}dx$.[2] Hence $Z = \sqrt{(1 + pp)}$ and the length of a curve joining the two points is $\int_a^b \sqrt{(1 + pp)}dx$. We have $dZ = \frac{p}{\sqrt{(1+pp)}}dp$ and the equation $N - \frac{dP}{dx} = 0$

[1] For the development of Euler's researches leading up to this work see Fraser (1994).

[2] The derivative p is given by Euler in the form $pdx = dy$. An oddity of his notation is that he writes dx^2 and dy^2, but always writes pp rather than p^2. Thus in §20 of Chapter 1 we have $\sqrt{(dx^2 + dy^2)} = \sqrt{(1 + pp)}dx$.

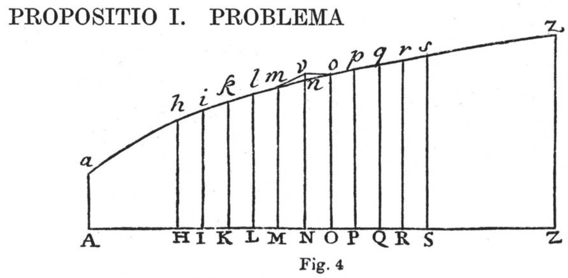

Figure 8.1 Euler (1744, Tabula I, Fig. 4).

reduces to $\frac{p}{\sqrt{(1+pp)}}$ = constant, or p = constant, which is the equation of the line $y = a + nx$, where a and n are constants.

In the fourth chapter of the *Methodus Inveniendi* Euler begins with a proposition, the intent of which is to assert a very wide interpretation for the methods he had introduced in the earlier chapters. He observes that if we have any equation between two variables x and y, we can always consider these variables as the orthogonal coordinates of a curve defined by the equation. Hence if we are given a function Z of x, y, and p (where $p = \frac{dy}{dx}$) we can apply the earlier methods to find the particular equation between y and x (the function y of x) that maximizes or minimizes the integral $\int_{a^0}^{a} Zdx$. In a corollary Euler elaborates on the significance of this finding:

> Therefore the method previously presented may be applied widely to find the equations between the coordinates of a curve, so that the expression $\int Zdx$ is a maximum or minimum. Indeed, it extends to any two variables, whether they belong to any given curve, or are only conceived of in analytical abstraction[3] (Euler, 1744, p. 130).

Euler is observing that logically his methods do not depend on any particular geometric coordinate system but are part of pure analysis.

In examples that follow in the chapter, Euler (1744, pp. 134–144) illustrates this conclusion by deriving the variational equation for examples involving curves given in non-orthogonal coordinate systems. The first example concerns the problem of the shortest distance between two

[3] "Methodus ergo ante tradita multo latius patet, quam ad aequationes inter coordinatas curvarum inveniendas, ut quaepiam expressio $\int Zdx$ fiat maximum minimumve. Extenditur scilicet ad binas quascunque variabiles, sive eae ad curvam aliquam pertineant quomodocunque, sive sola analytica abstractione versentur."

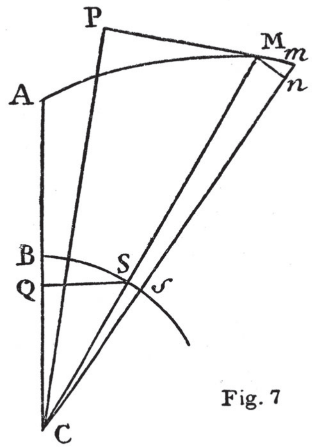

Figure 8.2 Euler (1744, Tabula I, Fig. 7).

points, where the problem is formulated in polar coordinates. In Figure 8.2, it is necessary to find the curve joining the points A and M of least length. Euler sets the angle ACM equal to x, and the radius CM equal to y. In the triangle nmM we have $Mn = ydx$ and $mn = dy$, so the infinitesimal pathlength Mm is $Mm = dx\sqrt{(yy + pp)}$. Thus, the length along the curve from A to M is

$$\int dx\sqrt{(yy + pp)},$$

where the integral is evaluated from $x = 0$ to the angle x corresponding to the point M. With $Z = \sqrt{(yy + pp)}$ and $dZ = Mdx + Ndy + Pdp$ we have

$$M = 0, N = \frac{y}{\sqrt{(yy + pp)}}, P = \frac{p}{\sqrt{(yy + pp)}}.$$

Because Z does not contain x we see immediately that a first integral of the equation $N - \frac{dP}{dx} = 0$ is $Z + C = Pp$, where C is a constant.[4] Given the expressions for Z and P above this equation may be simplified

[4] Euler is using the result that if $Z = Z(y, y')$ does not contain x then the equation $\frac{\partial Z}{\partial y} - \frac{d}{dx}\frac{\partial Z}{\partial y'} = 0$ is integrable. We have

$$\frac{dZ}{dx} = \frac{\partial Z}{\partial y}y' + \frac{\partial Z}{\partial y'}y'' + \frac{d(\frac{\partial Z}{\partial y'})}{dx}y' + \frac{\partial Z}{\partial y'}y'' = \frac{d(\frac{\partial Z}{\partial y'}y')}{dx}.$$

Hence $Z + C = \frac{\partial Z}{\partial y'}y'$, where C is a constant.

to $\frac{yy}{\sqrt{(yy+pp)}} = const = b$. The triangles Mnm and CPM are similar and we have the proportional equality $Mm : Mn = MC : CP$. Letting $Mm = dx\sqrt{(yy + pp)}$, $Mn = ydx$, $CM = y$ we obtain $CP = \frac{yy}{\sqrt{(yy+pp)}}$, which is a constant. Hence the perpendicular CP from C to the tangent to M is a constant, and the curve AM must be a straight line.[5] Since two constants are available in the integration of $Ndx - dP = 0$ the straight line AM is a solution to the problem.[6]

8.2.2 Invariance in calculus of variations and analytical dynamics

The examples just presented are at the center of some modern claims about Euler's intuitive familiarity with the concept of invariance. *Invariance* (or covariance) has different meanings in different areas of mathematics – algebraic forms, projective geometry, differential geometry, topology and functional analysis, to name a few. As far as variational equations are concerned, the relevant historical domain of research involved work in analytical mechanics and the calculus of variations in the second half of the nineteenth century. The subject of invariance came to the fore in the researches of Carl Jacobi in analytical dynamics. Jacobi began with the canonical equations of motion and considered transformations of the variables that would preserve the canonical form of these equations. He was able to show that such transformations can be given in terms of what became known as a generating function. The methods and ideas he pioneered were taken up further by researchers in celestial mechanics, most importantly by Henri Poincaré, the Swedish astronomer Ludwig Charlier, and the English mathematician Edward Whittaker. A key step was to show that a solution is effected by taking the generating function to be a solution of the Hamilton–Jacobi partial differential equation.

In the early years of the twentieth century German physicists working in quantum physics realized that canonical transformations and the associated Hamilton–Jacobi theory provided exactly the mathematical tools needed to investigate the physical systems that were of interest to them. The physicist Arnold Sommerfeld (1923, pp. 555–6) wrote: "Up to a

[5] In modern terminology CM and CP are the pedal coordinates of the point M on the curve AM. This terminology was not used by Euler.

