Mathematical Technique and Physical Conception in Euler's Investigation of the Elastica

by

CRAIG G. FRASER*

Introduction

In the 1730s and early 1740s Euler studied the statics of thin elastic bands or laminae, researches that he first began to publish in 1732 and which he carried out as a member of the Academy of Sciences in St. Petersburg. His investigation centred on determining the shapes such laminae assume in equilibrium when subject to various loadings. In the case of principal interest, the "elastica" or "elastic curve", external forces were supposed to act at the ends of the lamina while its weight was itself regarded as negligible. The differential equation of the elastica had been derived by Jakob I Bernoulli as early as the 1690s. Euler's analysis was based on the study of a certain differential equation; from a later mathematical viewpoint one would say that he was considering the graphical solution of an elliptic integral. In the course of his investigation he obtained the "Euler buckling formula" yielding the maximum load an elastic column can sustain without bending.

Euler presented his results in the first part of the appendix "De curvis elasticis" to his 1744 treatise on the calculus of variations, *Methodus inveniendi curvas lineas.*[1] Daniel Bernoulli had suggested in a letter to Euler of 1742 that the equation of the elastica could be obtained by assuming a certain expression – in modern terminology it would be called the strain potential energy of the elastica –

* Craig G. Fraser, Institute for the History and Philosophy of Science and Technology, Victoria College, University of Toronto, Toronto, Ontario, Canada M5S 1K7

Centaurus 1991: vol. 34: pp. 211–246

is minimized in equilibrium. In the opening sections of the appendix Euler verified Daniel's conjecture using methods from the calculus of variations. He proceeded to derive the same equation by known mechanical principles, thereby establishing the validity of the variational procedure. The greater part of the appendix is devoted to an investigation by direct methods, unconnected to variational mathematics, of the mechanics of elastic laminae. It seems clear that Euler was using the variational treatise as an opportunity to publish these supplemental researches in the theory of elasticity.

Euler's "De curvis elasticis" has been described in the historical literature, primarily in relation to the more general development of the theory of elasticity in the 18th century.[2] In addition to its importance as a contribution to mechanics his essay is historically noteworthy as a study in the interaction of mathematics and physics. His researches embody a style and a conception of mathematical physics characteristic of exact science in the early modern period, possessing numerous points of technical, conceptual and methodological interest. The present paper attempts to further our understanding of Euler's science by critically examining that part of his theory devoted to classification, with particular attention to these points of interest. In adding to the existing commentary on his essay it strives to represent the perspective that underlies his original investigation, emphasizing differences between his and the modern approach to the subject.

Theory of Elasticity 1694–1744

In order to understand the background to Euler's essay it will be useful to consider briefly the character of research on elasticity in the early 18th century. The elastic behaviour of rods and beams was of interest during the period in two related but distinct problems of statics. In the problem of fracture one attempted to determine the maximum load that a beam of given material and dimensions can sustain without breaking. Typically it was assumed that the beam is attached firmly to a wall and that the rupture takes place close to the wall (Figure 1(a)). In the problem of bending on the other hand one was concerned with determining the shape assumed

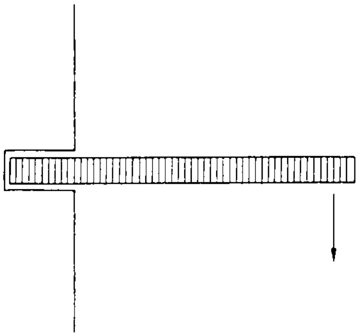

Figure 1(a)

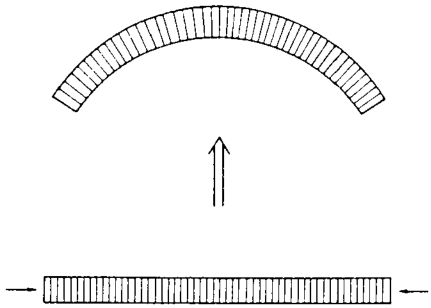

Figure 1(b)

by a rod or lamina in equilibrium when subject to given external forces. In the case of the elastica these forces were assumed to act at the ends of the rod and cause a bending of the rod, as shown in Figure 1(b). Galileo, Mariotte, Leibniz, Varignon and Parent obtained significant results on the first of these problems, while Jakob Bernoulli initiated study of the second. Euler's own investigation of the elastica grew directly from that of Bernoulli's.

The most significant characteristic of early work on elasticity was that it was carried out without the general theoretical perspective that is provided today by the concept of elastic stress. This concept, which underlies such basic modern formulas as the stress-strain relation $S = E\varepsilon$ ("Hooke's law") and the flexure formula $M = SI/c$, only originated in an explicit developed form in the 1820s in the treatises of Navier and Cauchy.[3] Although one can discern in the earlier writings (particularly those later in the century) some of the elements that enter into the modern understanding of stress, the essential idea – that of cutting a body by an arbitrary plane and considering forces per unit area acting across this plane – was absent.[4]

The divide that separates the modern theory and that of the early 18th century is illustrated by the problem of elastic bending.

Consider the derivation today of the formula for the bending moment of a beam. One begins by assuming that there is a neutral axis running through the beam that neither stretches nor contracts in bending. We apply elementary stress analysis and consider at an arbitrary point of the beam a cross-sectional plane cutting transversally the neutral axis. Elastic stresses distributed over the section are assumed to act across it. Calculation of their moment about the line that lies in the section, is perpendicular to the plane of bending and passes through the neutral axis leads to the flexure formula, $M = SI/c$, where M is the bending moment, I is the moment of area of the section about the line, c is the distance of the outermost unit of area of the section from the line and S is the stress at this outermost area.

In the problem of fracture Leibniz and Varignon obtained results that can be readily interpreted in terms of modern formulas and theory. Typically they assumed that the beam was joined transversally to a wall and that rupture occurred at the joining with the wall. Here the physical situation directly concentrated attention on the plane of fracture, something of concrete significance and no mere analytical abstraction. The conception then current of the loaded beam as comprised of longitudinal fibres in tension is readily understood today in terms of stresses acting across this plane.

In the problem of elastic bending by contrast researchers were much slower to develop an analysis that connected the phenomenon in question to the internal structure of the beam. Here there was nothing in the physical situation that identified for immediate study any particular cross-sectional plane. In all of Jakob Bernoulli's seminal writings on the elastica the central idea of stress fails to receive clear identification and development.[5]

An illuminating example of how elastic deformation was analyzed during the period is provided by an unpublished paper of Euler's, dating (apparently) from sometime in the 1730s.[6] Euler considered an "annulus" (a washer-like ring) that is disturbed from its equilibrium position and set as a consequence into motion. Figure 2 shows a part $abBA$ of the ring in its normal and stressed configuration. The segment $AaeE$ is regarded as being composed of concentric filaments. The inner line ae remains constant under deformation while the outer line AE is stretched to $A\varepsilon$. The triangle $eE\varepsilon$ shows

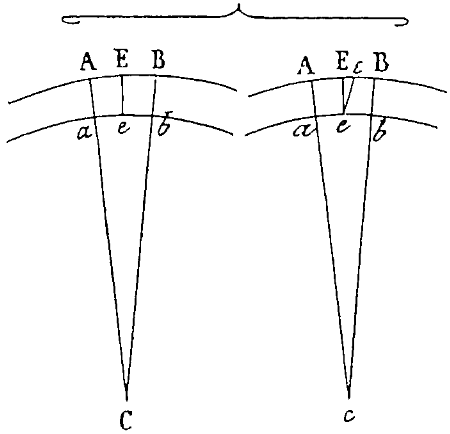

Figure 2. (from Euler's *Opera postuma* (1862), p. 130)

the stretching of the filaments as one proceeds outward from a to A. Euler sets $c = Aa$, $ds = ab$ and $dt = E\varepsilon$.

To obtain a measure of the elastic force Euler considers the material membrane $FGHJ$ (Figure 3) consisting of the extended part of a series of stretched filaments. $FJ = g$ is the magnitude of the extension and $FG = f$ is the width of the membrane. He supposes that the weight P is sufficient to sustain this stretching so that P/fg is a measure for the given material of the elastic force per unit of extension and per unit of width.

