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CHAPTER 12

The Calculus of Variations: A Historical Survey

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12.1. Introduction

The calculus of variations as a recognizable part of mathematics had its origins in Johann Bernoulli's challenge in 1696 to the mathematicians of Europe to find the curve of quickest descent, or brachistochrone. This problem as well as others in which it was required to determine a curve satisfying some minimal or maximal property proved to be amenable to the techniques of the new calculus. The development of the subject in the eighteenth century engaged the efforts of such leading analysts as Jakob Bernoulli, Leonhard Euler and Joseph Louis Lagrange. Bernoulli invented a general and successful method of solution. Euler in turn took this method and fashioned from it a coherent and extended mathematical theory, one based on certain standard differential equational forms. In 1762 the subject was radically reshaped as a result of Lagrange's introduction of the δ-algorithm. Lagrange's variational formulation provided the general mathematical framework for subsequent research in the field.

Euler and Lagrange showed that the solution to a variational problem must satisfy a certain differential equation, known today as the Euler or Euler-Lagrange equation. In an investigation of the second variation published in 1788 Legendre derived an additional criterion that must be satisfied by this solution. His derivation was based on a certain transformation of the variational integrand involving the integration of an auxiliary differential equation. He provided no method for integrating this equation, nor did he provide an analysis of the conditions under which the transformation is valid. In a seminal paper published in 1837 Carl Gustav Jacobi provided the outlines of precisely such a theory. There were two fundamental aspects to his theory. On the one hand it yielded a new and systematic way of transforming the second variation. On the other hand it furnished criteria for determining what later mathematicians would call conjugate points. In the thirty years following 1837, a range of authors elaborated and extended the results contained in Jacobi's paper. This line of research culminated in an 1868 paper by Adolph Mayer in which he provided a brilliant synthesis of the two fundamental aspects of Jacobi's theory.

In the 1870s the calculus of variations entered a new phase as German researchers began to investigate the subject in a rigorous way from the standpoint of the theory of a function of a real variable. In 1877 G. Erdmann published a paper giving conditions under which broken extremals, function whose derivatives are discontinuous at a finite number of points, are solutions to a variational problem. Two years later Paul du Bois-Reymond carried out a detailed study of the basic variational processes in terms of real-variable analysis. In the mid-1880s Ludwig Scheefer published research in which he subjected the traditional conditions of Euler, Legendre and Jacobi to very close critical scrutiny.

The leading figure in the new calculus of variations was Karl Weierstrass. In his lectures in the 1870s and early 1880s he pioneered a new method which provided conditions sufficient to ensure the existence of a maximum or minimum in single-integral variational problems. The basic idea of Weierstrass, the concept of a field of extremals, allowed one to develop the theory in reference to a much larger class of comparison variations. Hilbert presented an important simplification of Weierstrass's technique in his famous Paris address of 1900. Writings based on Weierstrass's field methods were published during the period 1895–1905 by Ernst Zermelo, Adolph Kneser, E. R. Hedrick, Oscar Bolza and E. J. B. Goursat. Major textbooks by Bolza (Bolza 1909) and Jacques Hadamard (Hadamard 1910) provided a masterful synthesis of contemporary achievements in the subject.

The present survey follows the general lines set forth in the preceding synopsis. Important topics which have been omitted or discussed very incompletely include the theory of multiple integrals, problems of constrained optimization, direct methods, the connections between differential geometry and the calculus of variations, and the various developments (existence theory, calculus of variations in the large, control theory) that have marked research in the present century. In the final two sections we discuss variational principles in mechanics and existence questions, but once again these are vast subjects whose history we can only touch upon.

12.2. Prehistory

Sometime around 150 BC the Greek mathematician Zenodorous wrote a treatise on *Isoperimetric Figures.* His results were presented half a millennium later by Pappus in the fifth book of the *Collection*, a work composed about 350 AD. Pappus's treatise was translated into Latin by Commandino in 1588 and was widely read and studied in the seventeenth century.

Pappus derived several results concerning the area of a circle and the area of polygonal figures of the same perimeter. He showed that the circle has a greater area than any regular polygon with the same perimeter. He also showed that given two regular polygons with the same perimeter, the one with the larger number of sides has the larger area. Finally, he showed that given two polygons of the same number of sides and the same perimeter, one of which is regular and the other is not, the regular polygon has the greater area. Pappus then proved some results concerning the volume of a sphere and the volumes of solids having the same surface area as the sphere. Galileo's *Two New Sciences* of 1637 contained a Latin treatise in which the study of constrained fall was placed on a new and sophisticated physical and mathematical basis. One of the problems he considered was to compare motion along the arc of a circle with the corresponding motion along a series of chords contained in the circle. He established that the time of fall along the path consisting of several chords decreases as the number of chords increases in number. Regarding the circle as the limit of polygonal chord paths, he concluded that the time of fall along the circle is less than the corresponding time along the chord joining the initial and final points.

The results of Pappus and Galileo involved a comparison of the circle and a polygonal figure. Neither researcher conceived of his respective problem in terms of a more general class of comparison curves. This limitation may be attributed to the absence of suitable mathematical methods for introducing and analyzing the behaviour of an arbitrary plane curve. With the invention of analytic geometry and calculus, it was possible to carry out a more extended investigation. The first genuine problem of the calculus of variations seems to have been formulated by Isaac Newton in his *Principia Mathematica* of 1687. In a scholium to Proposition 24 of Book II of this work Newton considered the problem of determining the volume of revolution that experiences the least resistance as it moves parallel to its axis through a resisting medium. He was able to obtain a condition on the minimizing curve that was given in terms of the tangent to the curve at each point.

Whiteside (in (Newton 1974, 466)) writes: "The immediate reaction of Newton's contemporaries to this scholium on its publication in the 1687 *Principia* was one of near-total incomprehension." An examination of Newton's private papers first published in this century show that he used techniques to solve this problem that are similar to the ones later developed by Jakob Bernoulli. Unfortunately, the published *Principia* contains only a statement of the solution. His methods seem to have remained unknown to his contemporaries and his work had little influence on the development of the subject.[1]

12.3. The Bernoullis, Taylor and Euler

The early Leibnizian calculus consisted of a sort of geometrical analysis in which differential algebra was employed in the study of "fine" geometry. The curve was analyzed in the infinitesimal neighbourhood of a point and related by means of an equation to its overall shape and behaviour.

The cycloid, an important curve which turned out to be the solution of several variational problems, is the path traced by a point on the perimeter of a circle as it rolls without slipping on a straight line. The cycloid has a simple description in terms of the infinitesimal calculus. Let the generating circle of radius r roll along the x-axis and let the vertical distance be measured downward from the origin along the y-axis (Fig. 12.1). An elementary geometrical argument revealed that the equation of the cycloid is

$$\left(\frac{ds}{dy}\right)^2 = \frac{2r}{y} \tag{12.1}$$

where $ds = \sqrt{dx^2 + dy^2}$ is the differential element of path length.