[6] Today we would integrate $\frac{yy}{\sqrt{(yy+pp)}}$ = constant and obtain a solution of the form $y = \frac{A}{\cos x + B}$ for the polar-coordinate equation of the straight line joining A to M.

few years ago it was possible to consider that the methods of mechanics of Hamilton and Jacobi could be dispensed with for physics and to regard it as serving only the requirements of the calculus of astronomic perturbations and the interests of mathematics." As a result of the rapid development of quantum theory the situation had changed dramatically. Sommerfeld continued: ". . . it seems [today] almost as if Hamilton's method was expressly created for treating the most important problems of physical mechanics."

In twentieth-century literature on mechanics, canonical transformations occupy a prominent place. However, the subject in the early years of the century was a peripheral one within the calculus of variations. It did not appear at all in Oskar Bolza's comprehensive *Vorlesungen über Variationsrechnung* of 1909, although this book did deal in detail with some parts of Hamilton–Jacobi theory. In the years following the publication of this book, the situation began to change, as mathematicians turned their attention to those transformations that were of such interest to celestial mechanicians and quantum physicists. One such figure was the Munich mathematician Constantin Carathéodory, who wrote a chapter on the calculus of variations for Phillip Frank and Richard v. Mises's 1925 *Die Differential- und Integralgleichungen der Mechanik und Physik*. Consider a dynamical problem described by canonical equations of motion for a given coordinate system. A transformation from this set of coordinates to a new set of coordinates is canonical if the equations of motion in the new coordinate system are also canonical. A key result proved by Carathéodory was that the canonical equations of motion are preserved under a specified class of transformations, given in terms of a suitable generating function. The invariance of the canonical equations under a canonical transformation is the fundamental key to the utility of Hamilton–Jacobi methods in describing dynamical systems.[7]

In 1935 Carathéodory published his *Variationsrechnung und Partielle Differentialgleichungen Erster Ordnung*, which contained his distinctive blend of partial differential equations, tensor analysis and the calculus of variations. The author displayed an incomplete grasp of the history, apparently unaware of Hamilton's dynamical memoirs of the 1830s that stimulated Jacobi's famous 1837 paper. He seemed to believe that Hamil-

[7] There are other forms of invariance that are of interest with respect to canonical transformations, such as the Poincaré integral invariants and Lagrange and Poisson brackets as canonical invariants (see Goldstein, 1950, pp. 247–58). The invariance that is germane to our discussion is the property of preserving the form of Hamilton's equations under transformation.

ton's contributions were solely to optics, and there is nothing to indicate any substantial familiarity with Jacobi's original papers.[8] The overall aim of the book was to investigate in a systematic way connections between partial differential equations and variational analysis. In part one on partial differential equations he developed the theory of canonical transformations, and presented a rather abstract and difficult-to-follow proof of the invariance of the Lagrangian equations of dynamics under a transformation (more precisely, he presented a general theorem from which this result is said to follow).[9]

Carathéodory was a cosmopolitan figure known for his facility with languages and broad appreciation of culture. From the 1930s until the end of his life he engaged in the study of the eighteenth-century history of the calculus of variations leading up to Euler's 1744 treatise as well as some of Euler's later memoirs on the subject. In his obituary of Carathéodory, Oskar Perron (1952, p. 42) noted, "he understood masterfully how to extract from the deficient methods of each period fruitful approaches for the exact treatment of the problems raised."[10] Carathéodory himself provided an elegant statement of the promise of history in an address he delivered in 1936 to a meeting of the Mathematical Association of America at Harvard University:

> It may happen that the work of the most celebrated men may be overlooked. If their ideas are too far in advance of their time, and if the general public is not prepared to accept them, these ideas may sleep for centuries on the shelves of our libraries. Occasionally, as we have tried to do to-day, some of them may be awakened to life. But I can imagine that the greater part of them is still sleeping and is awaiting the arrival of the prince charming who will take them home (Carathéodory, 1937, p. 233).

Carathéodory published two scholarly articles on the early history of the calculus of variations, but his main contribution was an introduction he wrote to Euler's *Methodus Inveniendi*, an undertaking carried out

[8] Carathéodory incorrectly gives the year of Jacobi's paper as 1836. He could not have been very familiar with the paper, since Jacobi's opening sentence consists of an acknowledgment of Hamilton's papers on mechanics of 1834 and 1835. Carathéodory's neglect of history here stands in contrast to his scholarly investigations at this time of the early-eighteenth-century history of the calculus of variations.

[9] In a review of the 1965 English translation of part one of Carathéodory's book, Richard Courant (1967) writes "The original [1935] book is a masterpiece of mathematical writing." An historical appraisal of its contents and influence remains to be written.

[10] "verstand er es auch meisterhaft, aus den unzulänglichen Methoden jener Zeit den fruchtbaren Kern herauszuschälen und wertvolle Ansätze zu finden zu einer exakten Behandlung der aufgeworfenen Probleme."

in what must have been difficult circumstances and connected to the planning for the re-publication of Euler's book in series one of his *Opera omnia*. Carathéodory completed this introduction by 1946, but volume 24 of the *Opera* with this introduction would not appear until 1952, two years after his death in February of 1950.

Carathéodory's account of Euler's book tended to have a modernizing quality, at times attributing to Euler twentieth-century methods and ideas that were understood to be implicit or at least foreshadowed in his analysis. This tendency is apparent in his account of Example 7 to Proposition III from Chapter 2 of Euler's 1744 book. The problem consists of maximizing or minimizing the integral $\int_a^b (x^2+y^2)^n \sqrt{(1+p^2)}dx$, where $p = \frac{dy}{dx}$.[11] Euler calculates N and P and considers the equation $N - \frac{dP}{dx} = 0$ for three cases: $n = 1$, $n = \frac{1}{2}$, and $n = \frac{3}{2}$. He shows in each case that the differential equation can be integrated by quadratures. In his solution Euler introduces complex numbers in order to integrate rational expressions in a way that had become fairly standard by that time. According to Carathéodory (1952, p. xxxvii), this problem today would be reduced by means of a conformal mapping to the problem of the shortest distance in the plane. While Carathéodory conceded that Euler did not notice this fact, he found it "astonishing" that in his introduction of complex numbers in the solution Euler followed a method that "in principle agrees with the one that we would use today."[12]

Carathéodory provided a classification of the different variational

[11] This problem for $n = 1$ was analyzed in parametric form by Tonelli (1923, pp. 430–435), who referred to Euler. Tonelli presented it as a minimization problem for moment of inertia. Carathéodory (1935, pp. 307–309) formulated the general problem in the complex plane where the real and complex parts are given parametrically. Euler presents the variational integral in the form $\int (xx + yy)^n dx \sqrt{(1 + pp)}$.