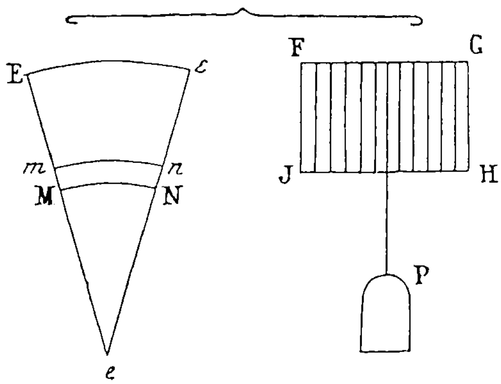

Figure 3 (from Euler's *Opera postuma* (1862), p. 130)

The stretched part of the ring segment *AaeE* is comprised of the triangle $eE\varepsilon$ (Figure 3). Consider the portion *MNnm* of $eE\varepsilon$ located a radial distance $eN = x$ from *e*. *MNnm* is composed of a series of concentric extended filaments bounded by *MN* and *mn*. Since $MN = (x/c)(E\varepsilon) = (x/c)dt$ the area of *MNnm* is $(xdxdt)/c$. The elastic force that gives rise to *MNnm* is therefore equal to $(Pxdxdt/cfg)$. Euler calculates the moment of this force about the point *e* to be $(Px^2dxdt)/c^2fg$. (Rather curiously he takes x/c instead of x as a measure of the moment arm *eM*.) By integrating this expression from 0 to *c* he obtains a value of the total "force of cohesion", $(Pcdt)/3fg$.[7] By relating this formula to the radii of curvature of *ab* in its normal and deformed states he arrives at an expression that he is able to use to investigate the vibratory motion of the ring.[8]

What is striking in Euler's treatment of this problem is the absence of anything that could be interpreted from a later perspective as stress analysis. The elastic forces that arise are regarded as being distributed over the plane in which they act, not over a transverse cross-section. These forces are also viewed as an absolute function of the displacement *dt*; thus Euler lacks the concept of elastic strain. The formula $(Pcdt/3fg)$ itself fails to relate in a satisfactory manner the bending moment to the cross-sectional structure of the ring.[9]

Although unsuccessful Euler's paper is of interest because of the detailed picture it presents of his understanding at this time of elastic phenomena. In his essay "De curvis elasticis" he would abandon any attempt at a direct physical analysis of elastic bending, concentrating instead on the mathematical investigation of relations that are treated methodologically as physical postulates.

Equation of the Elastica

The object of "De curvis elasticis" is a "lamina", a thin elastic ribbon or band that resists bending and retains its original length when deformed as a result of the application of an external force. Euler begins his investigation with a derivation of its equation of static equilibrium by means of variational techniques that had been introduced in the main body of the mathematical treatise. In his letter of 1742 Daniel Bernoulli had conjectured that the equation of the elastic curve could be obtained by minimizing the expression $\int sd/R^2$, where ds is an element of path-length along the curve, R is the radius of curvature at this element and the integration is taken over its length.[10] By analyzing the geometrical structure of the bent lamina at a point one could show that the elongation Δe of the outer fibre was inversely proportional to the radius of curvature R of the lamina at that point. Since it was known that the potential live force of the element ds, what in later mechanics would be called the strain potential energy of ds, was proportional to $ds(\Delta\varepsilon)^2$, it followed that it was also proportional to ds/R^2. Its total value over the entire lamina was therefore proportional to $\int ds/R^2$. Daniel Bernoulli's conjecture that this quantity must be a minimum in equilibrium was one of several variational laws of mechanics that were beginning to be formulated at this time.[11]

Since the lamina is assumed to retain its original length in deformation the variational problem falls mathematically within the class of isoperimetric problems. The integrand ds/R^2 of the resulting variational integral $\int ds/R^2$ contains second derivatives. Euler had in the *Methodus inveniendi* provided techniques that for the first time allowed one to derive the variational equations in such a case. Daniel Bernoulli's conjecture therefore provided an exemplary opportunity to apply his theory.[12]

Although the opening derivation provides a natural link with the mathematical treatise, it stands somewhat apart from the main investigation of "De curvis elasticis". It was presented by Euler because it represented a significant new result that indicated the interest and potential usefulness of a variational approach to mechanics.

In the direct derivation he considers a lamina AB that is built into a support at B (Figure 4). (Thus both B and the tangent to the lamina at B are regarded as fixed.) At the other end A a weightless rod is attached and from the end C of the rod a force P acts. (In the usual case P will act at A. The introduction of the rod allows more generally for the possibility of a non-negative moment of P about A.) The weight of the lamina is assumed to be negligible in comparison with P. The problem is to derive an equation that describes the lamina in equilibrium under this loading.

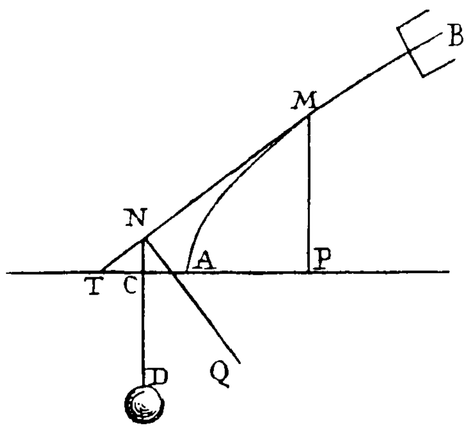

Figure 4 (from Euler's *Methodus inveniendi* (1744), p. 250)

Euler employs a Cartesian coordinate system with origin at A and oriented so that the y-axis coincides with the direction of the force P. Let M be a point on the lamina with coordinates x and y. Euler calculates the equation of the lamina in equilibrium according to principles laid down by Jakob I Bernoulli. The moment exerted by the external force about M equals $P(x+c)$, where c is the distance from A to the point C. The resisting "elastic force" (what we would today call the bending moment) of the lamina at M is regarded by Euler as given by the expression Ek^2/R, where Ek^2 is a quantity that measures the "stiffness" of the lamina and R is the radius of curvature of the lamina at M. (Ek^2 will depend on the type of material comprising the lamina as well as its dimensions.) In equilibrium these two expressions must be equal:

$$P(c+x) = \frac{Ekk}{R}. \tag{1}$$

The differential calculus yields the formula for R

$$R = -\frac{ds^3}{ds\,ddy}, \tag{2}$$

where dx is taken to be constant (that is, in modern parlance, x is the independent variable). Substituting this expression for R into

(1), multiplying each side of the resulting equation by dx and integrating Euler obtains

$$dy = \frac{-Pdx(\frac{1}{2}xx+cx+f)}{\sqrt{(E^2k^4-P^2(\frac{1}{2}xx+cx+f)^2)}}, \tag{3}$$

where f is a constant of integration.

Euler prefers to re-express (3) in terms of the general form he had previously obtained by means of the variational procedure. Thus the final equation of the elastic curve is taken to be

$$dy = \frac{(\alpha+\beta x+\gamma xx)\,dx}{\sqrt{(a^4-(\alpha+\beta x+\gamma xx)^2)}}, \tag{4}$$

where the quantities a, α, β and γ are connected to those of (3) by the relations

$$P = \frac{-2Ekk\gamma}{aa}, \quad c = \frac{\beta}{2\gamma}, \quad f = \frac{\alpha}{2\gamma}. \tag{5}$$

The direction of the force P is determined by the sign of the constant γ; for positive γ it points in the direction of the positive y-axis, for negative γ it points in the direction of the negative y-axis.

Euler omits any discussion of the physical reasoning required to obtain the expression Ek^2/R for the bending moment. Satisfactory treatment of this question would involve an investigation of the structure of the lamina in terms of something like stress analysis, which, as was noted earlier, was not available at this time. He at least recognized that it is necessary to consider two quantities in the constant of proportionality Ek^2: E, which depends on the nature of the material comprising the lamina, and k^2, which depends on the dimensions of the lamina.

In effect the lamina is treated as a line satisfying the property that its curvature at each point is proportional to $c+x$. The elastica therefore belongs with the catenary and the cycloid to a class of curves of the period that were solutions to mechanical problems. It should nonetheless be noted that the term "elastic curve", commonly used at the time, is something of an oddity. Whereas the catenary was obtained by considering tensions that act tangentially along a hanging cable, equation (1) is derived by considering the internal three-dimensional structure of the lamina. The concept of an elastic curve, a breadthless line satisfying (1), therefore abstracts from the very characteristic that serves ultimately to define it as a theoretical entity. (In modern engineering the term is used differently. It refers not to the object itself – a band or ribbon – but rather to the curve assumed under deformation by an axis through the centroids of the sections of the lamina.)

Equations corresponding to the forms (1) and (3) were first presented by Euler in a paper published in 1732. Throughout the 1730s he continued to work on the problem of elastic deformation, and published a memoir on the vibration of elastic rods in 1740. In the 1732 paper he acknowledged Jakob Bernoulli's pioneering work on the elastic curve. Versions of (1) and (3) had indeed originally appeared in Bernoulli's writings of 1694–95 in the *Acta eruditorum*. Jakob's significant innovation was to relate the bending moment to the radius of curvature and to express this radius in terms of a formula (namely (2) above) of the recently established Leibnizian differential calculus. He referred to the formula as a "golden theorem" and clearly viewed it as the key to unlocking the secret of the elastica. The spirit of his research, and what distinguished it from contemporary mechanical investigations, was a sense of intellectual excitement at the power of the new analysis to assist and even redefine the study of mechanical problems. It is this spirit that provided the impetus for Euler's own mathematical research in elasticity.