The cycloid was the solution to the brachistochrone problem. Consider a curve joining two points in a vertical plane and consider a particle constrained to descend along this curve. We must find the curve for which the time of descent is a minimum. Let us take the origin as the first point and let the coordinates of the second be $x = a$, $y = b$. We assume the particle starts from rest. By Galileo's law, the speed

[1] Goldstine (1980, 8) suggests that Newton's methods, bearing some similarities to Jakob Bernoulli's, may have been transmitted to the larger European community by James Gregory in his lectures at Oxford in the fall of 1694. There is however an obvious difficulty with this suggestion. Newton devised two solutions to the resistence problem. The solution that has similarities with Jakob Bernoulli's method was composed in 1685 and remained part of his private papers. The version conveyed to Gregory in the summer of 1694 was based on different ideas, as is evident in Goldstine's (Goldstine 1980, 19-21) detailed account.

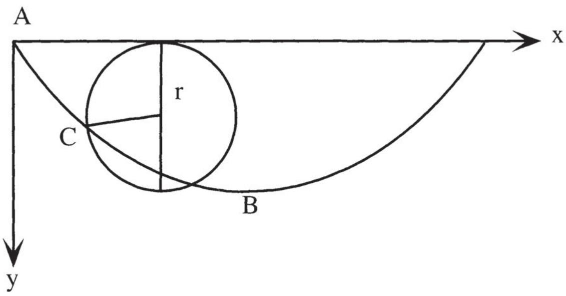

FIGURE 12.1

of a particle in constrained fall when it has fallen a distance y is $\sqrt{2gy}$, where g is an accelerative constant. We have the relations

$$\frac{ds}{dt} = \sqrt{2gy} \quad \text{or} \quad dt = \frac{1}{\sqrt{2gy}} ds = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}} \,. \tag{12.2}$$

Hence the total time of descent is given by the integral

$$T = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx. \tag{12.3}$$

The problem of the brachistochrone is to find the particular curve $y = y(x)$ that minimizes this integral.

Following Johann's Bernoulli's public challenge in 1696, solutions to this problem were devised by his elder brother Jakob, by Johann himself and by Newton and Leibniz. All of these men showed that the condition that the time of descent is a minimum leads to equation (12.1) and all—with the exception of Leibniz—concluded that the given curve is a cycloid. Johann's solution was based on an optical-mechanical analogy which is well known today from its description by Ernst Mach in his *Die Mechanik in Ihrer Entwicklung Historisch-Kritisch Dargestellt.* Although of considerable interest, his solution did not provide a suitable basis for the further development of the subject.

On the other hand Jakob Bernoulli's solution contained ideas that would develop into the calculus of variations. He considered three arbitrary infinitesimally close points C, G and D on the hypothetical minimizing curve and constructed a second neighbouring curve identical to the first except that the arc CGD was replaced by CLD (Fig. 12.2). Because the curve minimizes the time of descent, it is clear that the time to traverse CGD is equal to the time to traverse CLD. Using the dynamical relation $ds/dt \propto \sqrt{y}$ and this condition, Bernoulli was able to derive equation (12.1).

Jakob Bernoulli also investigated problems in which the minimizing or maximizing curve satisfied an auxiliary integral condition. The classical isoperimetric problem was the prototype for this class of examples. His idea was to vary the curve at two successive ordinates, thereby obtaining an additional degree of freedom, and to use the side constraint to derive a differential equation. Although Jakob died in 1705, some of his ideas were taken up by Brook Taylor in his *Methodus incrementorum* of 1715. Taylor skillfully developed and refined Jakob's conception, introducing some important analytical innovations of his own. Stimulated by Taylor's

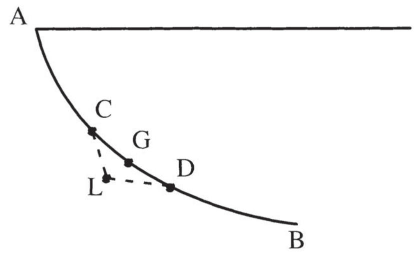

FIGURE 12.2

research and concerned to establish his brother's priority, Johann, then thirty-eight, also adopted Jakob's methods and developed them along more geometric lines in a paper published in 1719.

In two memoirs published in the St. Petersburg Academy of Sciences in 1738 and 1741, Euler extracted from the various solutions of Jakob and Johann Bernoulli, as well as the research of Taylor, a general approach to integral variational problems. These investigations were further developed and were made the subject of his classic treatise *Methodus inveniendi curvas lineas* (Euler 1744). Published when he was thirty-seven, this treatise was a remarkable work of synthesis in which he virtually created the "calculus of variations" (the name itself would come later) as a branch of analysis. He realized that the different integrals in the earlier problems were all instances of the single form

$$\int_a^b Z(x, y, y', \ldots, y^{(n)})\, dx \tag{12.4}$$

where Z is a function of x, y and the first n derivatives of y with respect to x. He derived a differential equation, known today as the Euler or Euler-Lagrange equation, as a fundamental condition that must be satisfied by a solution of the variational problem.

In Chapter 2 Euler developed his derivation of this equation (for the case $n = 1$) with reference to Figure 12.3, in which the line mno is the hypothetical extremizing curve. The letters M, N, O designate three points of the x-axis AZ infinitely close together. The letters m, n, o designate corresponding points on the curve given by the ordinates Mm, Nn, Oo. Let $AM = x$, $AN = x'$, $AO = x''$ and $Mm = y$, $Nn = y'$, $Oo = y''$. The differential coefficient p is defined by the relation $dy = p\,dx$; hence $p = dy/dx$. We have the following relations

$$p = \frac{y' - y}{dx}\,, \qquad p' = \frac{y'' - y'}{dx}\,. \tag{12.5}$$

The integral $\int_a^b Z\,dx$ was regarded by Euler as an infinite sum of the form $\cdots + Z_{\prime}dx + Z dx + Z'dx + \cdots$, where $Z_{\prime}$ is the value of Z at $x - dx$, Z its value at x and Z' its value at $x + dx$, and where the summation begins at $x = a$ and ends at $x = b$. (It is significant to note that Euler did not employ limiting processes or finite approximations.) Let us increase the ordinate y' by the infinitesimal "particle" nv, obtaining in this way a comparison curve $amvoz$. Consider the value of $\int_a^b Z\,dx$ along this curve. Since the curve is extremizing, the difference between this value

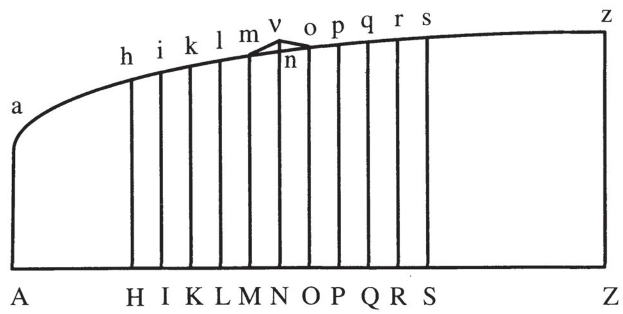

FIGURE 12.3

and the value of $\int_a^b Z\,dx$ along the actual curve will be zero. The only part of the integral that is affected by varying y' is $Z\,dx + Z'\,dx = (Z + Z')\,dx$. Euler wrote:

$$dZ = M\,dx + N\,dyP\,dp, \qquad dZ' = M'\,dx + N'\,dy' + P'\,dp'. \tag{12.6}$$

He proceeded to interpret the differentials in (12.6) as the infinitesimal changes in Z, Z', x, y, y', p, p' that result when y' is increased by nv. From (12.5) we see that dp and dp' equal nv/dx and $-nv/dx$. (These changes were presented by Euler in the form of a table, with the variables in the left column and their corresponding increments in the right column.) Hence (12.6) becomes

$$dZ = P \cdot \frac{nv}{dx}\,, \quad dZ' = N' \cdot nv - P' \cdot \frac{nv}{dx}\,. \tag{12.7}$$

Thus the total change in $\int_a^b Z\,dx$ equals

$$(dZ + dZ')\,dx = nv \cdot (P + N'\,dx - P').$$

This expression must be equated to zero. Euler set $P' - P = dP$ and replaced N' by N. He therefore obtained $0 = N\,dx - dP$ or

$$N - \frac{dP}{dx} = 0 \tag{12.8}$$

as the final equation of the problem.

