

Nonstandard Analysis, Infinitesimals, and the History of Calculus

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Introduction

Carl Boyer's *The Concepts of the Calculus; a Critical and Historical Discussion of the Derivative and the Integral* was published in 1939 and reprinted in several later editions. Its title harkened back to Ernst Mach's *Science of Mechanics* of 1883 and the conviction that a fuller understanding of a scientific subject was revealed through an examination of its history. Boyer's presentation of the history of calculus encapsulated the prevailing view, one that is still fairly common today. According to this, the first 130 years or so was a "period of indecision," in which infinitesimals were commonly used, series were deployed without attention to questions of convergence and divergence, and new results were prized above all else. In the nineteenth century, this new calculus was then put on a sound basis with the emergence of real analysis, the leading contributors to this development being such figures as Augustin Cauchy, Bernhard Bolzano, P. G. L. Dirichlet, Richard Dedekind, and Karl Weierstrass. The later development embodied an ethos of logical rigor that was absent in the discovery-oriented early years of the subject.

Although infinitesimals and divergent series were apparently banished from mathematics in the post-Cauchy era, they never quite disappeared. A whole new branch of analysis, the theory of summability, emerged at the end of the nineteenth century from the ashes of the older work on divergent series. Attempts to develop a logically rigorous theory of infinitesimals were also pursued by such mathematicians as Giuseppe Veronese, Paul du Bois-Reymond and Otto Stolz. (For a study of these researches see [Fisher 1981] and [Ehrlich 2006].) While the work of the infinitesimalists was noteworthy, they were not as successful as were the inventors of summability theory, and no viable and widely accepted non-Archimedean theory of analysis emerged during the nineteenth century.

About a quarter of a century after the appearance of Boyer's book, new approaches to analysis were advanced that employed infinitesimals. The best known and most influential of these theories was published between 1961 and 1966 by Abraham Robinson, a researcher in model theory in mathematical logic. Robinson's nonstandard analysis was based on a result known as the compactness theorem in model theory. Consider a formalized language that is rich enough to formulate all sentences that are true in the real numbers \mathbf{R} . The main idea in a somewhat simplified form goes as follows. (For details, see [Enderton 1972], 164–173.) If every finite subset of a given set K of sentences has a model, then by the compactness theorem K itself has a model. For each real number consider the sentence that asserts the existence of a number v such that $c < v$. Let K be the set consisting of all such sentences. It is evident that any finite subset of K will have

\mathbf{R} as a model, by selecting v greater than the maximum number in the subset. We conclude by the compactness theorem that K itself has a model \mathbf{R}^* , which will be a nonstandard extension of \mathbf{R} . The element v will be infinite, and an infinitesimal is obtained by taking its reciprocal. \mathbf{R}^* contains infinitesimal numbers, numbers that are smaller than any positive real number but are non-zero.

Robinson's nonstandard analysis embodied a formulation of calculus that used infinitesimals in accord with modern standards of rigor. Authors such as Jerome Keisler [Keisler 1976] wrote calculus textbooks based on non-Archimedean fields that dispensed with the framework of mathematical logic but nonetheless presented infinitesimal procedures in a rigorous way. Since the 1950s researchers have developed other approaches to non-Archimedean analysis, including Curt Schmieden and Detlef Laugwitz's Ω -calculus [Schmieden and Laugwitz 1958], John Conway's [Conway 1976] surreal numbers, and John Bell's smooth infinitesimal analysis [Bell 2008].

In the concluding chapter of his 1966 book Robinson reflected on the history of calculus and the relevance of nonstandard analysis for understanding past mathematics. This line of inquiry was also the subject of his essay "Metaphysics of the calculus," published in the next year. Robinson believed that it was necessary "to redraw" the older history of calculus in light of the rehabilitation of infinitesimals entailed by nonstandard analysis.

Robinson's views were embraced by philosopher Imre Lakatos in an essay written in 1966 and published posthumously in 1978, who used such terms as "revolutionary" and "epoch-making" to describe the significance of nonstandard analysis for understanding the history of calculus. Lakatos was passionately engaged in history and like Robinson believed that one's view of the history of calculus was being distorted through the lens of the modern Weierstrass foundation. Both Robinson and Lakatos focused on Cauchy, who was seen as a more traditional figure than historians such as Boyer had supposed and who continued to deploy infinitesimal methods in his work. According to the conventional historical view Cauchy initiated the modern rigorous formulation of the calculus, a project that was continued and elaborated in more critical detail by Weierstrass. Robinson challenged this view, calling attention to the fact that Cauchy continued to use the language of infinitesimals and employed arguments that were not rigorous by later standards. He maintained that the modern formulation of the calculus that became entrenched by the end of the nineteenth century should be credited to Weierstrass and his contemporaries and not to Cauchy, the latter having been part of the tradition that extended back to Leibniz. Although Lakatos was more qualified than Robinson in developing this argument, he accepted the broad outlines of the thesis. The view that the original formulation by the Leibniz school of calculus in terms of infinitesimals was somehow an unsatisfactory first step on route to the nineteenth-century foundation was of course also rejected.

Other mathematicians were motivated to revisit Cauchy's work on analysis in light of modern non-Archimedean theories. Included in this group were Gordon Fisher [Fisher 1978], and Laugwitz [Laugwitz 1987]. A substantial literature has

developed that examines Cauchy's treatment of various results in analysis. The most active worker here has been Laugwitz, who some two decades following the publication by Schmieden and him of the Ω -calculus commenced to publish a series of articles arguing that their non-Archimedean formulation of analysis is well suited to interpret Cauchy's results on series and integrals.¹

Since the 1960s there has been a new wave of writing about the history of eighteenth-century mathematics. Authors such as Henk Bos, Steven Engelsman, Niels Jahnke, Giovanni Ferraro, Craig Fraser and Marco Panza have charted the development of calculus without interpreting this development as a first stage in the inevitable evolution of an arithmetic foundation. These scholars also have not experienced any particular need to investigate the historical materials in light of recent modern developments in analysis. In their work the dynamics of the growing subject – the “metaphysics of the calculus” – is analyzed on its own terms in its original historical setting. The extent and detailed character of this literature highlight the fairly limited knowledge of the eighteenth-century subject that Robinson and Lakatos brought to their ruminations on the history of analysis.