Bernoulli's original derivation in 1694 was presented for the so-called rectangular elastica, where the external force acts at the end of the lamina ($c = 0$) and makes a right angle with the direction of the lamina at this point. (Physical examples were the half-slats that make up the ribbing of a barrel.) He succeeded in deriving its equations and even in carrying out some of the analytical-numerical work needed to determine the parameters associated with the curve. In a short note published in the *Acta eruditorum* in September 1694 Huygens criticized Bernoulli's memoir, noting among other things that he had failed to exhibit several possible configurations of the deformed elastica (Figure 5) as a consequence of his theory.[13] In his subsequent writings Bernoulli acknowledged this criticism and

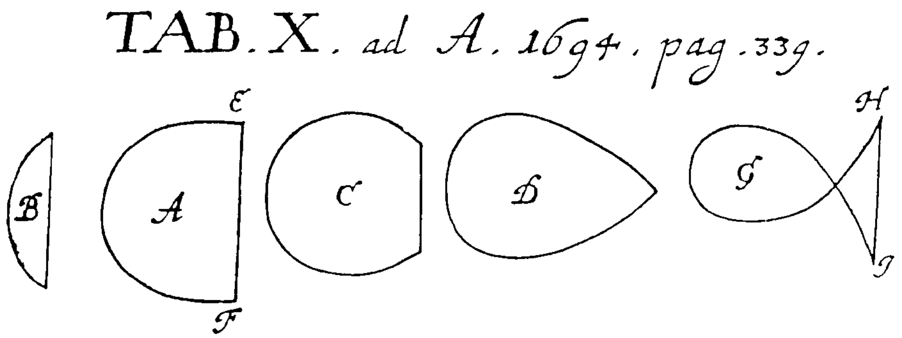

Figure 5 (from Huygens's note in the *Acta eruditorum* 1694)

briefly indicated how his analysis could be extended to cover other cases.[14] He was however primarily interested in such subjects as the proper position of the neutral fibre and the formulation of the laws of elastic stretch.

In the "De curvis elasticis" of 1744 Euler comments little on the background to his own work except to mention that the "very great man" Jakob Bernoulli had investigated elastic curves. It would seem clear that he was familiar with the *Acta eruditorum* of the 1690s, where Bernoulli's initial investigations had appeared. The criticisms of Huygens' published in that journal certainly provide the specific context for that part of his 1744 essay that deals with the statics of elastic laminae. Euler's achievement there was precisely to present a systematic mathematical analysis of equation (1) that provided a full response to Huygens's objection.

Classification of Elastic Curve

The focus of Euler's investigation is the different shapes assumed by the lamina when variable forces act at its ends. This concern is not motivated by any immediate technological application and indicates the theoretical orientation of his essay, a characteristic further evidence in his detailed study of the formal classes of elastic curve. Historically classification had tended among the exact sciences to be a special concern of mathematicians. Whereas physical researches were rooted finally in the analysis and prediction of empirical phenomena, mathematics required for its very identity as a subject the consideration of questions of organization and taxonomy. Problems of classification had occupied a prominent place in its history, from Euclid's study of irrationals in *Elements X* through to Descartes' grouping of plane curves and Newton's enumeration of the forms of the cubic.

Euler seeks in the section "Enumeratio curvarum elasticarum" to determine the "infinite variety of these elastic curves." He begins with the differential equation (4). He translates the origin of the axes a distance $\beta/2\gamma$ and sets $\gamma = 1$ in order to reduce (4) to the form

$$dy = \frac{(\alpha + xx)\,dx}{\sqrt{a^4 - (\alpha + xx)^2}}. \tag{6}$$

Euler notes that $a^4 - (\alpha + x^2)^2 = (a^2 - \alpha - x^2)\,(a^2 + \alpha + x^2)$ and sets $\alpha = a^2 - c^2$ so that (6) becomes

$$dy = \frac{(aa - cc + xx)\,dx}{\sqrt{(cc - xx)\,(2aa - cc + xx)}}. \tag{7}$$

The step from (4) to (6) is noteworthy in terms of what it reveals about Euler's mental outlook. He is regarding the elastic curve in much the same way one would a curve in analytic geometry. The translation of axes was routinely used there to simplify the curve's equation, and he seems to be employing this device for the same effect here. The constants in equation (4) however have a certain physical meaning; the coordinate transformation imposes a condition on the external force, namely that it now acts at the point A, with the corresponding moment about this point zero. This is in fact the case that he wishes to consider and he is introducing the transformation to arrive at it. Given that the elastic curve is an object with a definite physical identity, and given that a translation of axes can only assist in its description, it would have been preferable, as a point of logical exposition, to simply restrict consideration to this case, and to set β by fiat equal to zero in (4). (A logical possibility is that Euler is supposing that the system is rigidly attached to the coordinate axes and is being moved with them during the translation, while the applied force itself is contrived to remain stationary. Such an interpretation does not however seem plausible. Coordinate transformations were descriptive tools used for the analysis of mechanical systems; the axes were never regarded as an actual physical part of the systems. Furthermore, if he was envisaging the transformation this way one would expect some discussion of what has happened to the rod attached at the point *A*.)

Equation (7) is the basis for Euler's graphical study of the elastic curve or lamina (the two terms are used interchangeably). In Figure 6, *A* is the origin of the coordinate system and *AE* and *AB* are the directions of the positive *x* and *y* axes respectively. *M* is a typical point on the curve *AMC*. The external force *P* acts in the positive *y*-direction at *A*. Euler imagines that the points *A* and *B* may be regarded as being joined by a thread of tension *P* so that a force equal and opposite to *P* acts at *B*. (This way of envisaging the elastica had originated with Huygens [1694].) The two significant

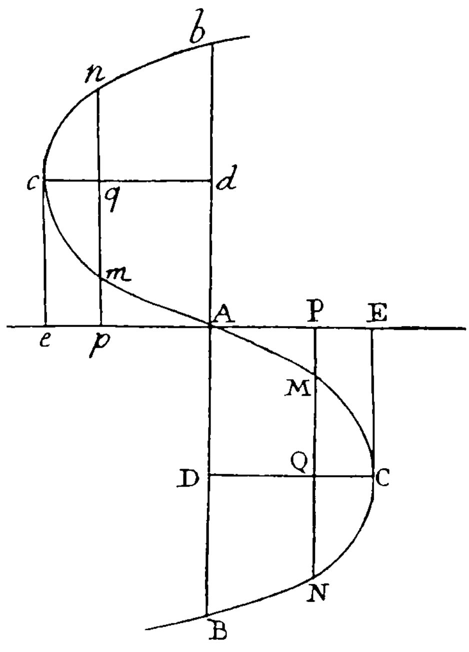

Figure 6 (from Euler's *Methodus inveniendi* (1744), p. 256)

quantities that he associates with the curve are P and the sine of the angle $\measuredangle PAM$ formed by the tangent at A and the x-axis:

$$P = \frac{2Ekk}{aa}, \quad \sin(\measuredangle PAM) = \frac{aa-cc}{aa}. \tag{8}$$

This expression for P is furnished directly by (5) and the value for $\sin(\measuredangle PAM)$ is derived by calculation from $\tan(\measuredangle PAM) = (a^2-c^2)/c\sqrt{2a^2-c^2}$, obtained by setting $x=0$ in (7).

Euler begins his study of (7) with several observations concerning its graph. Since dy/dx increases to infinity as x approaches c it is clear that the curve will consist of the hump AMC in the interval $0 \leq x \leq c$. Euler notes that if y and x are replaced by $-y$ and $-x$ in (7) the expression for dy/dx remains unchanged. He concludes that for $x \leq 0$ the curve will consist of the segment amc congruent to AMC but located on the opposite side of the y-axis. From the derivation of (7) it is apparent that A is a point of counterflexure, that is, that the curvature at A is zero. He proceeds to analyze the behaviour of the curve in the interval CB. To do this, he refers the origin to the point C and expresses x and y in terms of a $t-u$ coordinate system, where $CQ = t$ and $QM = u$. Here $x = c-t$ and $y = b-u$ and (7) becomes

$$du = \frac{(aa-2ct+tt)\,dt}{\sqrt{t(2c-t)(2aa-2ct+tt)}}.$$

For small t we have the approximate relation $du = a^2dt/2a\sqrt{ct}$, which leads upon integration to $u = a\sqrt{t/c}$. Thus the curve is parabolic in the neighbourhood of C, and Euler concludes that it will progress beyond C in the same manner that it advanced from A to C.

Euler summarizes these considerations by observing that the solution of (7) will consist of a periodic odd curve in which the segment ACB is symmetrical about the line DC. He proceeds to calculate certain important quantities associated with the curve. Using the binomial expansion he expresses the right sides of

$$dy = \frac{(aa-cc+xx)\,dx}{\sqrt{(cc-xx)(2aa-cc+xx)}} \text{ and } ds = \frac{aadx}{\sqrt{(cc-xx)(2aa-cc+xx)}},$$

as infinite series. Integrating term by term and noting that $\int_0^c dx/\sqrt{c^2-x^2} = \pi/2$ he obtains

$$AC = f = \frac{\pi a}{2\sqrt{2}}\left(1+\frac{1^2}{2^2}\cdot\frac{cc}{2aa} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{c^4}{4a^4}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{c^6}{8a^6}+\text{etc.}\right)$$

$$AD = b = \frac{\pi a}{2\sqrt{2}}\left(1-\frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{cc}{2aa}-\frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{5}{3}\cdot\frac{c^4}{4a^4} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{5}\cdot\frac{c^6}{8a^6}-\text{etc.}\right). \tag{9}$$

Euler remarks that these series may be used to calculate a and f from c and b and conversely, to calculate c and b from a and f. (It follows in particular that c and b may be derived for a lamina of given length $2f$ from a knowledge of the force $P = 2Ek^2/a^2$.)

The key to Euler's classification is the relationship with respect to magnitude of the constants a and c. The different species are obtained by considering the angle PAM as it changes in value from 90° to −90°.