Equation (12.8) is the simplest instance of the Euler differential equation, giving a condition that must be satisfied by the minimizing or maximizing arc. We would write

$$\frac{\partial Z}{\partial y} - \frac{d(\partial Z/\partial y')}{dx} = 0,$$

in modern notation. Euler also derived the corresponding equation when higher-order derivatives of y with respect to x appear in the variational integral. This derivation was a major theoretical achievement, representing the synthesis in one equational form of the many special cases and examples that had appeared in the work of earlier researchers.

Chapter 4 of Euler's treatise was of particular interest from the viewpoint of the conceptual foundations of analysis. The derivation of equation (12.8) was carried out in reference to Figure 12.3, in which the variables x and y are the orthogonal coordinates of a point on the minimizing curve. In Chapter 4, however, Euler calculated (12.8) in several problems in which the variables have quite diverse geometrical interpretations. For example, he formulated the problem of the shortest distance between two points using polar coordinates, derived (12.8) in terms of these variables and proved that the resulting curve is a straight line. His procedure showed that the reasoning involved in the derivation of (12.8) is very general and does not depend on the particular representation of x and y assumed in Figure 12.3. As Euler himself observed, the variables of the problem are abstract quantities and this figure is only a convenient geometrical visualization of an underlying analytical process.

12.4. Lagrange

Lagrange's first important contribution to mathematics, obtained when he was nineteen, consisted of his invention of the δ-algorithm to solve the problems of Euler's *Methodus Inveniendi.* He announced his new method in a letter of 1755 to Euler and published it in 1762 in the proceedings of the Turin Society. His δ-algorithm permitted the systematic derivation of the variational equations and facilitated the treatment of conditions at the endpoints. His innovation was immediately adopted by Euler, who introduced the name "calculus of variations" to describe the subject founded on the new method.

In his 1744 book Euler had noted the somewhat complicated character of his variational process and called for the development of a simpler method or algorithm to obtain the variational equations. Lagrange's new approach originated in his (tacit) recognition that the symbol d was being used in two distinct ways in Euler's derivation of (12.8). In (12.8) and the final step by which (12.8) is obtained, d was used to denote the differential as it was customarily used and understood in Continental analysis of the period (see (Bos 1974)). The differential dx was held constant; the differential of any other variable equalled the difference of its value at x and its value at an abscissa a distance dx from x. By contrast, the differentials dx, dy, etc., that appear in (12.6) were interpreted by Euler as the changes in x, y etc., that result when the single ordinate y is increased by the "particle" nv. Thus the "differentials" dy', dp, dp' equal nv, nv/dx, $-nv/dx$; the "differentials" dx, dy, dp'', etc., are zero.

The young Lagrange had the perspicacity to recognize this dual usage and invented the symbol δ to denote the second type of differential change. Using it, he devised a new analytical process to investigate problems of maxima and minima. Although the purpose of his method was to compare curves in the plane, it was nonetheless introduced in a very formal manner. The symbol δ has properties analogous to the usual d of differential calculus. Thus $\delta(x+y) = \delta x + \delta y$ and $\delta(xy) = x\delta y + y\delta x$. In addition, the d and δ are interchangeable, $d\delta = \delta d$, as are d and the integral operation $\int$.

The δ-process led to a new and very simple derivation of the Euler equation (12.8). It is necessary to determine $y = y(x)$ so that

$$\delta \int_a^b Z dx = 0 , \tag{12.9}$$

where $Z = Z(x, y, p)$ and $p = dy/dx$. Applying the δ operation to the expression Z, we obtain

$$\delta Z = N\delta y + P\delta y . \tag{12.10}$$

Note that here all of the ordinates are being varied simultaneously, and not just one, as had been the case in Euler's analysis. Because the δ and $\int$ are interchangeable, we have

$$\delta \int_a^b Z\, dx = \int_a^b \delta Z\, dx = \int_a^b (N\delta y + P\delta p)\, dx . \tag{12.11}$$

Because the d and δ are interchangeable, we also have $\delta p = \delta(dy/dx) = d(\delta y)/dx$. An integration by parts gives rise to the identity

$$\int_a^b P\delta p\, dx = \int_a^b P\frac{d(\delta y)}{dx}\, dx = P\delta y\Big|_a^b - \int_a^b \frac{dP}{dx}\delta y\, dx . \tag{12.12}$$

Hence the condition $\delta \int_a^b Z\, dx = 0$ becomes

$$P\delta y\Big|_a^b - \int_a^b \left(N - \frac{dP}{dx}\right) \delta y\, dx = 0 . \tag{12.13}$$

We suppose that δy is zero at the end values $x = a$, $x = b$. Equation (12.13) then reduces to

$$\int_a^b \left(N - \frac{dP}{dx}\right) \delta y\, dx = 0 . \tag{12.14}$$

From (12) we infer the Euler equation

$$N - \frac{dP}{dx} = 0 . \tag{12.15}$$

In his investigation Lagrange relied heavily on the algorithmic, algebraic properties of his new process. In his history Goldstine (Goldstine 1980, 112) writes: "To what extent Lagrange viewed his variation operator as a means for effecting a comparison of curves or as a purely formalistic construct I can form no precise reasoned opinion; but I incline toward that way since he seems to treat his variation in that way." We can discern in this early work of Lagrange the striking analytical philosophy, involving a rejection of diagrammatic aids and methods of proof, that would become a prominent feature of his mature mathematics. As his career progressed, his variational research would be closely associated with a conception of calculus as algebraic analysis (cf. Chapter 4) which was most systematically expressed in his famous textbooks of 1797 and 1806.

Euler took up Lagrange's method in his writings of the 1760s and 1770s. In a paper published in 1772 he presented what would become the standard interpretation of the δ-process as a means for comparing classes of curves or functions. We assume that y is a function of x and a parameter t, $y = y(x, t)$, where the given curve

$y = y(x)$ is given by the value of $y(x,t)$ at $t = 0$. We define δy to be $(\partial y/\partial t)\Big|_{t=0} dt$. One way of doing this, Euler explained, is to set $y(x,t) = X(x) + t \cdot V(x)$, where $y(x) = X(x)$ is the given curve and $V(x)$ is a comparison or increment function; hence we have $\delta y = dt \cdot V(x)$. In this conception the variation of a more complicated expression made up of $y(x,t)$ and its derivatives with respect to x is obtained by taking the partial derivative with respect to t, setting $t = 0$ and introducing the multiplicative factor dt. In later variational mathematics the parameter ε would often be used instead of t.