The new history has focused on uncovering the guiding principles and original outlook of calculus in the period from 1680 to 1820. Of course, classical analysis developed out of the older subject and it remains a primary point of reference for understanding the eighteenth-century theories. By contrast, nonstandard analysis and other non-Archimedean versions of calculus emerged only fairly recently in somewhat abstruse mathematical settings that bear little connection to the historical developments one and a half, two or three centuries earlier. Hence there is a natural tendency either to reject or to overlook nonstandard analysis in arriving at an appraisal of the early calculus. This point has been emphasized by Bos ([Bos 1974], 81-82):

I do not think that the appraisal of a mathematical theory, such as Leibniz's calculus, should be influenced by the fact that two and three quarter centuries later the theory is “vindicated” in the sense that it is shown that the theory can be incorporated in a theory which is acceptable by present-day mathematical standards.

If the Leibnizian calculus needs a rehabilitation because of too severe treatment by historians in the past half century, as Robinson suggests ([Robinson 1966], 260), I feel that the legitimate grounds for such a rehabilitation are to be found in the Leibnizian theory itself. I believe that, in order to prove its value as a mathematical theory, Leibniz's calculus does not need an adjustment to twentieth century requirements.

¹A balanced discussion of Cauchy historiography is provided by Umberto Bottazzini [Bottazzini 1990] in an introduction to an edited reprint of Cauchy's *Cours d'Analyse*. Concerning Lakatos's claim that the discovery of nonstandard analysis will be “epoch making” for the history of mathematics, Bottazzini writes ([Bottazzini 1990] p. lxxxvii), “Apparently this prophecy has not been fulfilled.” Teun Koetsier ([Koestier 1991], Chapters 3 and 4) also gives an illuminating account of the views of Laugwitz and Spalt in his monograph on Lakatos; the penultimate chapter on the equality of mixed partial derivatives is also very worthwhile.

In 1988 Joseph Dauben published “Abraham Robinson and nonstandard analysis: history, philosophy, and foundations of mathematics.” Here Dauben accepts the historians’ view that nonstandard analysis has limited relevance for an understanding of the early calculus, and he criticizes some of Robinson’s claims about Cauchy. He observes that Robinson’s larger purpose was to show that model theory and logic were not peripheral parts of mathematics but were linked to subjects of central concern to mathematicians. According to Dauben, Robinson’s claim that the history of calculus needed to be rewritten in light of the discovery of nonstandard analysis was simply “propaganda” in his efforts to establish this thesis. (In his biography of Robinson, Dauben ([Dauben 1995], 349-266), provides further discussion of Robinson’s views on the history of mathematics.)

In 2013 a group of researchers working on non-Archimedean analysis advocated views similar to those of Robinson, Lakatos and Laugwitz. In their article “Is mathematical history written by the victors,” ([Bair et al. 2013], abstract) assert that modern theories of infinitesimals “show the works of Fermat, Leibniz, Euler, Cauchy and other giants of infinitesimal mathematics in a new light,” that several of the procedures of the older calculus “were only clarified and formalized with the advent of modern infinitesimals.” Although analysis went through several distinct historical stages and involved many different strands of research, the authors believe that modern theories of infinitesimals provide a valuable touchstone to help us understand and interpret this history.

In considering the historiography of analysis we encounter two distinct strands of writings, one that is motivated by present-day mathematical perceptions to look at history, and the other focused primarily on understanding the development of ideas and methods in the context of their time. One must ask whether a given method originally developed in a particular way and with a particular purpose is in need of re-examination in light of modern theories. In the following study we focus primarily on two distinct but related topics. The first concerns the way in which infinitesimals were understood in the early years of the calculus, and the status of the logical critique of the calculus presented by contemporary observers such as Bishop Berkeley. We maintain that an understanding of the early developments requires attention to the dual geometric-algebraic character of the basic processes of the calculus. The question of the logical status of infinitesimals is of secondary interest, from either the perspective of a researcher in the early eighteenth century or an observer today. The second topic concerns the decisive shift to algebraic analysis that occurred in the writings of such figures as Euler and Lagrange in the second half of the century. The transition from algebraic analysis to the modern paradigm of Cauchy and Weierstrass constituted a fundamental change in outlook. Our central claim here is that the emergence of modern analysis in the nineteenth century should be viewed not in terms of the rejection of infinitesimals and divergent series but rather in terms of this broader change in perspective.

Leibniz, l'Hôpital and the early calculus

Infinitesimals in late seventeenth-century mathematical science have been the subject of a large body of recent historical writings, as is evident in the 2008 collection of essays “Infinitesimal Differences Controversies between Leibniz and his Contemporaries” [Goldenbaum and Jessepoh 2008] and the references contained therein. We will not attempt to survey the various issues explored by historians, but concentrate on the basic question of how infinitesimals functioned in the logical workings of the new subject. In particular, we examine the first book devoted to the new calculus, the Marquis de l'Hôpital's *Analyse des infiniments petits pour l'intelligence des lignes courbes* of 1696.

Bernard le Bovier de Fontenelle ([Fontenelle 1790-1792] 6:43), the permanent secretary of the Paris Academy of Sciences, had in 1727 included l'Hôpital among the “great” mathematicians who laid the foundations for an “epoch of almost total revolution in science.” Two decades earlier in an éloge to l'Hôpital Fontenelle ([Fontenelle 1790-1792], 6:31) observed that the popularity of the *Analyse* showed that “the revolution will become still greater.” He ([Fontenelle 1790-1792], 7:67) wrote again in 1720 in connection with this book that “there was in the mathematical world a well-marked revolution [une révolution bien marquée].” This rhetoric of revolution, less common and more significant in l'Hôpital's time than it is today, indicates the degree of excitement aroused by the new mathematical discoveries. (English translation by Cohen ([Cohen 1985], 213-214). Cohen regards Fontenelle's remarks of note because they are an early explicit statement by a contemporary observer of a revolution taking place in science.)