Species One. If c^2/a^2 is infinitesimal or very small then $\measuredangle PAM \simeq 90°$ and the segment ACB (Figure 6) is only slightly curved. Because c is small with respect to a, equation (7) may be approximated as $dy = adx/\sqrt{2(c^2-x^2)}$. Integration yields $y = (a/\sqrt{2})$ $\arcsin(x/c)$. Since the lamina is almost straight we may set $AB(= 2b) = 2f$. Since $x = 0$ when $(\sqrt{2}/a)2f = \pi$ we obtain $f = \pi a/2\sqrt{2}$. Substituting the value for P given by (8) into this last relation gives

$$P = \frac{Ekk}{ff}\cdot\frac{\pi\pi}{4}. \tag{10}$$

A notable implication of (10), which Euler mentions only in passing, is that a finite force is required to produce even an infinitesimal curving of the lamina. In later mechanics (10) would become known as "Euler's buckling formula" and the value of P in (10) as the "first Euler critical load."

Note: Euler also classifies as species (1) the degenerate case in which $a = \infty$ (force $P = 0$ and $f = \infty$) and the curve coincides with the y-axis and the lamina assumes its natural shape.

Species Two. If $0 < c^2/a^2 < 1$ then $\measuredangle PAM < 90°$ and the curve has the form of Figure 6. Since $c^2/2a^2 < \frac{1}{2}$ it is clear from (9) that f is finite. (9) also implies that $f > \pi a/2\sqrt{2}$. Hence $P > (Ek^2/f^2)(\pi^2/4)$ and so it is clear that the force drawing A and B together is greater than its value for the corresponding species-one curve in which $AC = f$.

Species Three. We have $a = c$ and $\measuredangle PAM = 0°$. This gives the rectangular elastica, originally derived by Jakob Bernoulli, whose equation is

$$dy = \frac{xxdx}{\sqrt{(a^4 - x^4)}}.$$

Euler mentions that he has shown elsewhere that b and f satisfy the "remarkable" relation $4bf = \pi a^2$ (a result derived from (9) by setting

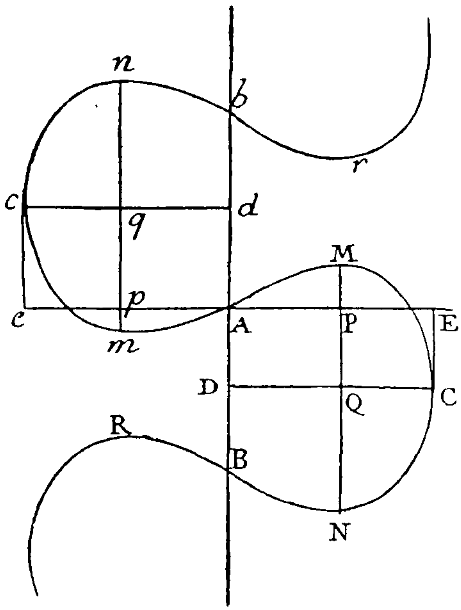

Figure 7 (from Euler's *Methodus inveniendi* (1744), p. 262)

$a = c$ and multiplying the two series.)[15] Beginning with an approximate value of f derived from (9) he uses (8) and this relation to evaluate P and b/a.[16]

Species Four. Here $a < c$, $\measuredangle PAM < 0°$ and the curve has the form of Figure 7. As c^2/a^2 increases in value the distance AB will decrease. An upper bound for c^2/a^2 will therefore be obtained by setting $AD = 0$ in (9) and solving for c^2/a^2. Euler calculates this quantity ("by methods familiar to everyone") as 1.651868 corresponding to an angle PAM ("found from the tables") of $-40°41'$. He observes that as c^2/a^2 approaches this limit multiple intersections of the humps of the curve will occur and that the diameters DC, dc will eventually merge with the axis AE.

Species Five. Here the limit $c^2/a^2 = 1.651868$ is reached, $AB = 0$ and the curve assumes the form of Figure 8. The angle at the knot MAN is 81°22′ (twice 40°41′).

Species Six. The ends of the lamina A and B are drawn apart by an increasing force and the lamina assumes the form $AMCNB$ of Figure 9. Here $2 > c^2/a^2 > 1.651868$ *and* $-90° < \measuredangle PAM < -40°41'$. (Although unremarked by Euler, multiple overlapping of the humps of the curve will occur as c^2/a^2 immediately exceeds 1.651868.)

The cases considered thus far may be regarded as the configurations successively assumed by the lamina AB as the force P increases in magnitude. They exhaust the possible shapes of the elastic curve in which the segment ACB is regarded as a lamina in tension under the action of forces P and $-P$ at A and B. Euler however considers several further possibilities .

Species Seven. Here $c^2/a^2 = 2$. Euler mentions the limiting case

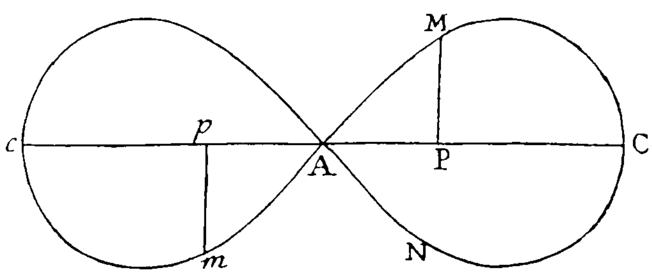

Figure 8 (from Euler's *Methodus inveniendi* (1744), p. 262)

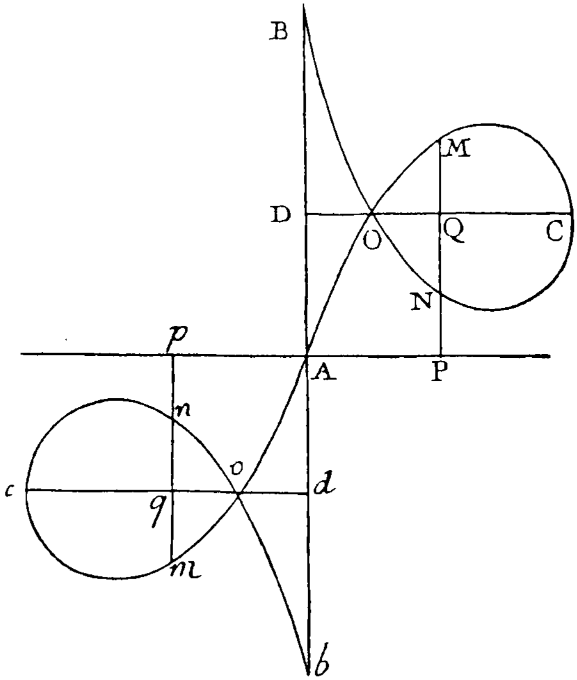

Figure 9 (from Euler's *Methodus inveniendi* (1744), p. 263)

where c tends to $\sqrt{2a}$ while the segment ACB remains constant. Because the series

$$1+\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\text{etc.},$$

is divergent it is clear from (9) that a (and therefore c) is zero and hence that the force P is infinite. (He does not show divergence although it is not difficult to establish.[17]) Thus when $c=\sqrt{2a}$ the lamina AB becomes a straight line drawn by an infinite force. (The alternative limiting case $x=0$ in which c tends to $\sqrt{2a}$ while AB tends to infinity is not considered by him. It is doubtful whether this case, which seems to require the functional viewpoint of later real analysis, would have constituted a possibility within his understanding of equations and their graphs.) He is however primarily interested in the non-degenerate limiting case that is obtained by beginning with the general curve of species (6), transferring the origin to the point D and taking the limit as c tends to $\sqrt{2}a$. If we assume a is non-zero then the arc AC tends to infinity. As a representative of species seven he therefore arrives at the curve of Figure 10, where $x = DQ$ and $y = QM$ are the coordinates of a typical point M on the curve. (For this single case he assumes that the positive y-axis is directed along DB.) Equation (7) becomes here

$$dy = \frac{(aa - xx)\,dx}{x\sqrt{(2aa - xx)}},$$

which is integrable and leads upon integration to

$$y = \sqrt{(cc - xx)} - \frac{c}{2}\,l\left(\frac{c + \sqrt{(cc - xx)}}{x}\right) (l = \text{natural logarithm}).$$

The curve will cut the axis DC at C ($x = c$) and at a second point O, determined by solving the equation $y = 0$ by means of tables. Euler arrives at the value $x = .2884191c$ for O. He calculates the

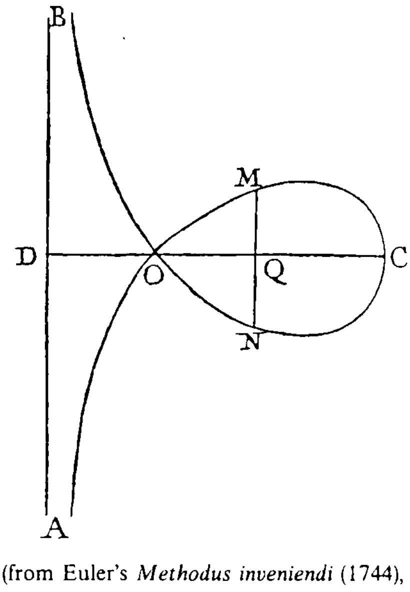

Figure 10 (from Euler's *Methodus inveniendi* (1744), p. 264)

knot angle MON and shows that it equals 112°56′48″. (It should be noted that Figure 10 which accompanies the text is poorly drawn; the knot angle in it is no greater than 90°. The distortion is even greater in Figure 11 which accompanies species eight.)