12.5. Legendre

Legendre initiated the study of the second variation in a memoir published in 1788. He considered the variational integral

$$I = \int_a^b f(x, y, y')\, dx \; . \tag{12.16}$$

Let us assume that a given function $y = y(x)$ makes the integral I a maximum or a minimum. Set $\delta y = w(x)$, where $w(a) = w(b) = 0$. The first and second variations I_1 and I_2 are by definition

$$(12.17)\qquad \begin{aligned} &\text{(a)} \qquad I_1 = \int_a^b \left(\frac{\partial f}{\partial y} w + \frac{\partial f}{\partial y'} w' \right) dx, \\ &\text{(b)} \qquad I_2 = \int_a^b \left(\frac{\partial^2 f}{\partial y^2} w^2 + 2 \frac{\partial^2 f}{\partial y \partial y'} w w' + \frac{\partial^2 f}{\partial y'^2} \right) dx \; . \end{aligned}$$

We will use the following standard abbreviations for the second partial derivatives:

$$P = \frac{\partial^2 f}{\partial y^2} \,, \quad Q = \frac{\partial^2 f}{\partial y \partial y'} \,, \quad R = \frac{\partial^2 f}{\partial y'^2} \; . \tag{12.18}$$

The difference in the value of I along the actual and comparison arcs is

$$\Delta I = I_1 + \frac{1}{2} I_2 + \text{higher-order terms} \; . \tag{12.19}$$

It is clear that I_1 will dominate in this expansion. Hence if a minimum is to occur, then I_1 must be zero for all admissible $w(x)$. The validity of the Euler equation for the problem now follows by means of Lagrange's procedure. Legendre recognized that it is also necessary to examine I_2 and to show that it is positive for all admissible $w(x)$. Let $v = v(x)$ be a function of x and consider the expression

$$\frac{d}{dx}(w^2 v) \; . \tag{12.20}$$

Because $w(a) = w(b) = 0$, the integral of (12.20) is zero:

$$\int_a^b \frac{d}{dx}(w^2 v)\, dx = 0 \; . \tag{12.21}$$

Thus if we add (12.21) to (12.17b), there will be no change in the value of the second variation:

$$I_2 = \int_a^b \left((P + v')w^2 + 2(Q + v) w w' + R w'^2 \right) dx \; . \tag{12.22}$$

The integrand is a quadratic expression in w and w'. Legendre observed that it will be a perfect square if

$$R(P+v') = (Q+v)^2 \ . \tag{12.23}$$

For $v(x)$ satisfying this differential equation, the second variation becomes

$$I_2 = \int_a^b R\left(w' + \frac{Q+v}{R}w\right)^2 dx \ . \tag{12.24}$$

It is evident that the given transformation is only possible if $R = \partial^2 f/\partial y'^2$ is nonzero on the interval $[a,b]$. Legendre inferred that the proposed solution will indeed by a minimum if we have

$$\frac{\partial^2 f}{\partial y'^2} > 0 \tag{12.25}$$

on the interval. Condition (12.25) would become known later as Legendre's condition.

Legendre extended his analysis to the case where the second derivative of y with respect to x appears in the variational integrand. Here the associated transformation involves the introduction of three auxiliary functions v, v_1, v_2 connected by three differential equations analogous to (12.23). Legendre's results raised several questions, some of which were discussed by Lagrange in his treatment of the subject in his *Théorie des fonctions analytiques* of 1797. In order to carry out the above transformation, it is necessary to integrate (12.23) and to obtain a solution $v = v(x)$. Legendre provided no general method for integrating this nonlinear differential equation. Lagrange showed using particular examples that the associated integral may not exist on the given interval. He also produced examples to show that if the size of the interval is not restricted in some way, then it is possible, for a given solution which satisfies Legendre's condition, to find comparison functions yielding a larger or smaller value of the variational integral. Having raised these questions, however, he did not proceed to develop any sort of theory to explain the conditions under which the second variation may be transformed.

12.6. Jacobi

12.6.1. Jacobi and his "school". The next figure in our history is the German mathematician Carl Gustav Jacobi, who published a seminal and very original paper in Crelle's Journal in 1837. The calculus of variations was only one of many parts of mathematics to which Jacobi made fundamental contributions. His research in elliptic functions, analysis, functional determinants, number theory and analytical dynamics established him as one of the leading mathematicians of Europe. Although born into a Jewish family, he converted to Christianity in order to pursue a career in mathematics. His most productive professional years were spent at the University of Königsberg from 1826 to 1843. In 1843 Jacobi obtained a position in Berlin, where he lectured and carried out research until his death in 1851.

In addition to his varied work in analysis and mechanics, Jacobi was an active teacher who exercised a strong influence on younger mathematicians of his day. Scriba (Scriba 1973, 51) writes: "Such were Jacobi's forceful personality and sweeping enthusiasm that none of his gifted students could escape his spell; they were drawn into his sphere of thought, working along the manifold lines he suggested, and soon represented a 'school.' C. W. Borchardt, E. Heine, L. O. Hesse, F. J. Richelot, J. Rosenhain, and P. L. von Seidel belonged to this circle; they contributed much to the dissemination not only of Jacobi's mathematical creations but also of the new research-oriented attitude in university instruction."[2] Adopting the model of the university philological seminar, Jacobi and the physicist Franz Neumann lectured at Königsberger directly on the subject of their research, a new practice that would subsequently be followed throughout the German university system.[3]

12.6.2. Jacobi's 1837 paper. In his 1837 paper Jacobi succeeded in establishing a systematic theory for the study of conditions required to ensure a maximum or minimum in the calculus of variations. His paper, in which proofs and justifications were omitted, became the basis for a vigorous programme of mathematical research. We begin our discussion with an examination of his initial fundamental insight. Following Lagrange's procedure, we integrate the equation $I_1 = 0$ by parts, obtaining

$$I_1 = \int_{x_0}^{x_1} Vw\,dx = 0 \ , \tag{12.26}$$

where

$$V = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) . \tag{12.27}$$

Because $w(x)$ is arbitrary, it is clear that the solution $y = y(x)$ to the variational problem will need to satisfy the Euler differential equation

$$V = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \ . \tag{12.28}$$

Jacobi's new idea was to express the relationship between the first and second variation in terms of the variational operation δ in a particular way. We have the following relationships between the variational integral I and its first and second variations I_1 and I_2:

$$I_1 = \delta I \ , \quad I_2 = \delta I_1 \ . \tag{12.29}$$

Hence if we express I_1 as

$$I_1 = \int_{x_0}^{x_1} Vw\,dx \ , \tag{12.30}$$

the second variation I_2 takes the form

$$I_2 = \delta I_1 = \delta\left(\int_{x_0}^{x_1} vw\,dx\right) = \int_{x_0}^{x_1} \delta V w\,dx \tag{12.31}$$

or simply

$$I_2 = \int_{x_0}^{x_1} \delta V w\,dx \ . \tag{12.32}$$

We have seen that the solution $y = y(x)$ to the variational problem must satisfy the Euler differential equation (12.28). The general solution to this second-order

[2] For details on "Die Jacobische Schule", see Klein (1926, 112-115).