As innovative as l'Hôpital may have been, the theory he detailed in his book owed a great deal to Johann Bernoulli, who had explained the new mathematical methods to the Frenchman during his stay in Paris in the early 1690s and in letters to him. The contents of the *Analyse* were very indicative of the technical concerns shared by researchers of the period, which remained strongly anchored in the study of curves and their properties. One area of calculus that is particularly revealing in highlighting fundamental conceptions is the investigation of maxima and minima, a subject that illustrates well how new ideas played out in a clearly defined mathematical problem context. This subject was taken up by l'Hôpital in section three of his book. We shall illustrate the theory with the example $y = x(a - x)$, where a is a constant, ([l'Hôpital 1696] §51, Example 4 of the *Analyse*, with $n = m = 1$), and use this example to evaluate some of the logical issues associated with the use of infinitesimals. These issues would later arise in polemical form in Bishop George Berkeley's *Analyst* of 1734.

If $y = x(a - x)$ then we have $dy = (x + dx)(a - x - dx) - x(a - x) = (a - 2x)dx - dx^2$. l'Hôpital reasoned that because dx^2 is infinitely small with respect to dx it follows that $dy = (a - 2x)dx$. Let us now consider the problem of finding the maximum of the curve given by $y = x(a - x)$. The situation was illustrated in several figures, of which Figure 1 applies in this case. (In the figure MR is dx and Rm is dy .) l'Hôpital reasoned in general that around a value x_0

for which y is a maximum, dy decreases as x approaches x_0 , changes sign at x_0 , and then increases in absolute value as x increases. At $x = x_0$ the corresponding differential dy must be 0 and at this point y is a maximum. In the example at hand we set $dy = (a - 2x)dx = 0$ and so $2x = a$ and there is a maximum at $x = a/2$.

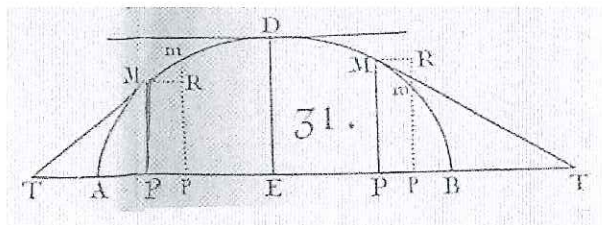


Figure 1: MR is dx and Rm is dy . Not only does Rm tend to zero as M approaches D , Rm also becomes arbitrarily small with respect to MR as M approaches D .

Authors such as Bishop Berkeley [Berkeley 1734] believed that there was a logical defect in the application of the differential calculus to find tangents to curves. The criticism in question would apply to the present example in the following way. Going back to first principles we have $dy = (a - 2x)dx - dx^2 = (a - 2x - dx)dx$. If $dy = 0$ then $(a - 2x - dx)dx = 0$. If $dx \neq 0$ then it follows that $a - 2x - dx = 0$. Setting $dx = 0$ we have $a - 2x = 0$ or $x = a/2$. This derivation seems to require us to suppose that both $dx \neq 0$ and $dx = 0$, an obvious paralogism.

To early practitioners of the calculus this sort of criticism would not have carried much force. In the differential calculus we are positing a partition of the x -axis into intervals of constant length Δx which extrapolates to an infinite partition of intervals of constant infinitesimal length dx . For a series of constant increments Δx the value of the corresponding increments Δy will vary, depending on the value of x . As one approaches a maximum it is evident that the increment $|\Delta y|$ becomes arbitrarily small with respect to Δx . That is, $\Delta y = (a - 2x - \Delta x)\Delta x$ (in absolute value) becomes arbitrarily small with respect to Δx . Suppose $x = c$ is not equal to $a/2$. Then $a - 2c \neq 0$ and $(a - 2x - \Delta x)\Delta x = (a - 2c - \Delta x)\Delta x = (a - 2c)\Delta x - \Delta x^2$ and it is evident that this expression cannot be made arbitrarily small with respect to Δx . On the other hand if $x = a/2$ then $\Delta y = -\Delta x^2$ and $|\Delta y|$ can be made arbitrarily small with respect to Δx by making Δx sufficiently small. Hence at a maximum we must have $a - 2x = 0$ or $x = a/2$.

Moving from increments to differentials, we conclude that at a value x that is a maximum $|dy|$ is less than dx/n for all positive integers n . If $x = c \neq a/2$ then $dy = (a - 2c)dx - dx^2$, and the condition $|dy| < dx/n$ for all n does not hold. On the other hand, if $x = a/2$ then $dy = -dx^2$. Since $dx < 1/n$ for all n it follows that this condition does hold.

In the general case where y is any expression involving x , we have $dy = p dx + (\text{second order terms})$. At a minimum or maximum we must have $p = 0$. For

at a minimum or maximum $|dy|$ becomes arbitrarily small with respect to dx . If $p \neq 0$ then $|dy|$ will not be arbitrarily small with respect to dx . Hence we must have $p = 0$ at a minimum or maximum.

The reasoning underlying the use of infinitesimals was robust, as Jean d'Alembert ([d'Alembert 1754], [d'Alembert 1765]) in his articles in the *Encyclopédie* tried to explain in some detail by appealing to the notion of limit. d'Alembert possessed the acutest critical sense of any of the commentators who addressed concerns about the foundations of the calculus. Berkeley's belief that what was actually going on involved a compensation of errors was an idea that lacked any real explanatory power. (For a lucid presentation of Berkeley's idea see [Youschkevitch 1971], 152-153.) The calculus worked not because there was a compensation of errors, but because the difference between $y'(x)dx$ and $y(x + dx) - y(x)$ was infinitesimally small with respect to dx .

On the other hand Berkeley and critics both before and after him did have a point, since the actual infinitesimals dx and dy clearly had a different status from the increments Δx and Δy . A quantity dx that is less than $1/n$ for all n but is not zero clearly is a magnitude very different from the traditional Archimedean magnitudes of classical mathematics.

Leibniz himself was very concerned about the status of infinitesimal entities, a concern that was expressed in greatest detail in his correspondence and manuscript writings. One commentator ([Reyes 2004], 171) has observed that "Leibniz's rhetorical engagement with the issues raised by infinitesimals reaches nearly unmanageable proportions." The large space Leibniz devoted to discussing infinitesimals indicates that he believed there were real foundational issues at stake, although he never developed a coherent theoretical strategy to deal with them. In this respect the modern development of a rigorous theory of actual infinitesimals may be of some relevance to an historical appraisal of the early calculus. If nothing else it provides some psychological support for the conviction that this calculus should not be viewed as a naïve and logically unsatisfactory form of what later would become real analysis.