Species Eight. We continue to place the origin of the coordinate system at the point D but now suppose that $c^2 > 2a^2$. Euler sets $c^2 = 2a^2 + g^2$ so that equation (11) becomes

$$dy = \frac{(xx - \frac{1}{2}cc - \frac{1}{2}gg)\,dx}{\sqrt{(cc-xx)(xx-gg)}}.$$

It is clear that the values of x must lie between $x = g$ and $x = c$. Euler infers the curve will assume the form of Figure 11, in which the line $x = g$ is tangent to the curve at the points G and H and there are infinitely many diameters dc, DC, dc. This conclusion

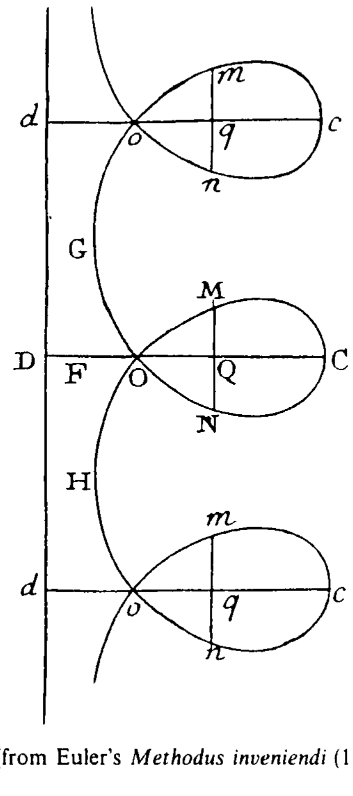

Figure 11 (from Euler's *Methodus inveniendi* (1744), p. 265)

seems to based on the fact that the given curve represents a physically plausible configuration of the bent spring. (It should be pointed out that series (9) which had hitherto constituted the numerical basis of his investigation are no longer valid; the series that replace (9) when one integrates from g to c are not nearly as tractable. Euler does not present any mathematical argument to show that the points H and G actually lie on the curve. It is also unclear in his analysis why the curve of species seven should be the limiting case of species eight when c equals $\sqrt{2}a$.) Euler observes that the knot angle MON in Figure 11 is greater than 112°56′48″.

Species Nine. Euler notes that if $g = c$ then the curve will be "reduced, vanishing into space." (He does not however remark that since $c^2 = 2a^2 + g^2$ this will happen when $a = 0$, i.e., when the force given by (8) becomes infinite.) Beginning with the general curve of species eight, he proceeds to move the origin to the midpoint of OC, sets c minus g equal to a constant and considers the limiting case that results when c and g tend to infinity. If $g = c - 2h$ and $x = c - h - t$ and we let c and g tend to infinity then (13) becomes

$$dy = \frac{tdt}{\sqrt{hh - tt}}.$$

It is clear that in this case the elastic lamina will be curved into a circle, the ninth and final species. (Although Euler does not remark it, it is apparent from $2a^2 = c^2 - g^2 = (c-g)(c+g)$ that a tends to infinity and hence that the force given by (9) tends to zero.)

Euler concludes the section on enumeration with a discussion of how the different species may be produced by hanging a weight from an elastic lamina, one end of which is built into a rigid support. He first supposes the weight acts at the free end A of the lamina and notes that species one to six arise according to the different angle that the tangent at A makes with the direction of the force. In order to exhibit all the species by means of a uniform mechanical process he returns to the original situation for which equation (1) was derived (Figure 4), in which the force P acts at the end of a rod of length h. The operative equation here is (3) (where the c that appeared in (3) is now renamed h). He proceeds to move the origin to the point C (with the lever still intact.) With this change of coordinates (3) becomes

$$dy = \frac{ds\,(mEkk + \tfrac{1}{2}Phh - \tfrac{1}{2}Pxx)}{\sqrt{E^2k^4 - (mEkk + \tfrac{1}{2}Phh - \tfrac{1}{2}Pxx)^2)}}, \qquad (11)$$

where m is the sine of the angle MAP. He notes that (11) has the same form as (7) and therefore must correspond to one of the elastic species identified above. By comparing constants in the two equations he is able to provide an enumeration of the different curves in terms of inequalities involving the constant h. (The eighth species is the limiting case obtained by setting Ph equal to a constant and letting h tend to infinity while P tends to zero.)

In effect what Euler is doing here is to consider an arbitrary point M on the elastica $AMCB$ and to imagine that its abscissa RM has become a lever with force P acting along the positive y-axis at the point R (Figure 12). The part MCB of the curve is now regarded as being maintained by the moment at M exerted by this force acting through the lever RM. R may be regarded as the origin of the coordinate system, a change that has no effect (being a translation along the y-axis) on the form of equation (7).

Euler could have dispensed with the unusual argument involving the rod and the comparison of equations (7) and (11) and simply observed that each of the species – including seven and eight – may be regarded as being given by means of the procedure just described.

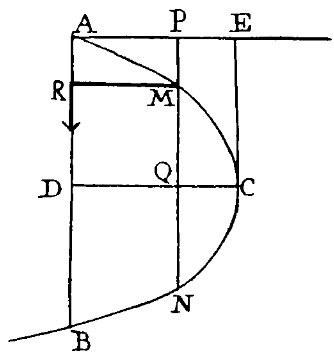

Figure 12 (not in Euler, adapted from Figure 5 above)

It should also be noted that the inequalities involving h which he derives could have been obtained directly by solving (7) for c^2/a^2 in terms of x and P. The content of his concluding discussion therefore seems to reduce to the observation that the class of elastic curves defined by the original equation (1) is entirely expressed in the enumeration of the nine species.

Discussion

There is a dual focus to Euler's study of the elastic curve. He is classifying solutions of equation (7) and in a certain logical sense his theory is independent of any mechanical interpretation. He nevertheless is also conceiving of the elastic curve as a material lamina that is being deformed by the action of an applied force P. The elastic curve is a hybrid, an object with both a mathematical and physical identity, each of which is unfamiliar and requires investigation, and it is being considered by Euler simultaneously from both viewpoints.

Frequently the differential equations of mathematical physics involve quantities like time or temperature that are relatively abstract. The variables of equation (7) by contrast have a direct, physical meaning – the distance along the lamina and its lateral displacement. The analytical description of the lamina possesses an unusual literalness that contributes to the identification in Euler's theory of mathematical and physical perspective.

In the approach adopted by Euler the mathematical viewpoint is largely determinative of the object under consideration. The precise physical situation of the lamina, for example, how it is positioned at the endpoints A and B, is not discussed and must be inferred from the analysis. (As species eight indicates, however, he is not above appealing to the physical model in order to reach mathematical conclusions about the nature of the curve.) Some of Euler's purely mathematical procedures invite comment. It is curious to see him approximating the curve at C by a parabola in order to investigate its behaviour about this point. In a rough or preliminary investigation the introduction of the approximative parabola, a simpler and better known object, would be natural. He could however have observed directly that for $y > b$ the differential equation $dy = -(a^2 - c^2 + x^2)dx/\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}$ (which also satisfies (1)) leads to an extension of AMC which is symmetrical about DC.

The preliminary analysis Euler undertakes to determine the general shape of the elastic curve would incidentally carry over without change to an investigation of the sine curve $dy = dx/\sqrt{a^2 - x^2}$. In this respect it is necessary to remember that a calculus of trigonometric functions was only beginning to be developed at this time and that "De curvis elasticis" was written before the refinement in analytical understanding that occurred with Euler's major treatises of the period 1748–1765.[18]

It should be noted that logically what is being classified by Euler are solutions to equation (7). Thus the simple example of a straight lamina AB of given length and stiffness in which the external load P is zero is not directly included in his classification (nor is it the limiting case of any of the species enumerated there). This example must be regarded instead as a segment of the infinitely long lamina that results when $P = 0$. The latter degenerate case, which really constitutes a distinct, unique possibility, is classified by Euler as species one and grouped with curves of small deflection, where the force P is of finite value.

In introducing his classification Euler had stated that he would list "the different kinds of curves in the same way in which the species of algebraic curves included in a given order are commonly enumerated." He was referring to Newton's study of 1704 of the forms of the cubic, which he had explicitly invoked at one point in the main body of the variational treatise in connection with the solution to a particular isoperimetric problem.[19]

Newton had developed a rather involved system of classification in terms of canonical forms, classes, genera and species. In total he arrived at seventy-two species of cubic curve. His scheme is illustrated by the third canonical form $y^2 = ax^3 + bx^2 + cx + d$. It was clear that the curve determined by this equation is symmetrical about the x-axis. By analyzing its roots Newton obtained five cases: roots all real and unequal, all real and two equal (in which there are actually two cases, according as the double root is the lesser or greater), all real and all equal, and one real and two unreal. Each of these cases gives rise to a distinct species for which he provided a diagram (Figure 13).

Although Euler's own classification was inspired by Newton, the principles of his enumeration are different. The elastic curve has a physical interpretation as a deformed lamina in tension. The different species are evolved naturally as solutions of a single differential equation of static equilibrium. The existence of the first species – mathematically indistinguishable from the second – is based on its physical significance as a model for a column undergoing compression.