[3] See (Todhunter; Turner 1971).

equation will contain two arbitrary constants α and β. Because the first-order term I_1 in (12.19) is zero, it is clear that the term involving I_2 will dominate in this expansion. It is not difficult to see that the third-order term can in general be made either positive or negative. Thus we will only have a genuine extremum if there is no $w(x)$ for which I_2 is equal to zero. We are therefore led naturally to consider conditions under which $I_2 = 0$. From (12.32) it is apparent that $I_2 = 0$ if

$$\delta V = 0. \tag{12.33}$$

Let $y = y(x, \alpha)$ be a solution of the Euler equation (12.28), where the notation indicates the dependency of the solution on the arbitrary constant α. We have $V(\alpha) = 0$. Let us regard α as a parameter and increase α by the increment $\delta\alpha$. We again have $V(\alpha + \delta\alpha) = 0$. We now consider a variational process in which $\delta y = (\partial y/\partial\alpha)\delta\alpha$. By subtracting $V(\alpha) = 0$ from $V(\alpha + \delta\alpha) = 0$, we see that $(\partial y/\partial\alpha)\delta\alpha$ is a solution of $\delta V = 0$. Similarly, if β is a second arbitrary constant appearing in y, then $\delta y = (\partial y/\partial\beta)\delta\beta$ will be a solution of $\delta V = 0$. Since $\delta V = 0$ is a second-order linear differential equation in δy, the general solution will be of the form $\delta y = \varepsilon \cdot u(x)$ where $u(x)$ is given by

$$u(x) = c_1 \frac{\partial y}{\partial \alpha} + c_2 \frac{\partial y}{\partial \beta} \tag{12.34}$$

where c_1 and c_2 are constants.

Jacobi's achievement was to devise a theory that connected solutions to the Euler equation to the analysis of the second variation. Thus he was able to use the function $u(x)$ given by (12.34) to obtain a solution to Legendre's differential equation (12.23). This was a remarkable result in itself. He also introduced a new transformation of the second variation, in which (12.34) also played a central role, that reduced I_2 to the form (12.24). Delaunay (Delaunay 1841) and Spitzer (Spitzer 1854) provided detailed derivations of this solution, including details that Jacobi omitted. Jacobi's theory was primarily directed at the more general case where the variational integrand contains higher-order derivatives of y with respect to x.

The considerable body of research stimulated by Jacobi's paper was devoted largely to an investigation of the transformation of the second variation. The most notable contributors here were Delaunay (Delaunay 1841), Spitzer (Spitzer 1854) and Hesse (Hesse 1857). However, in a brief passage in his paper Jacobi had also drawn attention to another aspect of the problem, what in later variational mathematics would be called the theory of the conjugate point. The essential question is the following. Possible minimizing arcs will be solutions to the Euler equation $V = 0$. Let its general solution $y = y(x, \alpha)$ contain the arbitrary constant α. We require that $y = y(x, \alpha)$ pass through the endpoints. Suppose that a neighbouring solution $y = y(x, \alpha + \delta\alpha)$ also passes through the endpoints. Because both $y = y(x, \alpha)$ and $y = y(x, \alpha + \delta\alpha)$ satisfy the end conditions, it follows that $\delta y = (\partial y/\partial\alpha)\delta\alpha$ is an admissible variation, i.e., one for which $\delta y(a) = \delta y(b) = 0$. By Jacobi's initial insight, for this choice of δy we have $\delta V = 0$. Hence the corresponding second variation I_2 is zero. It is clear in this situation that there can be no minimum, because the sign of the third variation can (in general) be made either positive or negative.

If we start at the initial point, we will eventually arrive at a second point through which it is possible to find two solutions of the Euler equation satisfying the associated end conditions. This second limiting point, the value of which cannot be reached or exceeded if we are to have a minimum, later became known as a conjugate point. Considered analytically, it is necessary to show that it is not possible to find functions of the form (12.34) which vanish at two points of the given interval $[a, b]$. By investigating (12.34), we are able to determine the limits within which it is possible to conclude that there is a solution to the variational problem.

Jacobi illustrated this restriction using the example of the elliptical motion of a particle moving about a centre of force, in which the trajectory is deduced from the principle of least action. In lectures delivered in the early 1840s and published posthumously in his *Vorlesungen* (Jacobi 1866, 46), he introduced an even simpler example, the case of a single particle constrained to move on the surface of a sphere but otherwise subject to no force. The principle of least action leads here to the conclusion that the trajectory must be a geodesic or path of shortest distance. Hence the particle moves on a great circle, i.e., the intersection of the surface of a sphere and a plane through its centre. If we begin at a given point A and traverse an angular distance of $180°$, we reach a point C conjugate to A. If the second point B is equal to or beyond C, then it is not difficult to see that there are comparison paths of equal or shorter distance.

12.7. Mayer

It is important to note that in the twenty-five years following the appearance of Jacobi's paper it was the transformation problem that received the primary attention of researchers. There was comparatively little concern during this period for the theory of the conjugate point. This was true of Hesse's 1857 paper, although Hesse did include a section in which he showed in the case of variational integrals of the form $\int_a^b f(x, y, y')\,dx$ that the nonexistence of a conjugate point on the given interval implies the validity of the transformation of the second variation. That is, if there is no point conjugate to a on the interval $[a, b]$, then the conditions that must be satisfied in order to transform the second variation are valid. Of course, the Jacobi theory was primarily directed at the more general case where the variational integral is of the form $\int_a^b f(x, y, y', \ldots, y^{(n)})\,dx$ for $n \geq 2$. Hesse did not try to extend his result to this case. It was not at all obvious in the general case how one would connect the transformation problem to the theory of the conjugate point.

A clear recognition of this theoretical question and its satisfactory resolution were the achievement of the Leipzig mathematician Adolph Mayer. He presented his results in his *Habilitationsschrift* of 1866 and an article two years later in Crelle's Journal. A full account of his result is beyond the scope of the present study. We can however indicate something of the general analytical setting within which he carried out his investigation. Following Clebsch (Clebsch 1858), Mayer formulated the fundamental variational problem as a Lagrange problem. Let us assume that there are n dependent variables $y_1, \ldots, y_n$. The variational integral is of the form

$$I = \int_a^b f(x, y, y', \ldots, y^{(n)})\,dx. \tag{12.35}$$

The variables y_i are assumed to satisfy m subsidiary differential equations of the form

$$\Phi_m(x, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0. \tag{12.36}$$

We wish to maximize or minimize I subject to (12.36). In 1806 Lagrange had shown how to obtain the variational equations by means of a multiplier rule. We multiply each of the equations $\Phi_m = 0$ by the multiplier function $\lambda_m(x)$ and form the expression

$$F = f + \sum_{i=1}^{n} \lambda_i \Phi_i \, . \tag{12.37}$$

Then the solution to the problem will satisfy (12.36) and the Euler equations corresponding to the variational integrand F:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0 \, . \tag{12.38}$$

The multiplier rule formulated in this way yields the traditional problem of maximizing or minimizing the integral $\int_a^b f(x, y, y', \ldots, y^{(n)})\, dx$ as a special case. This fact was noted explictly by Clebsch in his 1858 paper. Consider the case $n = 2$. Let $f = f(x, y_1, y_2, y_2')$ and assume $\Phi = y_1' - y_2$. Then $F = f + \lambda(y_1' - y_2)$. The Euler equations corresponding to F are

$$\begin{aligned} &\frac{\partial f}{\partial y_1} - \frac{d\lambda}{dx} = 0 \, , \\ &\frac{\partial f}{\partial y_2} - \lambda - \frac{d}{dx}\frac{\partial f}{\partial y_2'} = 0 \, . \end{aligned} \tag{12.39}$$

Eliminating the multiplier λ from these equations and letting $y_1 = y$, we obtain the standard Euler equation corresponding to the integral $\int_a^b f(x, y, y', y'')\, dx$:

$$\frac{\partial f}{dy} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0 \, . \tag{12.40}$$

In his paper Clebsch showed how the Jacobi transformation theory could be extended to the general setting of the Lagrange problem. Mayer's investigation of the transformation conditions and the theory of the conjugate point was carried out within the framework of Clebsch's analysis of the second variation. In the course of developing his results, he used specialized methods of Hamilton and Jacobi for integrating the variational equations. Mayer's 1858 paper was at the very highest level of contemporary mathematical work both from a technical and a theoretical point of view.