Differentials in the eighteenth-century calculus

The eighteenth-century conception of differentials and the rules that governed their use in calculus have been investigated by historians. ([Bos 1974] is the most detailed study of this sort.) In the Leibnizian calculus, the operation of differentiation possessed a dual character: algebraic/algorithmic on the one hand, and geometric on the other. The algebra comprised a set of rules that governed the use of the symbol d and was based on two postulates: $d(x + y) = dx + dy$ and $d(xy) = ydx + xdy$. Accompanying these rules there was also an order principle, according to which higher-order differentials in a given equation were to be neglected with respect to differentials of a lower order.

The differentials that appeared in a given problem could also be understood in another way: as the differences of values of a variable quantity at successive

points in the geometric configuration. The differential dx was set equal to the difference of the value of x at two consecutive points infinitely close together; higher-order differentials were set equal to the difference of successive lower-order differentials. Euclidean geometry was used to analyze the properties of the curve in terms of these differentials.

An example of the geometric treatment of infinitesimals is given by l'Hôpital's calculation in section 5 of the *Analyse* of the radius of curvature to a curve at a point. Let M be any point on the curve AMD (Figure 2). Let m be a point on the curve infinitely close to M . The normals to the curve at M and m intersect at the center of curvature C . The distance MC is the radius of curvature. Suppose $AP = x$ and $PM = y$ are the abscissa and ordinate of M . The lines MR and Rm parallel to AP and PM are the infinitesimal increments dx and dy of x and y . Using the differential algorithm l'Hôpital was able to obtain a formula for the radius of curvature involving the differentials dx , dy and the second order differential ddy . He also derived the same formula in a purely geometric way. Let n be a point on the curve infinitely close to m (Figure 3). l'Hôpital conceived of the portion Mmn as composed of the polygonal segments Mm and mn . The second differential of y , ddy , is given as $ddy = nS - mR = nS - HS = -Hn$. By means of similar triangles l'Hôpital arrived at the same expression for the radius of curvature that he had obtained earlier analytically.

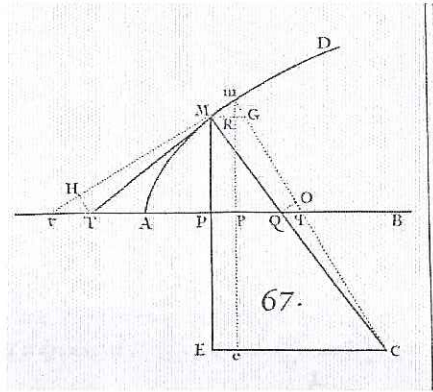


Figure 2: MC is the radius of curvature at the point M of the curve AMD .

In his mid-century treatises Euler, as part of his program of separating analysis from geometry (discussed in more detail below) made the algebraic conception of differentiation fundamental. In so doing he made the concept of the algorithm primary in his understanding of the foundations of the calculus. Some of the issues that arise in this shift in viewpoint are illustrated by his theory of differential expressions set forth in Chapters 8 and 9 of the first part of his 1755 *Institutiones Calculi*. Consider any formula containing dx , ddx , dy , ddy , Because these

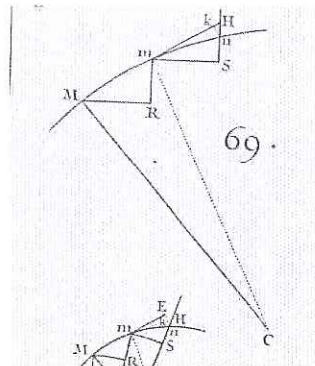


Figure 3: ddy is equal to $-Hn$.

quantities are no longer interpreted geometrically the meaning of the formula is unclear; its value will depend on whether dx or dy is held constant, an assumption that is not evident in the algebra. For example, the quantity ddy/dx^2 is zero if dy is constant; if dx is constant its value will vary according to the functional relation between x and y . Conversely, certain expressions, such as $(dyddx - dxddy)/dx^3$, may be shown to be invariant regardless of which variable is taken to be independent.

Euler's solution to the problem of indeterminacy in differential expressions was to introduce notation that made clear the relations of dependency among the variables. He did so by eliminating higher-order differentials as such, replacing them instead with differential coefficients. Rather than write ddy/dx^2 (dx constant) we define the differential coefficients p and q by the relations $dy = pdx$ and $dp = qdx$; ddy/dx^2 then becomes simply q . Euler provided rules and examples that showed how more complicated expressions can be reduced to ones containing only variables and differential coefficients. In addition to bringing order to the calculus, this emphasis on the differential coefficient was conceptually important in identifying the derivative as an independent object of mathematical study.

Shift to pure analysis

The fundamental notion of the early calculus was the operation of differentiation, understood as an algorithm or a process. As the subject developed, the notion of infinitesimal receded in importance, and the algorithm came to occupy a more prominent place. Johann Bernoulli redefined the integral as the operational inverse of differentiation, a definition that became standard in the eighteenth century. The original Leibnizian concept of an integral as a sum of infinitesimal elements was replaced by an object obtained by the inverse operation of differentiation applied to an analytic expression.

It is important to note that the primary foundational issue in eighteenth-century calculus did not involve infinitesimals at all. This issue concerned the relative place of geometric and analytic conceptions in the foundations of the subject, and the possibility of a conception of pure analysis as a subject autonomous from any geometric basis or interpretation. While there were other more specific foundational discussions, such as the debate over the function concept (which also did not involve infinitesimals), these discussions should be viewed from the perspective of the larger shift to pure analysis that occurred as the century progressed.

In the eighteenth century the various calculus-related parts of mathematics were heavily problem-oriented, rather than deductive or theorem-oriented. The emphasis of mathematics at this time was analytic in the sense that analysis is the appropriate mathematical method for investigating problems. However, by the middle of the century the term analysis had largely lost its original Pappusian meaning of “solution backwards.” (For a discussion of analysis and synthesis around 1700 see [Guicciardini 2009].) As is well known, during the early modern period analysis came to denote algebra and the use more generally of symbolic methods in the solution of problems. When one introduces a variable and derives an equation, one is assuming logically at the outset that the thing that is sought is at hand, even if one does not know its value. Hence methods in which the existence of the thing sought is first assumed as an unknown variable and its value is derived by means of some mathematical process are analytic. Analysis came to denote algebraic symbolic methods and was contrasted with geometric modes of solution.