[12] "...der in Prinzip mit demjenigen, den man heute benutzen würde, übereinstimmen." The following is a reconstruction of the train of thought that may have led Carathéodory to this improbable conclusion. In Section IV of his introduction Carathéodory began by describing a paper which Euler wrote near the end of his life and which was published in Euler (1810). Carathéodory refers to it only as E 731. In fact it was written some 35 years after the publication of the *Methodus Inveniendi*. The problem under consideration is a generalization of Example 7 from Chapter 2 of the 1744 book. Euler showed how the use of polar coordinates simplifies the integration. In the introduction Carathéodory then continued with a discussion of Example 7. Unmentioned by Carathéodory was the fact that he had analyzed in some detail a version of Example 7 in his 1935 book, where he took a complex number in polar form and transformed the real and imaginary parts to end up with an expression (in parametric form) formally similar to the one in Example 7. He was apparently impressed by the fact that in the 1810 paper Euler had used polar coordinates, that he himself used the polar representation of complex numbers in his 1935 book, and that Euler had in 1744 in his solution to Example 7 in the *Methodus Inveniendi* used complex numbers to integrate rational expressions. These considerations seem to have led Carathéodory to the above conclusion.

problems that Euler addressed. The problem of the shortest distance between two points in a polar-coordinate system, discussed above, was grouped with other examples from Chapter 4 of the *Methodus Inveniendi* under the title “Covariant transformation of variational problems.” He wrote that these examples “may be viewed as evidence for the covariance of Euler’s equations for arbitrary coordinate transformations.”[13] He concluded with the observation: “Thus, in Euler’s book, we find the first indications of a theory that has only been systematically developed in our time.”[14]

In his *History of the Calculus of Variations* Herman Goldstine (1980, p. 84) was inspired to include a whole section of the chapter on Euler under the heading “Invariance Questions.” Here are presented the examples from the first part of Chapter 4 of the *Methodus Inveniendi*. Goldstine acknowledged a suggestion he had received from André Weil that “Euler’s interest in this topic probably stemmed from Leibniz’s inquiries into the behavior of $\int y dx$ under coordinate transformations.” Goldstine stated, echoing Carathéodory, “It is truly in keeping with Euler’s genius that he should have worked at ideas that were only to be satisfactorily and completely discussed in modern times.”

8.2.3 Some critical reflections

Euler first formulated and proved fundamental theorems about variational integrals with a general integrand function $Z(x, y, y')$. The further elaboration of this theory took the form of the derivation and solution of the Euler variational equation for a range of problems. A variational problem is posed and gives rise to an integrand function $Z(x, y, y')$. It is necessary to find $y = y(x)$ such that $\int_a^b Z dx$ = an extremum. If we formulate the problem analytically in other variables u and v in a different coordinate system then the variational problem gives rise to an integrand function $W(u, v, v')$ and it is necessary to find $v = v(u)$ such that $\int_a^b W du$ = an extremum. One always begins with the problem, followed by an analytical description, followed by the Euler variational equation, followed by a solution.

[13] “können als Proben für die Kovarianz der Eulereschen Gleichungen bei beliebigen Koordinaten-Transformationinen bewertet werden.”

[14] “Somit finden wir im Eulerschen Buche die ersten Ansätze zu einer Theorie, die erste in unseren Tagen systematisch entwickelt worden ist.”

One could begin with the condition $\int_a^b Zdx$ = an extremum, independent of any motivating problem and ask if a transformation of variables from x, y to u, v leads to a differential equation that may also be solved. This is apparently what Carathéodory and Goldstine had in mind when they credited Euler with having the intuition or some inkling of modern ideas of invariance. The problem of the shortest distance between two points was formulated in a standard Cartesian coordinate system and reduced to a differential equation by the variational process. The same result was then obtained using a polar-coordinate system, and again reduced to a differential equation and solved. While Euler did not employ a transformation from Cartesian to polar variables, the ideas of transformation and invariance were implicit in his analysis. The difficulty with this interpretation is that it projects onto the original analysis a way of thinking about the subject that is not present either as a potential idea or as an unrealized intuition.

As a point of comparison consider canonical transformations in Hamilton–Jacobi theory. For any dynamical system one can take a set of coordinate variables and investigate the system from first principles using them. One would end up with a Hamiltonian and canonical equations for these variables. No transformations are involved. Why then are transformations useful? The answer is that one can find canonical transformations using a generating function, and in the new coordinates so obtained the equations will be canonical and may be easier to integrate. Indeed, by taking the generating function to be a solution of the Hamilton–Jacobi partial differential equation for the problem, one is able to transform the original coordinates to ones that are constant, and the problem is solved. The invariance of the canonical equations under transformations provides a coherent and effective tool for integrating the differential equations that describe the dynamical system.

The situation with the Euler equation is rather different. In textbooks on the calculus of variations, the Euler equation is typically obtained for a given problem using a suitable selection of variables. There seems to be no advantage in beginning with a set of variables and transforming to new ones to obtain a transformed Euler equation. One could simply formulate the problem directly in terms of the new variables.

It is certainly of mathematical interest to investigate the invariance of the Euler equation for a given set of variables, independent of how the variational integral was obtained or any geometric or physical significance one attaches to the variables. No one in the eighteenth century did this, and few modern textbooks do it. In 1969 the Austrian–American mathematician Hans Sagan published a textbook in which he gave a detailed account of the invariance of the Euler equations.[15] Having developed the basic variational theory in the usual way as part of real analysis, Sagan (1969, p. 108) makes the remarkable and incorrect assertion that the derivation of the Euler equation "was essentially based on the fact that the coordinates were cartesian coordinates." Nevertheless, he does provide a detailed and useful account of the issues – not altogether simple – arising in any transformation and states conditions under which the families of comparison arcs in the two systems are comparable. Sagan (1969, pp. 108–115) shows that under "fairly liberal conditions on the integrand, the extremal and the transformation itself" invariance will hold.

8.2.4 Euler and the foundations of analysis

In Leibniz's original paper of 1684 on the calculus he considered the problem of finding the path followed by a light ray in going from two points *A* and *B* across an optical interface. The time of transit is connected by an equation to a spatial coordinate variable. The relationship between the time and the spatial variable can then be expressed by a curve, and one is able to apply the differential algorithm that he developed for curves. In the early years of the next century mathematicians such as Pierre Varignon used a comparison orthogonal Cartesian graph in investigating curves given in polar coordinates. There was a pervasive use of geometrical diagrams and representations in investigating what today would be called functional relationships between variables. (See Fraser (2003) for more details.)

The *Methodus Inveniendi* was an important step in Euler's program to separate analysis from geometry, and here lies the significance of the first part of Chapter 4 of that work. An equation between two variables is the basic object of study, and could be conceived of and investigated independently of any particular geometrical interpretation or coordinate representation. Although the function concept is not explicitly intro-

[15] Hans Sagan received his PhD in 1950 in calculus of variations from the University of Vienna and had a career in the United States at North Carolina State University. He was a commentator for the collected works of Johan Radon, Hans Hahn, and Karl Menger. See "History of the Math Dept at NCSU," at `http://www4.ncsu.edu/~njrose/Special/Bios/Sagan.html`.

duced in 1744, it is certainly implicit in Euler's investigation, and would be the next step in the elaboration of his program of analysis.