12.8. Erdmann

The classical methods of the calculus of variations gave solutions satisfying certain smoothness requirements, in particular the requirement that the slope of the optimizing arc must vary continuously along the length of the curve. In a book published in 1871 the English mathematician Isaac Todhunter drew attention to "discontinuous" solutions, that is, solutions containing corners where the derivative changes suddenly in value. (The terminology of "continuous" and "discontinuous", which would remain with the subject, had its origins in the older eighteenth-century conception of continuity in terms of analytical form. A function was said to be "continuous" if it was given by a single analytical expression. This condition implied that the derivative (except at points of singularity) would vary smoothly.)

Although Todhunter initiated the serious study of discontinuous solutions, he failed to derive the analytical conditions satisfied by such arcs. In a paper published in 1876 the mathematician G. Erdmann succeeded in such an analysis. He did so using something called the variable-endpoint formula. In our discussion up to now we have assumed that the variation of the endpoints is zero. We now broaden the variational problem to include extremal arcs in which the second endpoint is allowed to vary in both the x and y directions. The formula for the variation of I is now

(12.41)

$$\delta I = \delta \int_a^b f\,dx = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \delta y\,dx + \frac{\partial f}{\partial y'}\delta y\Big|_{x=b} + \left(f - \frac{\partial f}{\partial y'}y'\right)\delta x\Big|_{x=b} .$$

Versions of this formula had appeared in textbooks of Lagrange (Lagrange 1806, lesson 22) and Lacroix (Lacroix 1806, 492-493)), and it was a standard result in the nineteenth century. If we require that the arc joining the endpoints be a solution to the Euler equation, then (12.41) reduces to

$$\delta I = \frac{\partial f}{\partial y'}\delta y\Big|_{x=b} + \left(f - y'\frac{\partial f}{\partial y'}\right)\delta x\Big|_{x=b} . \tag{12.42}$$

Consider the variational problem $\delta \int_a^b f(x, y, y')\,dx = 0$ and suppose that the solution $y = y(x)$ has a corner at $x = c$. It is clear that the integral from a to c and the integral from c to b must separately be optimal. Hence the Euler equation holds separately on each of these intervals. Let us consider a comparison arc produced by varying the point $\big(c, y(c)\big)$ in both the x and y directions. From the variable endpoint formula we obtain the condition

(12.43)

$$\frac{\partial f}{\partial y'}\delta y\Big|_{x=c^-} + \left(f - y'\frac{\partial f}{\partial y'}\right)\delta x\Big|_{x=c^-} - \frac{\partial f}{\partial y'}\delta y\Big|_{x=c^+} - \left(f - y\frac{\partial f}{\partial y'}\right)\delta x\Big|_{x=c^+} = 0 .$$

In this formula the signs + and − are used to indicate that the derivative y' in the relevant expressions is taken to the right or left, respectively, of c. Because $\delta x(c)$ and $\delta y(c)$ are arbitrary, we obtain the equations

$$\begin{aligned} \frac{\partial f}{\partial y'}\Big|_{x=c^-} &= \frac{\partial f}{\partial y'}\Big|_{x=c^+} , \\ \left(f - y'\frac{\partial f}{\partial y'}\right)\Big|_{x=c^-} &= \left(f - y'\frac{\partial f}{\partial y'}\right)\Big|_{x=c^+} . \end{aligned} \tag{12.44}$$

Equations (12.44) provide the conditions that must be satisfied by an optimizing arc at a corner c. They are known today as the Erdmann corner conditions.

There are two aspects of Erdmann's investigation with implications for the subsequent development of variational analysis. First, with the concept of a solution to include arcs with corners broadened, it was natural to extend this idea to the comparison arcs themselves and thereby to greatly enlarge the family of comparison curves.[4] The solutions that were given by traditional variational methods were "weak" extrema. Weierstrass would launch the study of "strong" extrema, ones derived from a variational process involving comparison curves whose slope may differ by any finite amount from the actual curve. (Because the class of comparison curves is enlarged, the requirement that the given curve is an extremal is a stronger condition.) Examples that were commonly used in the post-Weierstrass period to illustrate the distinction between strong and weak extrema involved comparison arcs with corners; some of these examples even originated with Erdmann's 1877 paper.[5]

A second important aspect of Erdmann's paper was his use of the variable-endpoint formula. As we shall see below, this formula would be involved directly in Weierstrass's derivation of necessary and sufficient conditions involving the famous excess function.

12.9. Weierstrass

12.9.1. Weierstrass's lectures. Weierstrass's contributions to the calculus of variations were a product of his middle and mature years. Although he began lecturing on the subject at the University of Berlin as early as 1865, his most significant results were presented in the summer lectures of 1879, when he was sixty-three years old. The edition which was eventually published in 1927 is based on these together with a second set of lectures given in 1883. Although this delay in publication limited the dissemination of his ideas, he exerted considerable influence on contemporary German variational research through his public lectures. Copies of his lectures circulated privately and his results were disseminated in published form by other researchers beginning in the middle 1890s.

More than any other researcher, Weierstrass established the logical outlook of the calculus of variations as a modern mathematical subject. The distinction between necessary and sufficient conditions appears clearly for the first time in his lectures. He carefully specified the continuity properties that must be satisfied by functions and their variations. In problems of constrained optimization he used theorems on implicit functions to ensure that the optimizing arc was embedded in a suitable family of comparison curves. As we observed above, traditional methods of the calculus of variations were devoted to the determination of weak solutions or extrema. Before, say, the 1860s researchers did not identify at the outset of their investigation the precise class of comparison arcs in a given variational problem. There was no prior logical conception concerning the nature of this class. However the δ-process introduced by Lagrange required that both the comparison arc and its slope at each point differ by only a small amount from the actual curve. This condition, which was imposed implicitly by the nature of the variational process, was evident in expression (12.17b) for the second variation, where both $\delta y = w$ and $\delta y' = w'$ were small quantities. Todhunter (Todhunter 1871, 269) in his essay on discontinuous solutions seems to have been the first to call explicit attention to this limitation on the class of comparison arcs: "If we assert that the relation [i.e., the Euler equation] does give a minimum, we must bear in mind that this means a minimum with respect to admissible variations ... our investigation is not applicable to such a variation as would be required in passing from the cycloid to the

[4]This possibilty was explicitly recognized by Todhunter in 1871; see the discussion below.