In the writings of Euler and Lagrange, analysis and synthesis no longer referred to the kind of mathematical method employed – which could be either geometric or algebraic – in a given mathematical investigation. Analysis was not primarily a method or an art, but rather an autonomous branch of mathematics, a subject area in its own right employing processes that were linguistic or symbolic in character. Above all, analysis avoided geometric modes of representation. Synthesis denoted a geometric conception of the mathematical object in which this object as a whole was taken as given and in which its properties were used in the course of the investigation. (For a more detailed exposition of the Eulerian-Lagrangian conception of analysis see [Fraser 2003].)

In the problems and examples of the early calculus geometric conceptions were ubiquitous. l’Hôpital’s book contained 156 figures. In the solutions of Newton, Huygens, and the Bernoullis detailed geometric knowledge of the particular problem situation at hand was often required for its solution. By contrast, analysis – understood as the application of a symbolic process given in terms of variables, operations and coordinate systems – entailed a more general and uniform approach. This fact is noted by Euler in 1736 at the beginning of his book on analytic mechanics:

But what obtains for all the works composed without analysis holds most of all for mechanics: even if the reader be convinced of the truth of the things set forth, nevertheless he cannot attain a sufficiently clear and distinct knowledge of them; so that, if the same questions be the slightest

bit changed, he may hardly be able to resolve them on his own, unless he himself looks to analysis and evolves the same propositions by the analytic method. Thus, I always have the same trouble, when I might chance to glance through Newton's *Principia* or Hermann's *Phoronomiam*, that comes about in using these, that whenever the solutions of problems seem to be sufficiently well understood by me, that yet by making only a small change, I might not be able to solve the new problem using this method. Thus I have endeavoured for a long time now, to get at the analysis behind those synthetic methods in order to draw out the same propositions that are more readily handled by my own analytic method, and so by working with this latter method I have gained a perceptible increase in my understanding. ([Euler 1736], Preface; translation from [Mahoney 1985], 198) and (Bruce, Euler Archive))

Euler's analytic philosophy of mathematics was embraced and developed to a new level of sophistication and purity by Lagrange. Although Lagrange's analytic tendencies were apparent from the very beginning of his career, his distinctive mathematical style really only became consolidated in the period 1770 to 1776, when he was in his late thirties and comfortably settled at the Berlin Academy. In these years the value of analysis becomes an explicit theme in his writings for the Academy on a range of subjects in pure and applied mathematics. The following is a selection from these writings, and is by no means complete. (Detailed citations for the references which follow are given in ([Taton 1974], 10-20).) In a paper of 1775 on the attraction of a spheroid Lagrange attempted to show that the method of "algebraic analysis" provided a more direct and general solution than the "synthetic" or geometric approach followed by Colin Maclaurin. In his study of the rotation of a solid he advanced an alternative to the mechanical treatment of d'Alembert and Euler, one that was "purely analytic", whose merit consisted "solely in the analysis" that it employed, and which contained "different rather remarkable artifices of calculation." In a memoir on triangular pyramids he noted that his "solutions are purely analytic and can even be understood without figures"; he observed that independent of their actual utility they "show with how much facility and success the algebraic method can be employed in questions that would seem to lie deepest within the province of Geometry properly considered, and to be the least susceptible to treatment by calculation."

The theme of analysis recurred in Lagrange's writings of the later 1770s and 1780s. In a memoir submitted to the Paris Academy in 1778 on the subject of planetary perturbations he offered a method for transforming the equations of motion that would "take the place of the synthetic methods proposed until now for simplifying the calculation of perturbations in regions beyond the orbit" and that "has at the same time the advantage of conserving uniformity in the workings of the calculus". In 1780 he published a memoir on a theorem of Lambert in particle dynamics. The result in question had been demonstrated synthetically, and Lagrange expressed concern that it might be regarded as one of "the small number [of theorems] in which geometric analysis seems to be superior to algebraic

analysis". His purpose was to present a simple and direct analytic proof. In the preface to his famous *Traité de la mécanique analytique*, completed around 1783, he announced that in it "no figures would be found", that all would be "reduced to the uniform and general progress of analysis." (See [Fraser 1983] for a discussion of this book.) Directness, uniformity and generality were qualities that Lagrange associated with analysis; he sometimes also mentioned simplicity. Analysis was noteworthy not simply for the results to which it leads, but also for the methods that it offered. In the writings cited above he was affirming the value of analysis in situations where an alternative geometric or mechanical treatment existed; it was the possibility of this alternative that led him to assert his own methodological preferences. One should also note the sheer preponderance of pure analysis in his work of the 1770s and 1780s in such topics as the theory of equations, Diophantine arithmetic, number theory, probability and the calculus, subjects in which explicit questions of approach or methodology did not arise.

In the 1780s a growing interest in the foundations of analysis was reflected in the decisions of the academies of Berlin and Saint Petersburg to devote prize competitions to the metaphysics of the calculus and the nature of the infinite. Lagrange's views on the subject were presented in their most developed form in his *Théorie des fonctions analytiques* of 1797. The subtitle signalled his intentions in this work: "Containing the principles of the differential calculus, freed from all consideration of infinitesimals, evanescent quantities, limits or fluxions and reduced to the algebraic analysis of finite quantities." (For studies of Lagrange's book see [Grabiner 1981] and [Fraser 2005].) Even before the arithmetic revolution of the nineteenth century, Lagrange's analytic approach led him to reject infinitesimals and kinematic conceptions as founding notions of the subject.²

Lagrange used the term "algebraic analysis" to designate the part of mathematics that resulted when algebra was enlarged to include calculus-related methods and functions. The central object here was the concept of an analytic function. Such a function $y = f(x)$ was given by a single analytic expression, constructed from variables and constants using the operations of analysis. The relation between y and x was indicated by the series of operations schematized in $f(x)$. The latter possessed a well-defined, unchanging algebraic form that distinguished it from other functions and determined its properties.