8.3 Euler and divergent series

8.3.1 Convergence and rigor

In the years following the publication of the *Methodus Inveniendi*, Euler pursued a range of subjects in analysis. A prominent area of investigation was infinite series, mainly but not exclusively power series. Detailed accounts were presented in his *Introductio in Analysin Infinitorum* of 1748 and in the multi-volume tomes on calculus that he published in the 1750s. During this period he began to investigate in a more serious way series that do not converge arithmetically but nonetheless have interesting properties. Euler's most productive effort in this direction was a paper on divergent series that he submitted first to the Berlin Academy in 1746, then to the St. Petersburg Academy in 1753, and that was finally published in 1760 in the memoirs of the St. Petersburg Academy for 1754–1755.[16]

Among modern mathematicians there is often a sense that Euler's use of divergent series was naïve and cavalier regarding convergence. Typical is C.N. Moore's (1932, p. 64) remark, "It is apparent that the procedure of Leibnitz and Euler in the case of the simple series $1 - 1 + 1 - 1 + \cdots$ is entirely out of harmony with present day notions of rigor in analysis." However, to one recent commentator Euler's reputation has been unjustly "tarnished" and should be redeemed "as a result of recent developments" in the theory (Kowalenko, 2011, p. 370). According to this view, the route to Euler's vindication is provided by the modern theory of *summability*. Summability, developed in the 1890s by Ernesto Cesàro and others, established a rigorous foundation for divergent series. This leads to the question: Was Euler a summability theorist ahead of his time?

8.3.2 Infinite series in the eighteenth century[17]

Early-eighteenth-century mathematicians had intuitive notions of convergence and divergence. Due to the prevailing view that mathematics was about *quantity*, one was free to obtain the "development" of any

[16] The chronology for the publication of this paper is given by Faber (1935, p. lxxv).
[17] In this section we follow Ferraro (2008) and Kline (1972).

quantity in the form of a power series (assumed to converge for at least some values of the variable) but could leave aside the question of convergence until applying the series to a geometric problem. This principle of "infinite extension" thus allowed for the formal manipulation of both finite and infinite series to occur prior to questions of convergence. The mid-century shift in calculus from its geometric foundation to a form of algebraic analysis gave a further boost to mathematicians' confidence in formulas. Under this philosophy, the general applicability of any method derived from the generality of its *object*. Since formulas were objectively given as part of algebra, their generality of usage was assured, even if this gave rise to divergent series. Formalism thus became untethered from geometry while remaining subtly connected with intuitive notions of quantity. As the most prolific practitioner of this new analysis, Euler's willingness to pursue formalism's implications for infinite series brought this tension to the foreground.

Consider the series $1 - 1 + 1 - 1 + \cdots$, which had been studied by several mathematicians before Euler. Clearly it did not become infinite. Grouping the terms $(1-1)+(1-1)+\cdots$ seemed to make the sum zero, but when writing $1-(1-1)-(1-1)-\cdots$ it appeared the sum was 1. In 1703, Guido Grandi argued that the series representation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

yields the answer $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$ upon letting $x = 1$. Leibniz concurred, but he made his case probabilistically: since the result was either 0 or 1 depending on the number of terms summed, one should take the answer to be the average. When Euler turned to this series, he began with the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

which he saw as a valid formal development of the "quantity" $\frac{1}{1-x}$. Upon letting $x = -1$, it yields $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \cdots$, in agreement with Grandi and Leibniz. But Euler went on to let $x = 2$, which gave him $-1 = 1 + 2 + 4 + 8 + \cdots$. This put him in new territory, as the series is no longer bounded but clearly divergent in the infinite sense. In a similar vein, the substitution of $x = -1$ into the development

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

led Euler to write $\infty = 1+2+3+4+\cdots$. Upon comparing these latter two results, Euler concluded that -1 must be greater than ∞, hence ∞ must serve as a kind of boundary between the positive and negative numbers. These results were not without criticism even in his own time. Ferraro (2008, p. 216) highlights the following comment Nicolaus Bernoulli made to Euler in 1743, "I cannot persuade myself that you think that a divergent series . . . provides the exact value of a quantity which is expanded into the series." Bernoulli's caution may ring true to modern mathematicians because of the hindsight afforded by the nineteenth century's turn from formalism to rigor (Fuss, 1843, pp. 701–702).[18]

8.3.3 Cauchy's new definitions

The decline of formalism stemmed mainly from its limitations as a means of generating useful results. Moreover, as methods began to change, an awareness of formalism's apparent difficulties and even contradictions lent momentum to efforts to rein it in. Euler had been confident that the "out-there" objectivity of algebra secured the generality of his formal techniques, but Cauchy demanded that generality be found within mathematical methods themselves. In his *Cours d'analyse* of 1821, Cauchy rejected formalism in favor of a fully quantitative analysis. Rather than the formulas themselves, the "quantities" became the individual values of the expressions when the variable took on a certain value. Hence the statement $f(x) = g(x)$ was not a formal relationship holding for indeterminate x but was a quantitative statement holding only for specific values of x. The statement $\frac{1}{(1-x)} = 1+x+x^2+x^3+\cdots$ was true only for the values for which the series converged. Otherwise (say, when $x = 2$) it was meaningless. Cauchy specified that given the sum of the first n terms of a series

$$s_n = \sum_{i=0}^{n-1} u_n,$$

[18] Commenting on the work of eighteenth-century mathematicians but referring specifically to Euler, Kline (1983, p. 307) writes, "Their efforts to justify their work, which we can now appraise with the advantage of hindsight, often border on the incredible." Wonderment at the reasonings of the mathematical masters of the past occurs not infrequently in modern historical commentaries. Consider the following comments of James Pierpont (1928, p. 32) on Lagrange's expansion of functions by Taylor series: "When a modern reader looks over reasoning like this and bears in mind that Lagrange was one of the greatest mathematicians of all time, he is amazed. The great gulf that separates mathematical reasoning of to-day from that of date 1813 is brought home very clearly to him."

if the partial sums s_n approach a limit s as n increases, then the series converges with s as its sum. If the limit does not exist, the series diverges and there is no sum. The question of convergence now came first, and the manipulations the formalists had assumed valid for both finite and infinite series now depended on whether the series converged. Niels Abel concurred with the new ideas and in a letter of 1828 to Bernt Holmboe declared, "Divergent series are in general something very fatal, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion one wishes and it is these series that have produced so many misfortunes and given birth to so many paradoxes" (Abel, 1881, pp. 256–7).[19]

It should be noted that mathematicians in England and Germany continued to study divergent series well into the middle decades of the nineteenth century. Eberhard Knobloch (2015, p. 501) has suggested that the theoretical predilections of English formalists such as George Peacock and Augustus De Morgan were aligned with Euler's conception of divergent series. Certainly, the English formal school was more favorably disposed to divergent series than were the French rigorists. (For a survey of these developments see Burkhardt (1910). Compare also Fraser (2003, pp. 325–7).) However, it is fair to say that the mainstream of analysis with its emphasis on rigor led as the nineteenth century progressed to the marginalization of divergent series as an area of mathematical investigation. The notion that rigor rescued mathematics from disarray has become a rather common view and might be considered a part of the *heritage* of the modern mathematician, leading to the discomfort over Euler's status. If one views the concepts "series" and "sum" as cumulatively improving entities gradually unveiled for us over hundreds of years, then one might well say that Cauchy's refinements do eclipse Euler's results.[20] However, Cauchy did not instill rigor by *introducing* convergence, as if Euler had failed to consider it. Cauchy changed the notion of what an infinite series is – no longer an algebraic object subject

[19] "Les séries divergentes sont en général quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration. On peut démontrer tout ce qu'on veut en les employant, et ce sont elles qui ont fait tant de malheurs et qui ont enfanté tant de paradoxes." These sentences are translated by Kline (1972, p. 973) as follows: "The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes."