[5]See, for example, Bolza (1904, 39, 73-74).

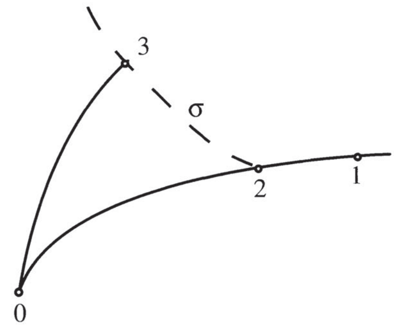

FIGURE 12.4

discontinuous figure: in such a passage $\delta p[=\delta y']$ would not always be indefinitely small. Of course it might be possible to give some special investigation for such a case, but certainly the case is not included in the ordinary methods of the Calculus of Variations."[6]

Weierstrass provided such an investigation, broadening the notion of a solution to include a much larger class of comparison arcs. At a general level his approach to the calculus of variations involved a very basic logical reorientation of the subject. In earlier variational research the nature of the mathematical objects was determined implicitly by the methods employed. By contrast Weierstrass began with objects defined constructively in terms of an explicit theory of a function of a real variable.

12.9.2. Weierstrass's excess function[7]. In Figure 12.4 the arc (0 1) is the solution to a given variational problem; i.e., it is an extremal (a solution to the Euler equation) for the problem passing though the points 0 and 1. (The term "extremal" was not used by Weierstrass; it originated with Kneser in 1900.) Consider a point 2 on this curve. Let 3 be a neighbouring point. Form the arc (3 2) joining these two points. Form the extremal curve (0 3) joining 0 and 3. The resulting arc (0 3 2 1) is a comparison arc to the given curve. Note that the slope of the segment (3 2) will in general differ by a finite amount from that of (0 1) at the point 2.

We use the notation I_{01} to denote the value of the given integral along the arc (0 1). Let the coordinates of the point 2 be (x, y). Let σ be a small positive quantity

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[6]Emphasis in the original. It should be noted that Todhunter (Todhunter 1861, 3) ten years earlier, in referring to the second variation, had observed in passing that both δy and δp are small. However, this was an isolated observation, not introduced in the course of any particular explanation of, or viewpoint on, the variational process.

[7]Throughout his work Weierstrass employed a parametric approach, in which the variables x and y are regarded as functions of a parameter t. Historically, the very large majority of all researchers had employed the ordinary theory, in which x is the independent variable and y is a function of x. Although the parametric approach has certain advantages from a geometric viewpoint, its analytical development is much less natural than the ordinary theory. During the period 1895–1905 when Weierstrass's ideas were becoming more widely known, researchers such as Bolza, Osgood and Goursat went to some effort to reformulate his results in terms of the ordinary theory. In our exposition of Weierstrass's methods we use the ordinary rather than the parametric theory, a decision that is based primarily on considerations of exposition. Because we are primarily interested in the essential variational ideas of Weierstrass's theory, we also omit the detailed analytical considerations involving functions of a real variable contained in the original lectures.

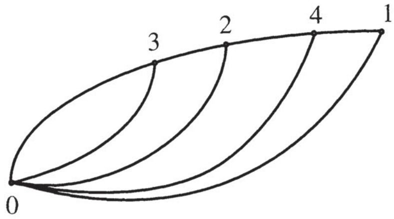

FIGURE 12.5

and let q be the slope of the arc (3 2). Note that q will in general differ by a finite amount from the value of y' at 2. Define $\delta x = -\sigma$ and $\delta y = -\sigma q$. The coordinates of the point 3 are then $(x + \delta x, y + \delta y)$. By the variable endpoint formula (12.42) we have

$$I_{03} - I_{02} = \frac{\partial f}{\partial y'}(-\sigma q) + \left(f - y'\frac{\partial f}{\partial y'}\right)(-\sigma) \ . \tag{12.45}$$

For small σ we also have

$$I_{32} = f(x, y, q)\sigma \ . \tag{12.46}$$

The variation of the integral is therefore

$$I_{03} + I_{32} - I_{02} = \left\{ f(x, y, q) - f(x, y, y') - \frac{\partial f}{\partial y'}(x, y, y')(q - y') \right\} \sigma \ . \tag{12.47}$$

Weierstrass's excess function is defined as

$$E(x, y, y', q) = f(x, y, q) - f(x, y, y') - \frac{\partial f}{\partial y'}(x, y, y')(q - y') \ . \tag{12.48}$$

By means of (12.48) Weierstrass was able to introduce a new necessary condition, one that is valid when the comparison family of arcs is enlarged to include curves whose slope differs by a finite amount from the actual curve. In order for the given integral to be a minimum, we must have

$$E(x, y, y', q) \geq 0 \tag{12.49}$$

for all points x, y and for all values of the slope q. Weierstrass called (12.49) the fourth necessary condition; it supplemented the conditions already derived by Euler, Legendre and Jacobi.

In his lectures Weierstrass showed that a modified version of condition (12.49) is sufficient to ensure a minimum. To do so, he employed a slightly more complicated construction, one however that was a natural development of the preceding analysis. As before, let (0 1) be an extremal joining the endpoints 0 and 1 (Fig. 12.5). This curve is given analytically as $y_0 = y_0(x)$. Consider the comparison arc $y = y(x)$, indicated by the curve $(0, 3, 2, 4, 1)$ in Figure 12.5. The points 2 and 3 are points on this curve with coordinates (x, y) and $(x + dx, y + dy)$, respectively. Let $(0, 2)$ and $(0, 3)$ be extremals joining 0 and 2 and 0 and 3. Define

$$S(x) = I_{02} + I_{21} \ , \tag{12.50}$$

where I_{02} is the variational integral I evaluated along the extremal $(0, 2)$ and I_{21} is the value of this integral evaluated along the segment $(2, 1)$ of the comparison arc

$(0, 2, 3, 4, 1)$. We have

$$S(b) = \int_a^b f(x, y_0, y_0')\, dx \;, \qquad (12.51)$$
$$S(a) = \int_a^b f(x, y, y')\, dx \;.$$

Then the variation of I is

$$\Delta I = \int_a^b \big(f(x, y, y') - f(x, y_0, y_0')\big)\, dx = -\big(S(b) - S(a)\big) \;. \qquad (12.52)$$

If we can show that $S(x)$ is a decreasing function of x on $[a, b]$, it will follow that $\Delta I \geq 0$. We therefore calculate dS/dx and investigate the condition $dS/dx \leq 0$. We have

$$dS = S(x + dx) - S(x) = I_{03} + I_{32} - I_{02} \;. \qquad (12.53)$$

Let $p(x, y)$ denote the slope of the extemal $(0, 2)$ at the point 2. We assume that $p(x, y)$ is a well-defined function of the coordinates x, y of 2. By the variable-endpoint formula (12.42)

$$\begin{aligned} I_{03} - I_{02} &= \frac{\partial f}{\partial y'}(x, y, p)\, dy + \left(f(x, y, p) - p \frac{\partial f}{\partial y'}(x, y, p) \right) dx \\ &= \left(f(x, y, p) + \frac{\partial f}{\partial y'}(x, y, p)(y' - p) \right) dx \;. \end{aligned} \qquad (12.54)$$

Also

$$I_{32} = f(x, y, y')\, dx \;. \qquad (12.55)$$

Hence

$$\begin{aligned} dS &= -\left(f(x, y, y') - f(x, y, p) - \frac{\partial f}{\partial y'}(x, y, p)(y' - p) \right) dx \\ &= -E(x, y, p, y')\, dx \;. \end{aligned} \qquad (12.56)$$

Thus if the condition

$$E(x, y, p, y') \geq 0 \qquad (12.57)$$

is satisfied for all comparison arcs $y = y(x)$, then dS/dx is always negative and the solution $y_0 = y_0(x)$ renders the given variational integral a minimum.