It is moreover questionable whether the sort of classificatory scheme Newton employs for algebraic curves is applicable to the solution of a transcendental equation of type (7). Thus the multiple overlapping that occurs in the transition from elastic species four to five and from five to six would seem to suggest an infinity of species of the Newtonian variety.

A modern, more topological approach would tend to collapse

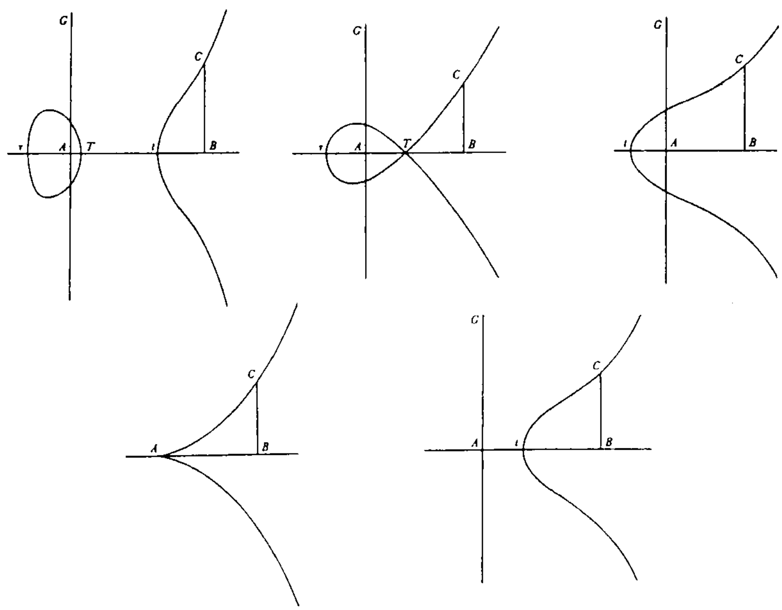

Figure 13 (from Newton's "Enumeratio linearum tertii ordinis" (1704)) together Euler's species one to four and retain the rest, resulting in six classes in all, three of which (five, seven and nine) would be limiting cases. There will of course be a conventional element in any attempt at classification. One seldom encounters in modern mathematics the detailed study of the curves of the higher transcendental functions. Apart from the question of classification Euler's theory of the elastica is of enduring mathematical interest as an exceptional example of graphical analysis.

Comparison with Modern Theory

Euler's analysis of the elastic curve is an impressive piece of mathematical science. The equation of equilibrium is investigated without any simplifying assumption and the different kinds of curve are exhaustively enumerated. Careful attention is devoted to determining the particular angles and lengths associated with each figure. Numerical relationships involving infinite series are calculated by means of the most advanced techniques of contemporary analysis.

While the mathematical sophistication of his investigation is clear, its character as a contribution to applied science is less so. It will be useful to consider this question by comparing his theory with the modern approach to the problem of elastic bending.

In engineering statics today one develops a theory for a given object – a beam, strut, or column – by laying down a coordinate system along its unstressed configuration and analyzing distortions (usually small) from this position. In Euler's approach the elastic curve provides a unified model for all of these objects and the coordinate system is oriented once and for all with respect to the direction of the external force P.

Euler's goal – to formally enumerate the different curves that are solutions to (7) – failed to become a prominent concern of subsequent research in the theory of elasticity. The artifice of a lever arm to mediate the action of the end weight is introduced in order to provide a privileged mechanical procedure by means of which all the species of elastic curve may be exhibited. Once again it is part of his special theoretical stance that this should be a subject of interest.

The guiding principle of the classification concerns the way in which the shape of the curve alters as the angle PAM decreases in value. Euler is aware that for a lamina of given length these shapes will be successively assumed as the result of the action of an increasing force P. He explicitly notes that series (9) may be used (at least in principle) to calculate the deflection c from a knowledge of the external force. This fact is however not pursued, and he does not in particular investigate the dependence of the lateral bending (quantity $c/2f$) on the magnitude of the load (quantity $2Ek^2/a^2$). His analysis should here be contrasted with the modern treatment of the elastica where the primary concern is correlating $c/2f$ as a function of P and where the question of classification does not arise at all.[20]

Perhaps nothing more clearly indicates the distinctive character of Euler's investigation than his failure to consider higher-order configurations of the lamina. As will be explained in more detail in the next section, formula (10) ($P = Ek^2\pi^2/4f^2$) provides a measure of the minimum load needed to induce bending in the column AB, the latter being regarded as the part ACB of the periodic curve depicted in Figure 6. The shape of the bent column may however consist of more than one arch of the given curve. Such a possibility, which was first investigated by Lagrange [1771], is of considerable physical interest. If the column contains n loops of the curve then the expression for the load given by (10) becomes $P = n^2Ek^2\pi^2/4f^2$. The column is evidently substantially stronger, a result which confirms theoretically the known practical value of side bracing. All of these considerations would seem to be directly inferrable from Euler's analysis. His own failure to do so may be explained by his particular mathematical perspective. Although the multi-arched column involves a new physical situation, the underlying mathematical object – the periodic odd curve of small amplitude – is the same.

Column Problem

Euler follows the part of his essay on classification with a short section titled "De vi columnarum" ("On the force of columns"). Here he returns to the subject of species-one curves and notes the applicability of equation (10) ($P = Ek^2\pi^2/4f^2$) to a column fixed at its base and loaded at the other end by a weight P. For a slightly bent column it is apparent that $2f$ is equal to the height. It therefore follows from (10) that the maximum weight a column can sustain without bending is inversely proportional to the square of its height. Euler notes the relevance of this fact to wooden columns.

For (10) to be valid it is necessary that the upper end of the column be pinned; it is by no means clear that Euler was aware of this fact – the figure that is included with the text shows the upper end free. He seems however to have been primarily interested in the proportionality of the load to the square of the height and in comparing columns of varying heights. From this perspective it is indifferent whether the upper end is pinned or free.

Equation (10) would become celebrated in later engineering column theory as Euler's buckling formula. His derivation provides a remarkable example of how purely mathematical development can lead to a result of substantial physical interest and significance. There are nevertheless two respects in which his treatment of the bending of a column demonstrates a rather limited appreciation of its character as an engineering problem. First, he does not appear to have recognized that the value for P given by (10) is in fact critical. Although he is aware that as the force P increases the lateral displacement c also increases he shows no awareness that it would be of interest to investigate the precise correlation between P and c for a given range of values. (The infinite series (9) would have provided the numerical basis needed to carry out such an investigation.[21]) In particular, nowhere does he note what is of primary concern from an engineering viewpoint, that a very small fractional increase in P above the value in (10) will lead to a very substantial increase in $c/2f$.

Second, although Euler is aware that the constant Ek^2 in equation (1) ($P(c+x) = Ek^2/R$) measures stiffness, he is unable to relate it theoretically to the dimensions of the lamina. For a prismatic beam of width b and thickness h he conjectures incorrectly that the bending modulus Ek^2 is proportional to Ebh^2. (It is in fact proportional to Ebh^3. He may have been recalling the faulty derivation contained in his unpublished "De oscillationibus annulorum elasticorum" (Euler, 1862). It is also possible that he confused the section modulus which appeared in contemporary formulas on the fracture of beams with the bending modulus Ek^2 of his own equation (1).) In other subjects of the essay – the determination of Ek^2 by experiment and the curvature of laminae in which Ek^2 is variable – he avoids any theoretical consideration of the actual physical nature of elastic bending, positing instead relationships in which functions and constants appear that are to be determined by experiment or experience.

In a memoir published by the Berlin Academy in 1759 Euler resumed his investigation of the column problem.[22] Here he moves even further from the theory of elastic bending as it had originally appeared in the writings of Jakob Bernoulli. He suggests that his earlier results are also applicable to non-elastic materials, provided only that equation (1) remains valid. Instead of being something that is derived this equation is now regarded as a primitive relation of the theory, prescriptive of a certain class of physical phenomena. The moment of stiffness Ek^2 is itself treated as a function of the distance from the top of the column and the equation of bending is investigated for various functional forms as a problem in differential equations.[23]

The memoir of 1759 also contains Euler's discussion of a "paradox" he associates with the column formula (10). He observes that whereas the smallest force applied perpendicularly to the end of a beam will produce some deflection, a certain finite weight (given by (10)) is needed to produce even the slightest bending of the column. A value of the load below that of (10) will produce no bending, while values greater will give rise to a progressive increase in the size of the bending. He detects here the appearance of a violation of the law of continuity, which would seem to require for a continuous range of values of the load a continuous measure of bending.

To explain this paradox Euler considers in closer detail the derivation of equation (10). He suggests that if we assume $\theta = \tan(\sphericalangle DAC)$ (Figure 6) is greater than zero but ignore third and higher powers of θ then (10) becomes

$$P = \pi\pi \cdot \frac{Ekk}{\ell\ell} \sqrt{(1+\theta\theta)}, \tag{10'}$$

where ℓ is the length of the column. (In fact, this formula is incorrect;

the expression that one may derive from series (9) is $P = (\pi^2 Ek^2/\ell^2)\sqrt{1+(\theta/2)^2}$.[24] This error however has no effect on the argument that Euler is developing.) Hence if we allow θ to be imaginary we obtain values of the load that are smaller than those given by (10). A deflection of the column, albeit an imaginary one, is produced by these values.