[20] This line of thought may even be natural to, say, a calculus teacher who needs to justify why convergence matters.

to formal manipulation, but a relationship between definite numerical series subject to convergence first. Euler and Cauchy had distinct notions of "series," so it is dubious to charge Euler with mere negligence.[21] One might instead pivot to the idea that Euler's reputation was unfairly stained by "victors" who rewrote mathematics at his expense. Consider Hardy's (1949, p. 17) remark: "Mathematics after Euler moved slowly but steadily towards the orthodoxy ultimately imposed on it by Cauchy, Abel and their successors, . . . after Cauchy, the opposition seemed definitely to have won." The belief that Euler's successors treated him poorly might lead the historian today to assess his theory of series in a more sympathetic way.

8.3.4 Summability theory

The predominant mathematical trend in the nineteenth century was to support Cauchy's thinking about infinite series. The sum of a convergent series *existed* and the sum of a divergent series did not *exist*. To be sure there were English and German exceptions, and their researches provided impetus to continue to think about divergent series. Also notable was evidence for the utility of divergent series, such as in the asymptotic approximation of certain functions. Research in complex analysis (then called the theory of analytic functions) suggested a possible need. Consider an analytic function represented by a power series on an open disk but not for values on the disk's boundary: a new concept of "sum" would allow for the assignment of a value for the function on the boundary.

Alongside these new research concerns, shifts occurred in the foundational methods of mathematics. Mathematicians began to see their theories not as descriptions of reality but as syntactic structures whose theorems were logical derivations from axioms and definitions. Support for this conception came from the non-Euclidean geometries, which showed that the postulates of Euclid were not necessary to produce a consistent geometry. Kline (1972, p. 1097) writes: "The mathematicians slowly began to appreciate that mathematics is man-made and that Cauchy's definition of convergence could no longer be regarded

[21] Fraser (1989, pp. 323–4) notes, "The significant change in the theory of infinite series, however, was not so much that classical analysis brought rigour to the subject by paying attention to convergence, but that an arbitrary series whose individual terms were specified at will now became, subject to convergence over some domain, implicitly an object of mathematical study. The understanding of what an infinite series was had undergone a substantial transformation."

as a higher necessity imposed by some superhuman power." The first suggestive attempts to produce a new theory of series came from Ferdinand Georg Frobenius (1880) and Otto Hölder (1882). In 1890, Ernesto Cesàro provided the first modern concept of the sum of a divergent series, known as the theory of summability.[22]

According to the theory one is free, in principle, to select any desired procedure P for the summation of a series. This choice is merely conventional and has no intrinsic connection to the series. There is no sense of finding the "true" sum or any anchoring to an external reality. Hence the sum of a series "exists" (or not) only relative to the procedure P; if it does exist, it is called the P-sum. In practice, P should give fruitful consequences. For example, it is desirable that the P-sum agree with the usual sum for a convergent series – this is called *regularity*. Consider, then, the summation of the previously considered series $1-1+1-1+\cdots$ according to the following method. Given the series $s_n = u_0+u_1+\cdots+u_n$, if

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

exists and is equal to s, then we call s the (C, 1) sum of the series (C stands for Cesàro). Now considering our series $s_n = \sum_{i=0}^{n}(-1)^i$ we see that the "sum of the partial sums" is

$$s_0 + s_1 + \cdots + s_n = \begin{cases} 1+0+1+\cdots+1 & = \frac{n+2}{2} \quad \text{for even } n \\ 1+0+1+\cdots+0 & = \frac{n+1}{2} \quad \text{for odd } n \end{cases}.$$

But in either case we get

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = \frac{1}{2}$$

so the (C, 1)-sum of the series is $\frac{1}{2}$. This agrees, of course, with Euler's answer.

For a second example (Hardy, 1949, pp. 7–8), consider another definition as follows. If the power series $\sum a_n x^n$ converges in the usual sense for small x and defines a function $f(x)$ (subject to some additional conditions) and $f(1) = s$, then we call s the $\mathfrak{E}$-sum of the series $\sum a_n$ (the symbol $\mathfrak{E}$ is for Euler). But then for

$$f(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

[22] This exposition follows Ferraro (1999).

we have $s = f(1) = -1$, so the $\mathfrak{E}$-sum of the series $1 + 2 + 4 + 8 + \cdots$ is -1, which again agrees with Euler's answer.

Despite these advances, Abel's distrust of divergent series seems to have remained fairly widely held. In the preface to Hardy's *Divergent Series* (1949, p. vii) J.E. Littlewood remarked: "In the early years of the century the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity." However, by the time of Hardy's volume, summability was well established, making divergent series mundane and leading to a reassessment of past results. In fact, Hardy (1949, p. 47) states that Leibniz, without actually specifying a definition, nevertheless employed precisely the (C, 1) procedure, and that the E-procedure is based on Euler's principles. His remarks are those of a mathematician who tends to see distant predecessors, in Fried's (2018) parlance, as "mathematical colleagues." Thus Hardy (1949, p. 15) says, "It is a mistake to think of Euler as a 'loose' mathematician, though his language may sometimes seem loose to modern ears; [it] somehow suggests a point of view far in advance of the general ideas of his time . . . language which might almost have been used by Cesàro or Borel."[23]

Hardy's perception would be echoed in historians that followed him. In the article on Euler in the *Dictionary of Scientific Biography* A.P. Yushkevich (2008, p. 473) writes:

> But he also was a creator of new and important notions and methods, the principal value of which was in some cases properly understood only a century or more after his death. Even in areas where he, along with his contemporaries, did not feel at home, his judgment came, as a rule, from profound intuition into the subject under study. His findings were intrinsically *capable of being grounded in the rigorous mode of demonstration* that became obligatory in the nineteenth and twentieth centuries [emphasis added].

Euler biographer Ronald Calinger (2016, p. 93) observes that "In some cases it would take a century for scientists to grasp the proper use of his procedures." On the subject of infinite series Kline (1972, p. 453) lauds Euler's formal manipulations.

It is certainly true that Euler displayed a remarkable mathematical inventiveness and employed a range of methods in his study of divergent series. Kline (1972, p. 1110) concludes that Euler's results awaited

[23] Hardy's remarks here are also quoted by Varadarajan (2006, p. 130), who adds, "It is therefore clear that Euler had an understanding of the issues involving divergent series that was very much ahead of his time."

only a rigorous confirmation, which vindicated them. He states directly: "With hindsight we can see that the notion of summability was really what the eighteenth- and early nineteenth-century men were advancing." These comments seem to ascribe to Euler an unaccountable prescience that saw the "right" foundational theory on some imagined future horizon. However, as we shall now try to show, the approach of a modern mathematician differs substantially from that of an eighteenth-century formalist.

8.3.5 Different kinds of definition

The contrast in understanding between an eighteenth-century mathematician such as Euler and a modern researcher appears most strikingly in the notion of *definition*. On this point, Hardy concedes the methodological gulf between Euler and himself. Regarding eighteenth-century mathematicians he says:

> They had not the habit of definition; it was not natural to them to say, in so many words, 'by X we *mean* Y'. . . Mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - \cdots$?' but 'What *is* $1 - 1 + 1 - \cdots$?', . . . This habit of mind led them into unnecessary perplexities (Hardy, 1949, p. 6).