Lagrange's contemporary Simon Laplace was also an adherent of algebraic analysis, writing ([Laplace 1835], 465) that "by abandoning oneself to the oper-

²In his later didactic works Lagrange displayed a certain degree of self-consciousness about the history of the mathematics that he himself had played such a prominent role in shaping. The early geometric calculus had been characterized by a range of detailed procedures and methods that were tailored to the particular problems at hand. In some remarks on the development of the calculus of variations in the preceding century [Lagrange 1806] observed: "But in the current state of analysis we may regard these discussions as pointless, for they concern forgotten methods, which have given way to others that are simpler and more general. However, such discussions may yet retain some interest for those who like to follow step by step the progress of analysis, and to see how simple and general methods arise from particular questions and complicated and indirect procedures."

ations of Analysis . . . one is led, by the generality of this method and by the inestimable advantage of transforming the reasoning into mechanical procedures, to results often inaccessible to synthesis. Such is the fruitfulness of Analysis that it suffices to translate particular truths into this language in order to see emerge from their very expression a multitude of new and unexpected truths." (English translation in ([Hawkins 1977], 121).)³ The intellectual attitudes of Lagrange and Laplace were historically specific, rooted in modes of reasoning that were common in the second half of the eighteenth century but which gradually lost currency as the nineteenth century progressed.

Algebraic analysis as an approach to mathematics was valued by other prominent researchers of the period as a powerful tool of investigation throughout science. Nicolas de Condorcet ([Condorcet 1847], 467) wrote in 1786 of Euler that he "sensed that algebraic analysis was the most comprehensive and certain instrument one can employ in all sciences, and he sought to render its usage general." (Translation from ([Rider 1990], 115).)⁴ The French philosopher Auguste Comte, although a somewhat later figure, revered Lagrange and believed that he had brought mathematics to an almost completed state. In volume one of his *Cours de Philosophie Positive* Comte [Comte 1830] wrote, "Today, in fact, mathematical science is less important even for the very real and very valued knowledge that directly makes it up, than for constituting the most powerful instrument that the human mind can employ in the investigation of the laws of natural phenomena" ([Comte 1830], 112). Comte divided mathematics into an abstract part, which consisted of algebra and all of analysis, and a concrete part and asserted that "only the abstract part is purely instrumental" ([Comte 1830]. 113). According to Comte, "the sense [*l'esprit*] of mathematical analysis consists in considering magnitudes only from the point of their relations, and independently of any idea of determinative value" ([Comte 1830], 215). (For a discussion of Comte's philosophy of mathematics see [Fraser 1990].) In this conception questions about the status of infinitesimals or the meaning of imaginary numbers were very much secondary, and the primary emphasis was on operations, functions, relations and the active process of working to solutions.

³Thomas Hawkins ([Hawkins 1977], 122) has referred to the kind of reasoning employed by Lagrange and Laplace as "generic," and contrasted their conception of generality with the more critical approach of Cauchy and Weierstrass. Hawkins does so in a historical study of the work of the four men on systems of linear differential equations with constant coefficients, but the observation would apply to their work taken as a whole.

⁴Historian of physics John Heilbron [Heilbron 1990], looking at the later eighteenth century from the perspective of physical science, has called attention to a distinct instrumentalist orientation to the application of mathematics. Heilbron ([Heilbron 1990], 3) observes that "in the second half of the century, analysis and algebra, which, in contrast to geometry, had an instrumentalist bias, became the exemplar of the mathematical method."

Divergent series and changing philosophical perspectives

For some insight into the historiographical issues raised by modern theories of infinitesimals it is worthwhile to look at other case studies involving the evaluation of older theories in light of later mathematical developments. A good example is Euler's work on divergent infinite series in the 1750s and 1760s and the emergence of summability theory at the end of the nineteenth century. In some respects this is a more promising case study than nonstandard analysis and the early calculus, because the chronological and conceptual separation between the early and later developments is smaller. (The historical thread is well known; for an overview see [Kline 1972], Chapter 47.)

In the eighteenth-century infinite series were conceived of as things that were given as part of objective reality; they were not defined entities. A completely general infinite series $\sum_{i=1}^{\infty} a_i$ where the a_i are arbitrary real numbers was never considered by Euler and his contemporaries. Infinite series were not created through definition but always come from somewhere, generated by an analytic process of some sort. As Euler [Euler 1760] put it, "infinite series find a place in analysis inasmuch as they arise from some closed expression." For example, the infinite series $1 - x + x^2 - x^3 + \dots$ is generated from the expression $1/(1+x)$, since $(1+x)(1-x+x^2-x^3+\dots) = 1$. Setting $x = 1$ Euler concluded that the divergent series $1 - 1 + 1 - 1 + \dots$ has the sum $1/(1+1)$ or $1/2$.

The logical difference in outlook between Euler and modern researchers has been recognized by such authorities as G. H. Hardy, who writes in his book *Divergent Series* ([Hardy 1949], 5-6):

...it does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by X we mean Y '. There are reservations to be made ... but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we define $1 - 1 + 1 - \dots$?' but 'What is $1 - 1 + 1 - \dots$?',

A similar observation is made by Konrad Knopp ([Knopp 1928], 457), who writes:

In our exposition, *the symbol for infinite sequences* was created and then worked with; it was not so originally, these sequences *were there*, and the question was, what could be done with them. [italics in the original]

Historian of mathematics Giovanni Ferraro ([Ferraro 2000], 95) has elaborated on this point:

Today we say that the series $1 - 1 + 1 - 1 + \dots$ is not summable according to the usual definition but is summable by Cesaro's method. This means that the "real" sum of the series does not exist: the sum depends on the chosen definition; one can imagine many different definitions and, therefore, many different sums of the same series. In a sense, modern definitions generate the object of study. Eighteenth century mathematicians viewed the mat-

ter differently. Definitions were clarifications of the unique and necessary “truth” that already existed in Nature. A mathematical theory was not a matter of the logical deduction of axioms and definitions, subject only to the principle of contradiction; rather, it was an idealisation derived from physical reality.