It is indicative of Euler's peculiar mathematical sensibility that he thinks such an explanation is appropriate here. To the extent that an account of the paradox would seem necessary it must presumably refer to the physical situation of the column. Thus a force applied at the end of a beam actually causes the deflection, whereas the given weight is only consistent with or sustaining of the bending of the column. Furthermore, a weight however slight will sustain bending if the moment of elasticity Ek^2 is assumed sufficiently small.

Conclusion

The context of Euler's investigation was very much established by the scientists who had preceded him. Thus the equation of the elastica was derived by Jakob Bernoulli, the problem of determining its bent forms was formulated by Huygens, the theme of classification originated with Newton, and the possibility of a variational treatment was suggested by Daniel Bernoulli. Euler's distinctive achievement derived from the determination and solid analytic sense with which he pursued and developed the ideas of these men.

In treating the basic equations of elasticity as physical postulates that are subject to experimental determination and mathematical development Euler followed an approach that would be characteristic of later positivistic physics. However, whereas the positivists would emphasize the role of mathematics as an operational tool in obtaining numerical values, there is in Euler's theory a much more fundamental interpenetration of mathematical and physical conception.[25] The distinctiveness of his approach is evident in the way in which he conceptualizes the problem of the elastica, in the style of his investigation as well as in the very idea of the elastic curve itself.

A curious divide opened in the nineteenth century between mathematics and theoretical physics, so that as physics progressively employed more mathematics it simultaneously distanced itself epistemologically and foundationally from this subject. Modern theories of material mechanics whatever their level of mathematical maturity must reflect this separation. Euler's "De curvis elasticis" provides an uncommon example of exact science produced before the separation had taken place.

Acknowledgements

I would like to thank Thomas Archibald for helpful discussions as well as for his comments on a draft of the present article. I am grateful to James Casey for directing my attention to modern research on column theory. I have benefited from discussions with Emily Grosholz on various questions concerning philosophical and historical aspects of mathematical physics. Finally I thank an anonymous referee for advice on a point of exposition.

REFERENCES

Ball, W. W. R.

1890–1891: "On Newton's classification of cubic curves". *Proceedings of the London Mathematical Society* 22, 104–143.

Bernoulli, Jakob

1694: "Curvatura laminae elasticae. Ejus identitas cum curvatura lintei a pondere inclusi fluidi expansi. Radii circulorum osculantium in terminis simplicissimis exhibiti; una cum novis quibusdam theorematis huc pertinentibus, &c.". *Acta eruditorum*. Also in Jakob Bernoulli's *Opera* V.1 (1744), 576–600.

1695: "Explicationes, annotationes et additiones ad ea quae in actis superiorum annorum de curva elastica, isochrona paracentrica, & velaria, hinc inde memorata, & partim controversa leguntur; ubi de linea mediarum directionum, aliisque novis". *Acta eruditorum*, 537–553. In *Opera* V.1 (1744), 639–663.

1706: "Veritable hypothèse de la résistance des solides, avec la démonstration de la courbure des corps qui font ressort". *Mémoires de l'académie des sciences de Paris 1705*, 176–186. In *Opera* V.1, 976–989.

Cannon, J. T. and Dostrovsky, S.

1981: *The evolution of dynamics: Vibration theory from 1687 to 1742*. New York: Springer.

Euler, L.
1732: "Solutio problematis de inveniendi curva quam format lamina utcunque elastica in singulis punctis a potentiis quibuscunque sollicitata". *Commentarii academiae scientiarum Petropolitanae 1728* V. 3, 70–78; in *Opera omnia* Ser. 2 V. 10 (1948), 1–16.
1743: "Theoremata circa reductionem formularum integralium ad quadraturam circuli". *Miscellanea Berlinensia* 7, 91–129; in *Opera omnia* Ser.1 V. 17, 1–34.
1744: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti.* Additamentum I, "De curvis elasticis", occupies pp. 245–320. The work is reprinted as volume 24 of Euler's *Opera omnia,* Ser. 1 (1952). Additamentum I (pp. 231–308) is supplied by C. Carathéodory with notes containing references to Euler's other works.
1759: "Sur la force des colonnes". *Mémoires de l'académie des sciences de Berlin 1757* V. 13, 252–282. In *Opera omnia* Ser. 2 V. 17 (1982), 89–118.
1780: "Determinatio onerum, quae columnae gestare valent". *Acta academiae scientiarum Petropolitanae 1778 I,* 121–145. In *Opera omnia* (1982) Ser. 2 V. 17, 232–251.
1862: "De oscillationibus annulorum elasticorum". *Opera postuma* V. 2, pp. 129–131, reprinted in *Opera omnia* Ser. 2 V. 11, (1948), 378–382.

Fraser, C. G.
1990: "Lagrange's analytical mathematics, its Cartesian origins and reception in Comte's positive philosophy". *Studies in History and Philosophy of Science* 21, 243–256.

Fuss, P. H.
1843: *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle* V. 2.

Grattan-Guinness, I.
1990: *Convolutions in French mathematics,* 3 volumes, Basel: Birkhaüser.

Heyman, J.
1972: *Coulomb's memoir on statics,* Cambridge: University Press.

Huygens, C.
1694: "Excerpta ex epistola G. H. Z. [Christiani Huygenii de Zuylichem] ad G. G. L. [Goth. Gul. Leibnitium]". *Acta eruditorum,* 339–341. Reprinted in Jakob I Bernoulli's *Opera* V. 1 (1744), 637–638, and in Huygens *Oeuvres* V. 10, 671–672.
1905: *Oeuvres complètes de Christiaan Huygens V. 10 Correspondance 1691–1695.*

Katz, V. J.
1987: "Calculus of the trigonometric functions". *Historia Mathematica* 14, 311–324.

Lagrange, J. L.
1771: "Sur la force des ressorts pliés". *Mémoires de l'académie de Berlin 1770* V. 25, 167–203. Reprinted in Lagrange's *Oeuvres* V. 3, 77–110.

Leibniz, G. W.
1850: *G. W. Leibniz Mathematische Schriften* V. 2, ed. C. I. Gerhardt.

Linsenbarth, H.
1910: *Abhandlungen über das Gleichgewicht und die Schwingungen der ebenen elastischen Kurven von Jakob Bernoulli (1691, 1694, 1695) und Leonh. Euler (1744), Ostwald's Klassiker der exakten Wissenschaften* V. 175, Leipzig. Contains a German translation of Euler's "De curvis elasticis" with notes.

Maupertuis, P. L. Moreau de
1742: "Loi du repos des corps". *Histoire de l'académie royale des sciences 1740. Avec les mémoires de mathématique et de physique pour la même année*, 170–176 (Mémoires), Paris. Revised version published in Maupertuis's *Oeuvres* V. 4 (1768), 43–64.
Newton, I.
1704: "Enumeratio linearum tertii ordinis". Published as the first of two appendices "Two treatises of the species and magnitude of the curvilinear figures" to his *Opticks*, pp. 138–169. Later editions include an English version published by John Harris, in his *Lexicon technicum. Or, an universal dictionary of arts and sciences* V. 2 (London, 1710), article "Curves". Harris's edition is reproduced in facsimile in D. T. Whiteside (ed.), *The mathematical works of Isaac Newton* V. 2 (Johnson Reprint: New York, 1967), 137–161. (Whiteside (pp. xxv–xxvi) provides here a discussion of the various editions of the "Enumeratio".)
Oldfather, W. A., Ellis, C. A. and Brown, D. M.
1933: "Leonhard Euler's elastic curves". *Isis* 20, 72–160.
R. V. Southwell
1941: "Elastic stability, the strength of compression members (struts)", Chapter 13 of *An introduction to the theory of elasticity for engineers and physicists*, second edition. Oxford: Oxford University Press.
Timoshenko, S. P.
1953: *History of strength of materials*, New York: McGraw Hill.
Todhunter, I. and Pearson, K.
1886: *A history of the theory of elasticity and of the strength of materials*, Volume 1, Cambridge: University Press.
Truesdell, C.
1960: "The rational mechanics of flexible or elastic bodies 1638–1788", published as V. 11 part 2 of Euler's *Opera omnia Series 2*.
Youschkevitch, A. P.
1971: "Leonhard Euler". *Dictionary of Scientific Biography* V. 4, 467–484.

NOTES

1. Euler's "De curvis elasticis" has been translated into German by Linsenbarth [1910]. The notes that accompany the translation are reproduced and supplemented by D. M. Brown in the English translation of Oldfather [1933].
2. Euler's theory is described in the major survey histories of elasticity: Todhunter and Pearson [1886, pp. 33–39], Timoshenko [1953, pp. 30–36], and Truesdell [1960, 203–219]. Truesdell's history is the most detailed and is noteworthy for the emphasis it places on Euler's theory as a model for modern research. A final study is Heyman [1972, 106–109], which emphasizes the marginality of Euler's researches to the subject of engineering design.
3. The researches of Navier and Cauchy are described by Todhunter and Pearson [1886,

Chapters 3 and 5]. A more recent account is presented by Grattan-Guinness [1990, V. 2, 969–1047].