Similarly, Konrad Knopp (1928, p. 457) puts his finger on the issue: "In our exposition, *the symbol for infinite sequences was created* and then worked with; it was not so originally, these sequences *were there*, and the question was, what could be done with them" (emphasis in original). For mathematicians of the modern era, a symbol has meaning only when one assigns it such. Definitions are acts of the will, which construct *a priori* the very objects of study and thus cannot be "true" or "false." One defines basic terms (or primitives) implicitly via the axioms, while other terms are defined explicitly simply as symbolic abbreviations. A theory consists of symbols arranged in a syntactic structure, whose elements can take on any interpretation one wishes; hence, for example, one is free to define a summation procedure P and see what happens.

This practice was foreign to Euler. Eighteenth-century mathematicians did make definitions, yet they were of the "Euclidean" sort, where the appropriate response to their question "What is X?" was a *descriptive* definition that gave an accurate account of X's true nature. Objects of mathematical study were taken to have a real existence in the world. The purpose of a definition was to specify and clarify an object to enable its investigation.[24] Thus while a P-sum is self-consciously "man-made," an Eulerian sum is an attempted observation of nature. Euler's thinking was therefore more in line with Cauchy's than with that of a modern summabilist. Both Euler and Cauchy wielded the "Euclidean" notion of definition as elucidation of "the unique and necessary 'truth' that already existed in nature" (Ferraro, 2008, p. 222). For Euler, objective truth was a necessary criterion for a proper definition.

8.3.6 Euler's definitions

Euler's formalism derived from the objective reality of the rules of algebra. Consider his treatment of the statement $d(\log x) = \frac{dx}{x}$. Leibniz held that this is meaningful only for positive real x, but Euler disagreed:

> For, as this calculus concerns variable quantities, that is, quantities considered in general, if it were not generally true that $d \cdot lx = \frac{dx}{x}$, whatever value we give to x, either positive, negative, or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains (Euler, 1751, p. 143, translation by Fraser, 1989, p. 331).[25]

For Euler, the applicability of the calculus stemmed from the general character of its formulas and rules, which were true and given as part of the subject of mathematics. Yet with analysis detached from its original anchoring in geometry, the status of its *objects* of study was not entirely clear. Euler's results on divergent series had strained the connection between formalism and "quantity" and required clarification. But Euler did not make an arbitrary definition subject only to logical consistency; this would have been unthinkable. Instead, he sought a concept of "sum" that would avoid disputes and provide a basis for further research; it must *not* be arbitrary. Euler dealt with the matter in his *Institutiones calculi differentialis* (1755) and *De seriebus divergentibus* (1760). His finding that -1 exceeded ∞ *was* sensibly quantitative, he said, on the grounds that infinity, in analogy with zero, was a transition from positive to negative. This could be defended by the "law of continuity and geometry." But

[24] Naturally Hardy does not consider such an act to be a "definition," as the meaning was different by his time.

[25] "Car comme ce calcul roule sur des quantités variables, c. à d. sur des quantités considérées en général, s'il n'etoit pas vrai généralement, qu'il fût $d \cdot lx = \frac{dx}{x}$, quelque quantité qu'on donne à x, soit positive ou négative, ou même imaginaire, on ne pourrait jamais se server de cette régle, la verité du calcul differential étant fondée sur la généralité des regles, qu'il renferme."

upon seeing that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

yields $1 = 1+4+12+32+\cdots$ for $x = 2$ but gives $\infty = 1+2+3+4+\cdots$ for $x = 1$, it now seemed that even 1 must be greater than ∞, which was a clear difficulty. Hence Euler (1755) admitted that a basically quantitative understanding of divergent series could not be sustained. In his *Institutiones calculi* he extended the notion in a more formal direction:

Let us say, therefore, that the *sum* of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense, the sum of the infinite series $1-x+x^2-x^3+\cdots$ will be $\frac{1}{(1+x)}$, because the series arises from the expansion of the fraction, whatever number is put in place of x.

This statement agrees with the above comment about $d(\log x)$. He continues:

If this is agreed, the new definition of the word sum coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections (Euler, 1755, pp. 78–9, translation by Bromwich, 1908, p. 266).[26]

Barbeau and Leah (1976, p. 142) interpret these passages to mean that Euler "distinguishes between convergent and divergent series along modern lines . . . Thus his assignment of a sum to a divergent series is a matter of conscious decision, made on pragmatic grounds and defensible by the consistency of mathematical analysis." They note that while Euler often is perceived as misunderstanding infinite series, his ideas were vindicated by Hardy. It is certainly true that Euler was creative and versatile in his investigation of such series. Yet there is a clear difference between Euler's thinking and Hardy's. Euler was not seeking an artificial

[26] "Dicamus ergo seriei cuiusque infinitae summam esse expressionen finitam, ex cuius evolutione illa series nascatur. Hocque sensu seriei infinitae $1+x+x^2+x^3+$ &c. summa revera erit $= \frac{1}{1-x}$, quia illa series ex huius fractionis evolutione oritur; quicunque numerus loco x substituatur. Hoc pacto, si series suerit convergens, ista nova vocis summae definitio, cum consueta congruet; & quia divergentes nullas habent summas proprie sic dictas, hinc nullum incommodum ex nova hac appellatione orietur. Denique ope huis definitionis utilitatem serierum divergentium tueri, atque ab omnibus iniuriis vindicare poterminus." Difficulties with this definition in the context of some of Euler's posthumously published work on trigonometric series are discussed by Faber (1935, p. lxiv). See also Knobloch (2015, p. 501).

way to retain the use of divergent series but was trying to capture what a "sum" really was, to clarify the nature of the objects of his study. For Euler the sum of a series was a function if and only if the series resulted from the formal development of that function by the principle of infinite extension. He defended his definition:

> If therefore we change the accepted notion of sum to such a degree that we say the sum of any series is a finite expression out of whose development that series is formed, all difficulties vanish of their own accord. For first that expression from whose expansion a convergent series arises displays the sum, this word being taken in its ordinary sense; and if the series is divergent, the search cannot be thought absurd if we hunt for that finite expression which expanded produces the series according to the rules of analysis[27] (Euler, 1760, p. 212, translation by Barbeau and Leah, 1976, p. 148).

Again, a cursory reading might see Euler's comments as making an arbitrary definition. But there is a distinction: Euler's definition is grounded in the "rules of analysis". As he saw it, the true meaning of "sum" had to accommodate divergent series. He believed that the use of divergent series could never lead to an error (Ferraro, 2008, p. 225).