The perspective of eighteenth-century researchers on divergent series reflected the broader conception of the nature of mathematics that was prevalent during this period. Sergei Demidov ([Demidov 1982], 37), writing of the failure of Euler and d’Alembert to understand each other’s point of view in the discussion of the wave equation, observes:

A cause no less important of this incomprehension rests, in our opinion, on the understanding of the notion of a solution of a mathematical problem. For d’Alembert as for Euler the notion of such a solution does not depend on the way in which it is defined ... rather the solution represents a certain reality endowed with properties that are independent of the method of defining the solution. To reveal these properties diverse methods are acceptable, including the physical reasonings employed by d’Alembert and Euler.

A biographer of d’Alembert ([Grimsley 1963], 248) has noted his insistence on “the elementary truth that the scientist must always accept the essential ‘givenness’ of the situation in which he finds himself.” The sense of logical freedom that developed in later mathematics – expressed, for example, in Richard Dedekind’s famous statement of 1888 that numbers are free creations of the human mind – reflects aspects of the modern subject that were absent in the eighteenth century.

The situation with respect to the calculus of infinitesimals parallels the history of divergent series. With the establishment of an arithmetic basis for calculus in the nineteenth century, differentials understood as free-standing infinitesimal entities were banished from rigorous analysis. There is a conceptual gulf between Euler’s proof of the theorem on the equality of mixed partial differentials and the modern Weierstrassian theorem on the equality of mixed partial derivatives ([Fraser 1989], 319-321). The emergence of nonstandard analysis in the second half of the twentieth century brought with it the resurrection of infinitesimals, and seemed to confer new respectability on the work of the calculus pioneers. Of course, the logical setting of nonstandard analysis was alien to the outlook of the eighteenth century. In the earlier period the calculus of differentials was understood as something that was given to the researcher from without, a domain of objects that could be investigated geometrically or analytically. By contrast, in nonstandard analysis the infinitesimal arises within a system of definitions and imposed axioms. Two of the most prominent features of the early calculus – the tension between analytic and geometric modes of representation and the central place occupied by the algorithm – are not reproduced at all in nonstandard analysis as defining characteristics of the subject.

Functions of a Complex Variable

In the algebraic analysis of Euler and Lagrange no distinction was made between the subjects of real analysis and complex analysis. In his *Introductio in Analysin Infinitorum* Euler ([Euler 1748], §2) wrote, “A variable quantity includes all numbers, positive and negative, whole and fractional, rational, irrational and transcendental. Even zero and imaginary numbers are not excluded in the meaning of a variable quantity.” The algebraic understanding of the calculus reinforced the implicit assumption that the extension of the calculus to the complex domain raised no new issues. While there was a substantial amount of work done in the eighteenth century on problems that one today would classify as part of complex analysis, the various results never coalesced into a distinct branch of mathematics. One could by the rules of algebra manipulate expressions containing imaginaries, and it was clear in specific problems that the formalism possessed a geometric interpretation. However, the idea that one must distinguish two theories, for functions of a real and a complex variable, never arose in the 18th century. In reference to algebraic analysis in the eighteenth century, Jahnke ([Jahnke 2003], 107) writes, “...when a formula demanded an interpretation of a variable as being complex this was tacitly assumed. For Euler and Lagrange, a clear distinction between real and complex analysis as developed in the 19th century was unthinkable.” (The point in question is also discussed by [Fraser 1989], 327–328.)

Imaginary or impossible numbers, and more generally analytic expressions containing what were sometimes referred to as “imaginary constants,” entered calculus at an early stage in its development. A pioneer in this field was Johann Bernoulli, some of whose results were transmitted by Leibniz in an article of 1703 in the *Acta Eruditorum*. In this piece Bernoulli ([Bernoulli 1703], 30) considered an expression of the form $dt/\sqrt{-1}t$, which he referred to as the “differential of an imaginary logarithm.” Bernoulli called the integrals of such expressions “imaginary logarithms” or “quadratures of imaginary hyperbolas.”

In an article published in 1712 in the *Acta Eruditorum* Johann Bernoulli considered imaginary hyperbolas of the form $y = 1/(x - \sqrt{-1})$ and integrated such expressions to obtain $z = \log(x - \sqrt{-1})$. As we noted in §6, Johann redefined the notion of the integral, from a quantity that was a sum of area elements, to something more like what one would call a primitive or antiderivative. It is worth noting that examples such as the one just presented would have provided motivation for such a shift. It is difficult to conceive of the area under the imaginary hyperbola $y = 1/(x - \sqrt{-1})$ or to conceive of the integral of this expression in terms of a summation process. On the other hand from the differential calculus it was known that the differential of $\log x$ was dx/x . Hence $z = \log(x - \sqrt{-1})$ was an antiderivative of $1/(x - \sqrt{-1})$ and could be taken as its integral, even though $y = 1/(x - \sqrt{-1})$ had no direct geometric meaning.

Cauchy was the single most important figure in the nineteenth century in establishing complex analysis as an independent branch of mathematics. His contributions to the field were developed over thirty years in a series of memoirs

devoted to integration in the complex domain. Complex analysis emerged in his work as a subject with its own theorems, problems and applications. Argand and Gauss had shown that the field of complex numbers could be interpreted as points in a two-dimensional continuum. A function of a complex variable became a relation among variables connecting points in two such continua. The framework that Cauchy had developed for real analysis, involving neighborhoods, limits and pointwise definition of the derivative, was immediately generalizable to the complex domain.

With the transition from the algebraic calculus of the eighteenth century to the new analysis of the nineteenth century it became possible for a theory of functions of a complex variable to emerge as a coherent branch of mathematics. The construction of such a theory marked a major event in the history of mathematics. It is worth noting the very minor role infinitesimals and matters of ontology played in this historic shift.⁵