4. Truesdell [1960, 43] provides a useful list of the properties of what is called the "stress vector". It is clear however that such an abstract summary can only partially characterize the concept of stress as it appears in specific historical applications of the theory of materials.
5. Timoshenko [1953, 26–27] seems to attribute more to Bernoulli than may be actually found in his writings. Thus the formula he presents on p. 27 was only published by Euler in 1780 (see note 23, this article.)
6. The essay was eventually published in 1862. Truesdell [1960, 143] dates the paper to 1727, when Euler was only twenty and before he had left Basel. My own sense for the chronology of Euler's researches suggests that a somewhat later date would be more plausible. Thus his work in 1727 (outlined by A. P. Youschkevitch [1971, p. 468]) was unrelated to the contents of the paper. He was at this time investigating isochronous curves in resisting media, reciprocal algebraic trajectories and the arrangement of masts on a ship. There is nothing in his treatise *Dissertatio physica de sono* (1727) concerning the problem of solid elastic deformation. The problem of the elastica first appears in a memoir of 1732. (According to Truesdell [1960, 148] an early manuscript version of this memoir exists in the Basel University Library, but the manuscript (to judge from its title) deals only with that part of the published work concerning flexible bodies.) The subject of the *vibration* of an elastic solid is only first mentioned in Euler's correspondence with Daniel Bernoulli in 1735 and does not receive publication until 1740. Since a vibrating elastic solid is the subject of "De oscillationibus ...", and since this paper appears to be something of a preliminary study, it would seem reasonable to suppose that it was written sometime in the middle 1730s.

   (Truesdell reports that an examination by G. K. Mikhailov of the paper and handwriting of the original confirms his conjecture that "De oscillationibus ..." dates from Euler's Basel period. In the absence of more detailed information concerning the location and nature of the papers involved and the methodology employed in the dating it is difficult to evaluate this piece of evidence. Given however that Euler was professionally active from 1727 to 1783 (dictating much of his work in his later years) a date of around 1735 would seem to be consistent with an early determination of his handwriting.)
7. The text that is reprinted in the *Opera* has $Pc^2dt/3fg$ rather than $Pcdt/3fg$. (The latter is what actually follows from the preceding step of the derivation.) That this is a misprint is evident from the fact that Euler immediately sets $dt = (a-b)cds/ab$ and obtains $Pcc(a-b)ds/3abfg$, the formula he actually works with in the paper. (That the c^2 is a typographical error is also clear from the fact that Euler always denotes the square of a single-letter variable a by aa rather than a^2.)
8. A detailed critical account of Euler's analysis is presented by Cannon and Dostrovsky [1981, 37–43].
9. Through a series of "corrections" it is possible to obtain the modern formula for the bending moment of a prismatic beam from Euler's procedure. In calculating the moment of the elastic force we use x rather than x/c as the moment arm; we replace dt by the strain dt/ds; we incorporate the thickness h of the ring into the final formula; finally, we interpret $E = P/fg$ as "Young's modulus". With these changes Euler's formula becomes $(Ec^2/3)(dt/ds)$, which is the flexure formula for a prismatic beam in which the neutral axis

is assumed to lie on an outer surface. (It is on the basis of an argument something like this that Truesdell [1960, 145] arrives at a very high evaluation of Euler's paper.)

10. Bernoulli's letter was dated 20 October 1742 and sent from Basel. It is reprinted in Fuss [1843, 499–507].
11. See P. L. Moreau de Maupertuis's "Loi du repos des corps" [1742], as well as Euler's own "De motu proiectorum in medio non resistente, per methodum maximorum ac minimorum determinando", published as the second appendix of his *Methodus inveniendi* [1744, 311–320].
12. The notes due to Linsenbarth in the Oldfather translation [1933, 152–153] are helpful in following Euler's variational derivation.
13. Huygens's comments were contained in a letter to Leibniz, written from Holland and dated 24 August 1694. The letter is reprinted in Huygens's *Oeuvres* [1905, 664–670, especially pp. 665–6]. It is also published in Leibniz [1850, 189–193]. Excerpts were translated from French into Latin and published in the *Acta eruditorum* of 1694 (see Huygens [1694]). This last note is reprinted in Jakob Bernoulli's *Opera* [V. 1, 1744, 637–638] and in Huygen's *Oeuvres* [1905, 671–672]. (Huygens's original sketch is reproduced in the *Oeuvres*. Figure 5 is the illustration that appears in the *Acta eruditorum*, which is of course what Euler would have seen.)
14. Truesdell [1960, 101] describes Bernoulli's response to Huygens's letter.
15. The result, which belongs to the pre-history of the theory of elliptic integrals, was published in Euler [1743, 91] (in *Opera omnia* Ser. 1 V. 17, p. 34). Related memoirs are cited by Carathéodory in a note in *Opera omnia* Ser. 1 V. 24 (p. 247). Todhunter [1886, 37–38] establishes the result using gamma functions, and Linsenbarth (in Oldfather [1933, 154]) presents a proof using elliptic integrals.
16. Euler makes here a calculational error that is corrected in the translation of Oldfather [1933].
17. To establish divergence we prove that the series majorizes the harmonic series $1+\frac{1}{4}+\frac{1}{4\cdot 2}+\frac{1}{4\cdot 3}+\ldots$ Consider the induction hypothesis:

$$\frac{1^2\cdot 3^2\cdot 5^2\cdot\ldots\cdot(2n-1)^2}{2^2\cdot 4^2\cdot 6^2\cdot\ldots\cdot(2n)^2}>\frac{1}{4\cdot n}. \qquad (*)$$

(*) holds for $n=2$. It is straightforward to show that

$$\frac{(2n+1)^2}{(2n+2)^2}>\frac{n}{n+1} \qquad (**)$$

Multiplying (*) and (**) we obtain

$$\frac{1^2\cdot 3^2\cdot 5^2\cdot\ldots\cdot(2n-1)^2\cdot(2n+1)^2}{2^2\cdot 4^2\cdot 6^2\cdot\ldots\cdot(2n)^2\cdot(2n+2)^2}>\frac{1}{4\cdot(n+1)}.$$

Hence the result is established.

18. For a study of the early development of the trigonometric functions, see Katz [1987].
19. See Euler [1744, 195–196], in *Opera* Ser. 1 V. 24, 184–185.
20. See for example Southwell [1941, 429–435]. Compare also note 21, this article. (An opposing perspective is expressed in the tables presented by Truesdell [1960, 212–213], which seem to refer more to what a modern specialist might derive from Euler's theory

than to what he actually does. Thus for example Truesdell's equations (173), (176), (186) and (187) appear nowhere in Euler.)

21. Let $a_0\,(=2\sqrt{2}f/\pi)$ and $P_0\,(=E^2k^2\pi/4f^2)$ denote the species-one limiting values obtained from (9) and (8) by letting c tend to zero. Assume now that we retain terms of order c^2/a^2 in (9):

$$f = \frac{\pi a}{2\sqrt{2}}\left(1+\frac{c^2}{8a^2}\right). \tag{9'}$$

Consider the deflection $c=(2f)e$, where e is small. (9′) becomes

$$f = \frac{\pi a}{2\sqrt{2}}\left(1+\frac{e^2f^2}{2a^2}\right).$$

This equation leads to a quadratic in a which may be solved approximately as $a = a_0+\delta a_0$, where $\delta = -\pi^2e^2/16$. Since $P=2Ek^2/a^2$ it follows (in approximation) that $P=P_0+\varepsilon P_0$, where $\varepsilon = -2\delta = \pi^2e^2/8$. Hence a fractional increase in P_0 of $\pi^2e^2/8$ corresponds to a value for $c/2f$ of e. We may therefore conclude that for small increments a given fractional increase in the load P_0 corresponds to a deflection that is an order of magnitude larger. (Note: The values of P and $c/2f$ obtained from (9′) are in good agreement (for small $c/2f$) with the ones in Southwell's book [1941, 434] that are computed from tables of elliptic integrals.)

22. The memoir is titled "Sur la force des colonnes". It is noteworthy that Euler uses the term "force" rather than "résistance", the word in modern French that denotes the strength of a material structure.

23. Much later, in a treatise published in 1780, Euler would finally suceed in relating the moment of stiffness Ek^2 to the transverse dimensions of the column. The relevant part is § 14–21 of Euler [1780]. Pearson remarks [1886, 44] that to his knowledge this memoir is the first in which the formula appears. Significantly, Euler's treatment suggests a familarity with the contemporary analysis of fracture problems.

24. In the following we ignore terms of the form $(c/a)^n$ for $n>2$. We have $\theta = \tan(\measuredangle DAC) = dx/dy\ (x=0) = (c\sqrt{2a^2-c^2}/(a^2-c^2)$. Hence $\theta = \sqrt{2}c/a$. We have from (9) $f=(\pi a/2\sqrt{2})(1+(c^2/8a^2))$. It follows that $1/a^2 = (\pi^2/8f^2)(1+\frac{1}{2}(\theta/2)^2)$. Hence $P=2Ek^2/a^2 = (\pi^2Ek^2/4f^2)(1+\frac{1}{2}(\theta/2)^2) = (\pi^2Ek^2/4f^2)\sqrt{1+(\theta/2)^2}$.

25. For a comparative discussion of Comte's positivism and 18th-century exact science see Fraser [1990].