Euler was trying to study the objects that nature had thrust upon him, by making and testing hypotheses. An example of this heuristic is his consideration of the series

$$s = x - (1!)x^2 + (2!)x^3 - (3!)x^4 + \cdots$$

which diverges for all x except 0. This series formally satisfies the differential equation $ds + \frac{sdx}{x^2} = \frac{dx}{x}$; however, the equation can be integrated to obtain the solution

$$s = e^{\frac{1}{x}} \int_x^0 \frac{e^{\frac{-1}{t}}}{t} dt.$$

Euler maintained that the infinite series must be the expansion of this solution. Letting x equal 1 we obtain a value for the divergent hypergeometric series of alternating factorials:

$$1 - 1! + 2! - 3! + \cdots = e \int_0^1 \frac{e^{\frac{-1}{x}}}{t} dt,$$

[27] "Si igitur receptam summae notionem ita tantum immutemus, ut dicamus cuisque seriei summam esse expressionem finitam, ex cuius evolutione illa ipsa series nascatur, omnes difficultates, quae ab utraque parte sunt commotae, sponte evanescent. Primo enim ea expressio, ex cuius evolution nascitur series convergens, eius simul summam, voce hac vulgari sensu accepta, exhibit, neque, si series fuerit divergens, quaestio amplius absurda reputari poterit, si eam indagemus expressionem finitam, quae secundum regulas analyticas evoluta illam ipsam seriem producat."

with the integral on the right being approximately 0.59637.[28] This estimate for the sum of the series was confirmed by other methods Euler introduced to find its value (the "Euler summability method" and partial fractions; see Barbeau and Leah, 1976, pp. 149–53). Thus, he saw that his definition of sum was supported by results from different approaches, confirming the objectivity and correctness of his proposal to define the sum in this way.[29]

Despite this success, Euler's attempt to capture the true nature of divergent series encountered difficulties. In 1797 J.F. Callet pointed out that given

$$\frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 + \cdots,$$

when $x = 1$ we have

$$\frac{2}{3} = 1 - 1 + 1 - 1 + \cdots,$$

which does not agree with Euler's previously determined value $\frac{1}{2}$. In some of his posthumously published writings Euler himself drew attention to "paradoxes" that arise in the study of infinite series (see the footnote 18 on page 236). These kinds of problems would eventually lead to Cauchy's attempt at a novel definition for the "true" sum of a series. While Cauchy was initiating a new epoch in the history of calculus, a belief in objective truth was something he continued to share with the older researchers. By contrast the summabilists, whose work traced the fault lines of nineteenth-century upheavals in the foundations of mathematics, introduced definitions that had no concern with truth beyond logical coherence.

28 Interestingly, although the power series diverges, nonetheless its successive partial sums give rather good approximations of the integral for a given x. For example, if one lets $x = 0.1$ and takes the successive partial sums of s one obtains a good approximation to the integral from the seventh sum to the fourteenth sum; the sums remain close until the twentieth sum after which the factorials begin to dominate and the expansion diverges. Euler did not comment on the asymptotic character of the expansion although it is not unlikely that he was aware of it. Considerations related to asymptoticity arise in the memoir Euler (1750); see Faber (1935, pp. cxi–cxii) and Barbeau and Leah (1976, p. 150).

29 While philosophical analysis is beyond the scope of this chapter, it could be maintained that the contents of Euler (1760) are characterizable in terms of the "quasi-empiricism" identified by philosopher Hilary Putnam (1975). Another article that touches on Euler's work on divergent series in relation to the philosophy of mathematics and mathematics education is Schroter (2018).

8.4 Conclusion

There is more than one way to view the mathematics of the past. Ivor Grattan-Guinness (2004) identifies a disjunction between *heritage* (our tracking of a particular concept's journey along the "royal road" from the past to the present) and *history* (our attempt to explain why a certain mathematical development happened). The "heritage" approach evaluates past mathematics in light of recent theories, looking for similarities that reveal the gradual unveiling of a mathematical concept. Conversely, "history" instinctively looks for differences and discontinuities. Michael Fried (2018) extends Grattan-Guinness's idea by positing a spectrum of viewpoints. He describes *mathematicians*, who see their work as entirely coextensive to that of earlier mathematicians; *mathematician–historians*, trained mathematicians to whom modern mathematics provides the privileged perspective with which to assess the past; and *historians of mathematics*, to whom "the past is a problem" that stands in contrast with the present. Fried comments, "Faced with a mathematical text, historians of mathematics try not to coordinate the text with the mathematics of the present, but to set it out from the present; they try to make it not more familiar but rather more strange, more foreign . . . bring out its identity." Fried notes that these sentiments are echoed in political philosopher Michael Oakeshott's concern for the past as past, and with each moment of the past in so far as it is unlike any other moment.

Therefore, one's assumptions and mode of thought become paramount. When studying past mathematics, should we take it to be an historically evolving subject or understand it as the unfolding of a timeless whole? The mathematics teacher or theorist may view it in the latter way, at least psychologically; mathematics (all of it) is what mathematicians do, so it must possess common styles of thought, procedures, inferences, and rules for mathematical advance. By contrast, history looks for differences. A historian can remain agnostic on philosophy of mathematics while an historically minded mathematician generally may not feel so inclined. Each mode of thought has distinct objectives.

Historically speaking, then, the perception of Euler as a visionary may obscure the actual character of his work and its foundational import. The significance of the examples in Chapter 4 of the *Methodus Inveniendi* derives from their place in Euler's evolving program to separate analysis from geometry, not in any glimpse of some future notion of invariance.

The significance of Euler's formal approach to divergent series is not in the way it foreshadowed modern theories of summability, but rather in the latitude it provided him to obtain actual numerical values for divergent series. The modern conception of mathematical invariance or summability, as well as the associated philosophical commitments, are very different from Euler's own beliefs and outlook.

Claims that Euler grasped invariance, or was a summabilist, thus are anachronistic. On one hand, anachronistic approaches find productive uses in the classroom. After all, we are able to recognize from our point of view that both Euler and Hardy were engaged in studying "divergent series." Hence we may draw on "heritage" for didactic purposes – say, to teach about convergence. Indeed, there is some appeal to tracing the "history of a concept" – a directional journey from a "period of indecision," to the clear present. On the other hand, for one who takes our modern concepts and methods to be correct, it is easy to slip from a view that Euler *ought* to have used them to a claim that he *did* somehow use them. It is then that anachronism reaches the end of its utility: a more historical lens is required to help us see the "past as past" and understand Euler's achievements in their own context.

References

Abel, Niels Henrik (1881). *Oeuvres complètes de Niels Henrik Abel, Nouvelle Edition Vol. 2*. Christiania: Grøndahl & Søn.

Barbeau, Eduard J. and P.J. Leah (1976). Euler's 1760 paper on divergent series. *Historia Mathematica* **3**, 141–60.

Bolza, Oskar (1909). *Vorlesungen über Variationsrechnung*. Leipzig and Berlin: B.G. Teubner.

Boyer, Carl B. (1939). *The Concepts of the Calculus: A Critical and Historical Discussion of the Derivative and the Integral*. New York: Columbia University Press.

Bromwich, Thomas J. (1908). *An Introduction to the Theory of Infinite Series*. London: Macmillan.

Burkhardt, Heinrich (1910). Über den Gebrauch divergenter Reihen in der Zeit von 1750–1860. *Mathematische Annalen* **70**, 169–205.

Calinger, Ronald S. (2016). *Leonhard Euler: Mathematical Genius in the Enlightenment*. Princeton, NJ: Princeton University Press.

Carathéodory, Constantin (1925). Variationsrechnung. In *Die Differential- und Integralgleichungen der Mechanik und Physik*, Philipp Frank and Richard von Mises (eds). Braunschweig: Friedrich Vieweg & Sohn. Second edition (1930), pages 227–79.