Interlude: Infinitesimals and mathematical science

It is of course important to remember that even today one employs infinitesimals in engineering and physics, where they play a valuable role in the derivation of physical laws and formulae that is free of any concerns in the foundations of mathematics. Even within mathematics proper researchers following Weierstrass continued to use infinitesimals, and one has to be cautious about attributing very much foundational import to this fact. An illustrative example is a paper of Ernst Schröder's from 1870, where he derived the first result in what later became known as the subject of complex dynamics. We follow Daniel Alexander's [Alexander 1994], 6-8) presentation of Schröder's theorem. Let $\phi(z)$ be a complex function which is analytic on a neighborhood of a point x which satisfies $\phi(x) = x$ with $|\phi'(x)| < 1$. Let $\phi^n(z)$ denote the n (th) iterate of $\phi(z)$, that is the n -fold composition of $\phi(z)$ with itself. Then for all z in some neighborhood D of x

$$\lim_{n \rightarrow \infty} \phi^n(z) = x. \quad (1)$$

In terminology that became standard later, x is an attracting fixed point for the function $\phi(z)$. To prove this result we begin by taking the Taylor expansion for $\phi(z)$ about x :

$$\phi(z) = x + \phi'(x)(z - x) + \dots, \quad (2)$$

where $0 < |\phi'(x)| < 1$. If z is sufficiently close to x then the term $\phi'(x)(z - x)$ dominates in this expansion. Hence for infinitely small ϵ

$$\phi(z) = x + \phi'(x)\epsilon. \quad (3)$$

⁵For an indication of Euler's work involving functions and complex variables see [Euler 1983]. The nineteenth-century history of complex analysis is described by Bottazzini [Bottazzini 2003] and Bottazzini and Gray [Bottazzini and Gray 2013]. Smithies [Smithies 1997] is very good on Cauchy's contributions.

Taking ϕ of each side of this equation we have

$$\phi^2(z) = \phi(x + \phi'(x)\epsilon) = \phi(x) + \phi'(x)\phi'(x)\epsilon = x + \phi'^2\epsilon, \quad (4)$$

or

$$\phi^2(z) = x + \phi'^2\epsilon. \quad (5)$$

Continuing this process we obtain

$$\phi^n(z) = x + \phi'^n\epsilon. \quad (6)$$

Because $|\phi'(x)| < 1$ it follows that

$$\lim_{n \rightarrow \infty} \phi^n(z) = x. \quad (7)$$

Alexander ([Alexander 1994], 7) suggests that Schröder's use of infinitesimal arguments "is perhaps symptomatic of his isolation from the German mathematics mainstream, since it suggests that Schröder was unaware of Karl Weierstrass' (1815-1897) rigorous delta-epsilon approach to mathematics." While this may be true, it would not be difficult to recast the derivation in a form that satisfies the Weierstrassian criteria for rigor. The appearance of infinitesimals in the derivation was very different from their use by such contemporary mathematicians as Paul du Bois-Reymond, who were consciously attempting to create a viable mathematical theory of infinitesimals. In devising a satisfactory proof that captured the idea of the result, Schröder displayed the pragmatic sense of the working mathematician for whom fine points of mathematical rigor were not of primary concern.

Conclusion

The transition from algebraic analysis of the eighteenth century to Cauchy-Weierstrass analysis of the nineteenth century marked a much greater discontinuity than did the emergence of nonstandard analysis out of classical analysis in the second half of the twentieth century. Nonstandard analysis is an offshoot of modern analysis and sits solidly on the modern side of the conceptual gulf opened up by the Cauchy-Weierstrass foundation. In this respect Robinson and to a lesser degree Lakatos were mistaken in their assessment of Cauchy. Rather than try to understand Cauchy as someone who developed within a given intellectual and historical milieu, they approach the history from an essentially artificial point of view. The claim that Cauchy should be seen as a conservative figure whose approach was aligned with the Leibnizian infinitesimalist tradition constituted a misguided exercise in historical revisionism. In fact, Cauchy was a revolutionary figure in the foundations of calculus, not simply because of his contributions to rigor, but also for developing a more conceptual approach that made the numerical continuum the fundamental object of reference.

The work of Euler, Lagrange and their contemporaries represented a new stage in the history of calculus, one that was very different from both the geometric

approach of the pioneers as well as the modern approach initiated by Cauchy. In the writings of Euler and Lagrange analysis was a subject separate from geometry, although it was also a very effective tool for the investigation of curves and surfaces. This separation of the formalism of the calculus from geometry is not part of the logical landscape of the modern subject. A theorem about a function defined on some interval of real numbers under specified conditions of differentiability has a geometric interpretation implicit in its very formulation.

Lagrange's algebraic analysis was made up of functional relations, algorithms and operations on expressions composed of variables and constants. The values that these variables received, their arithmetic or geometric interpretation, were of secondary concern. In real analysis, by contrast, the basic object of study is the numerical continuum. The formalism of the modern subject is in a fundamental sense interpreted – given meaning – as a theory of functions defined on intervals of real numbers. In this respect classical real analysis resembles the calculus of the early eighteenth century, when the calculus was a kind of fine geometry, a way of representing and investigating the curve. The early calculus was interpreted in the geometry of curves in the same way that the modern calculus is interpreted in real analysis.

The geometric continuum of the early calculus possessed a particulate structure, different from that of the numerical continuum of modern analysis, and a calculus of infinitesimals was consistent with such a conception. The various accounts of infinitesimals from the eighteenth century (described for example in [Boyer 1939], Chapter 6) may be viewed as reasonable attempts to formulate a coherent system of magnitudes. The calculus united an algebra of differential expressions and a geometry of discrete infinitesimals, and the resulting conception formed the basis of a powerful and successful mathematical theory.

The relevance of modern non-Archimedean theories to an historical appreciation of the early calculus is a moot point. It is doubtful if it is possible or advisable to reconstruct a past mathematical subject in a way that conforms to modern theories and is also consistent with how a practitioner of the period would have worked. Such a project runs the obvious risk of imposing one's own interests and conceptions on the past subject. It is possible that such an endeavor will end up with something of intellectual interest and mathematical value, but it is unlikely that it will constitute a significant contribution to history. Ivor Grattan-Guinness in his essay "The mathematics of the past: distinguishing its history from our heritage" (2004) has drawn a distinction between history and heritage, between an historical account of the development of past mathematics, and a cultural celebration of past mathematical achievements for today's researchers and students.⁶ The early calculus and modern nonstandard analysis both belong to

⁶Referring to the way mathematicians look at the past, Grattan-Guinness observes ([Grattan-Guinness 2004],abstract), "Old results are

modernized in order to show their current place; but the historical context is ignored and thereby often distorted. By contrast, the historian is concerned with what happened in the past, whatever be the modern situation."

the heritage of mathematics, but attempts to connect them as parts of history have been less successful.

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