Carathéodory, Constantin (1935). *Variationsrechung und Partielle Differentialgleichungen erster Ordnung*. Leipzig and Berlin: B.G. Teubner.

Carathéodory, Constantin (1937). The beginning of research in the calculus of variations. *Osiris* **3**, 224–40.

Carathéodory, Constantin (1952). Einführung in Eulers Arbeiten über Variationsrechnung. In *Leonhardi Euleri Opera Omnia* Ser. 1 V. 24, viii–lxii. Bern: Orell Füssli.

Carathéodory, Constantin (1965). *Calculus of Variations and Partial Differential Equations of the First Order. Part I: Partial Differential Equations of The First Order*. Translation of the first part of Carathéodory (1935) by Robert B. Dean and Julius J. Brandstatter. San Francisco, London, and Amsterdam: Holden Day Inc.

Courant, Richard (1967). Review of Carathéodory (1965). *Mathematical Reviews*, MR0192372.

Euler, Leonhard (1744). *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti*. Lausanne and Geneva: Marcum–Michaelem Bousquet & socios. Reprinted as *Leonhardi Euleri Opera Omnia Ser. 1 V. 24* (1952), Bern: Orell Füssli. There is a German translation of Chapters 1, 2, 5, and 6 by Paul Stäckel in *Ostwalds Klassiker der exakten Wissenschaften* Nr. 46, Abhandlungen über Variations-Rechnung. Erster Theil Joh. Bernoulli (1696), Jac. Bernoulli (1697), Leonhard Euler (1744), Leipzig: Wilhelm Engelmann (1894). There is an English translation by Ian Bruce at http://www.17centurymaths.com/contents/Euler'smaxmin.htm.

Euler, Leonhard (1750). Consideratio progressionis cuiusdam ad circuli quadraturam inveniendam idoneae. *Commentarii Academiae Scientiarum Petropolitanae* **11** (1739), 116–127. In *Leonhardi Euleri Opera Omnia* Ser. 1 V. 14 (1925), Leipzig and Berlin: B.G. Teubner, pages 350–363.

Euler, Leonhard (1751). De la controverse entre Mrs. Leibnitz & Bernoulli sur les logarithmes des nombres negatifs et imaginaires. *Memoires de l'Académie des Sciences de Berlin* 5 (1749), 139–171. In *Leonhardi Euleri Opera Omnia* Ser. 1 V. 17 (1914), Leipzig and Berlin: B.G. Teubner, pages 195–232.

Euler, Leonhard (1755). *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*. St. Petersburg: Imperial Academy of Sciences. Reprinted as *Leonhardi Euleri Opera Omnia Ser. 1 V. 10* (1911), Leipzig and Berlin: B.G. Teubner.

Euler, Leonhard (1760). De seriebus divergentibus. *Novi Commentarii Academiae Scientiarum Petropolitanae* **5**, 205–237. In *Leonhardi Euleri Opera Omnia Ser. 1 V. 14* (1925), Leipzig and Berlin: B.G. Teubner, pages 585–617.

Euler, Leonhard (1810). Solutio problematis ob singularia calculi artificia memorabilis. *Mémoires de l'académie de St.-Pétersbourg* **2** (1807/8), 3–9. In

Leonhardi Euleri Opera Omnia Ser. 1 V. 25 (1952), Bern: Orell Füssli, pages 280–285.

Faber, Georg (1935). Übersicht über de Bände 14, 15, 16, 16* der ersten Serie, *Leonhardi Euleri Opera Omnia Ser. 1 V. 16,2* Carl Boehm (ed.). Basel: Orell Füssli, pages vii–cxii.

Ferraro, Giovanni (1999). The first modern definition of the sum of a divergent series: An aspect of the rise of 20th century mathematics. *Archive for History of Exact Sciences* **54**, 101–35.

Ferraro, Giovanni (2008). *The Rise and Development of the Theory of Series up to the Early 1820s*. New York: Springer.

Fraser, Craig (1989). The calculus as algebraic analysis: some observations on mathematical analysis in the 18th century. *Archive for History of Exact Sciences* **39**, 317–335.

Fraser, Craig (1994). The origins of Euler's variational calculus. *Archive for History of Exact Sciences* **47**, 103–141.

Fraser, Craig (2003). Mathematics. In: *The Cambridge History of Science. Volume 4: Eighteenth-century Science*, Roy Porter (ed). Cambridge: Cambridge University Press, pages 305–27.

Fried, Michael (2018). Ways of relating to the mathematics of the past. *Journal of Humanistic Mathematics* **8** (1), 3–23.

Fuss, Paul Heinrich (1843). *Correspondance mathématique et physique de quelque célèbres géomètres du XVIIIème siècle*, St. Pétersbourg: Académie Impériale des Sciences.

Goldstein, Herbert (1950). *Classical Mechanics*. Reading, MA: Addison-Wesley.

Goldstine, Herman H. (1980). *A History of the Calculus of Variations from the 17th through the 19th Century*. New York: Springer-Verlag.

Grattan-Guinness, Ivor (2004). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica* **31**, 163–185.

Hardy, Godfrey H. (1949). *Divergent Series*. Oxford: The Clarendon Press.

Kline, Morris (1972). *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press.

Kline, Morris (1983). Euler and infinite series. *Mathematical Magazine* **56**, 307–14.

Knobloch, Eberhard (2015). Euler transgressant les limites l'infini et la musqiue. In: *Leonhard Euler Mathématicien, Physicien et Théoricien de la Musique*, Xavier Hascher and Athanase Papadopoulos (eds). Paris: CNRS Editions, pages 491–505.

Knopp, Konrad (1928). *Theory and Application of Infinite Series*. London and Glasgow: Blackie & Son Limited.

Kowalenko, Victor (2011). Euler and divergent series. *European Journal of Pure and Applied Mathematics* **4**, 370–423.

Moore, Charles N. (1932). Summability of series. *The American Mathematical Monthly* **39**, 62–71.

Oakeshott, Michael (1933). *Experience and its Modes*. Cambridge: Cambridge University Press.

Perron, Oskar (1952). Constantin Carathéodory. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **55**, 39–51.

Pierpont, James (1928). Mathematical rigor, past and present. *Bulletin of the American Mathematical Society* **34**, 23–53.

Putnam, Hilary (1975). What is mathematical truth? Chapter 4 of *Mathematics, Matter and Method*. London and New York: Cambridge University Press.

Sagan, Hans (1969). *An Introduction to the Calculus of Variations*. New York: McGraw Hill.

Schroter, Andrew (2018). In defence of Platonism in the mathematics classroom. *Philosophy of Mathematics Education Journal* **33** (online).

Sommerfeld, Arnold (1923). *Atomic Structure and Spectral Lines*. London: Methuen & Co. Ltd. (Second edition 1928).

Tonelli, Leonida (1923). *Fondamenti del calcolo delle variazioni. Volume secondo*. Bologna: Nicola Zanichelli Editore.

Varadarajan, Veeravalli Seshadri (2006). *Euler through Time: A New Look at Old Themes*. Providence, RI: American Mathematical Society.

Yushkevich, Adolf P. (Youschkevitch) (2008). Euler, Leonhard. *Complete Dictionary of Scientific Biography, vol. 4*, pages 467–484. New York: Charles Scribner's Sons. Gale Virtual Reference Library. Originally published in 1974.