12.9.3. The field concept. To carry out the preceding sufficiency proof, Weierstrass assumed that the minimizing arc $y_0 = y_0(x)$ could be embedded in a strip or region ("Flächenstreifen") of the plane containing $y_0(x)$ and covered by a family of solutions to the Euler equation. This family satisfies the property that there is a unique member joining the initial point 0 and any subsequent point in the region. In his 1900 *Lehrbuch der Variationsrechnung* Kneser introduced the formal term "field of extremals" to designate such a family of curves. In adopting this usage, Kneser was evidently inspired by the well-known field concept in physics. This idea had originated in the writings of Faraday and was subsequently developed by Maxwell and his followers. By the end of the century the concept of a field had become a standard part of theoretical physics. Of course, a field of extremals in the calculus of variations is a purely mathematical construction; the field notion operates at the level of analogy with actual physical fields, serving as a conceptual tool in the formulation of sufficiency proofs. In certain particular cases, for example, the projectile motion of a particle acted upon by a central force and governed by a variational law, the possible physical trajectories of the particle coincide with the extremals of the mathematical field; in this case the mathematical concept possesses a direct physical realization (although even here the lines of force are not the same as the extremals of the field). In general, however, the variational field concept is more abstract, essentially something which subsists by analogy with the physicists' construct.

12.10. Refinement of Weierstrass's methods

12.10.1. Hilbert's invariant integral. Weierstrass's methods underwent two significant modifications at the hands of subsequent researchers. The most important of these was Hilbert's introduction in 1900 of the invariant integral to simplify the derivation of the sufficiency condition (12.57).[8]

Suppose that $y_0(x)$ is a solution to the Euler equation passing through the given endpoints. Let

$$I = \int_a^b f(x, y_0, y_0^1)\, dx. \tag{12.58}$$

We assume that $y_0(x)$ is embedded in a field of extremals, so that at each point in some region containing $y_0(x)$ there is a well-defined function $p(x, y)$ giving the slope at (x, y) of the unique extremal that passes through the initial point and (x, y). Let $y(x)$ be a comparison arc which coincides with $y_0(x)$ at $x = a$, $x = b$; $|y(x) - y_0(x)|$ is small but $|y'(x) - y_0'(x)|$ need not be.

Consider the integral

$$I^* = \int_a^b \left(f(x, y, p) + \frac{\partial f}{\partial y'}(x, y, p)(y' - p) \right) dx\ . \tag{12.59}$$

Hilbert realized that I^* is path-independent, i.e., its value is independent of the particular functional path $y = y(x)$, so long as $y = y(x)$ coincides with $y_0(x)$ at the endpoints. Along the curve $y_0 = y_0(x)$ we have $y_0'(x) = p\big(x, y_0(x)\big)$ and so it follows that $I^* = I$. Hence the variation ΔI is

$$\begin{aligned} \Delta I &= \int_a^b \big(f(x, y, y') - f(x, y_0, y_0')\big)\, dx \\ &= \int_a^b \Big(f(x, y, y') - f(x, y, p) - \frac{\partial f}{\partial y'}(x, y, p)(y' - p)\Big)\, dx\ , \end{aligned} \tag{12.60}$$

i.e.,

$$\Delta I = \int_a^b E(x, y, p, y')\, dx\ . \tag{12.61}$$

Thus if the condition

$$E\big(x, y, p(x, y), y'\big) \geq 0 \tag{12.62}$$

is satisfied for all comparison arcs $y = y(x)$, then it follows that the solution $y_0(x)$ renders the given variational integral a minimum.

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[8]For accounts of Hilbert's research in the calculus of variations, including his introduction of the invariant integral, see Goldstine (1980, 314-330) and (Thiele 1997).

The key to Hilbert's derivation was his recognition of the invariance of I^*. Let us write the integrand of (12.59) in the form

$$\begin{aligned}&\left(f(x,y,p)+\frac{\partial f}{\partial y'}(x,y,p)(y'-p)\right)\,dx\\ &\quad=\left(f(x,y,p)-p\frac{\partial f}{\partial y'}(x,y,p)\right)\,dx+\frac{\partial f}{\partial y'}(x,y,p)\,dy\ .\end{aligned}\tag{12.63}$$

Hilbert observed that the condition of exact differentiability, expressed in terms of the partial differential equation,

$$\frac{\partial}{\partial y}\left(f(x,y,p)-p\frac{\partial f}{\partial y'}(x,y,p)\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}(x,y,p)\right)\ ,\tag{12.64}$$

is equivalent to the validity of the Euler equation

$$\frac{\partial f(x,y,y')}{\partial y}-\frac{d}{dx}\frac{\partial f(x,y,y')}{\partial y'}=0\tag{12.65}$$

along curves of the field, for which the relation $dy/dx = p(x,y)$ holds.

Although Hilbert did not explain how he arrived at the idea for the invariant integral, he may well have pursued the following line of thought. Expression (12.61) for the total variation had been derived by Ernst Zermelo (Zermelo; Hahn 1895) and was also present implicitly in the lectures of Weierstrass; these authors used a parametric approach, in which each comparison curve of the family receives its own parameterization. By rewriting Weierstrass's result in ordinary form, as Hilbert did in equation (12.61), it immediately became apparent that the integral I^* is path-independent. Hilbert then verified the path-independence a posteriori in terms of the validity of the Euler equations and used this fact to simplify the derivation of Weierstrass's sufficiency condition.

The disadvantage of Hilbert's formulation was that it was not immediately apparent how one would generalize it to more complicated examples; the extension is even unclear in the simple case where the variational integrand contains the second derivative of y with respect to x. Adolph Mayer (Mayer 1904) succeeded in generalizing the derivation in a formal sense by means of multiplier rule. However, the idea of the invariant integral really only became a viable notion when it was embedded in a larger theoretical approach that had its origins in the writings on mechanics of Hamilton and Jacobi (see 12.11). In an article published in an American journal, Oskar Bolza (Bolza 1906) introduced the concept of a "field integral" and used it to establish the invariance of the Hilbert integral in a natural and general way. Bolza's method was in essence the following. Consider the field integral

$$S(x,y)=\int_a^b f(x,y,y')\,dx\ ,\tag{12.66}$$

where it is assumed that S is evaluated along the unique extremal from the initial point to the point (x,y). We have $S\big(b,y_0(b)\big)=I$. If we let $\delta y = dy$ and $\delta x = dx$, we obtain from the endpoint formula (12.42) the relation

$$dS=\frac{\partial f}{\partial y'}\,dy+\left(f-y'\frac{\partial f}{\partial y'}\right)\,dx\ .\tag{12.67}$$

In this formula the quantity y' is equal at (x,y) to the slope $p(x,y)$ of the extremal that passes through this point. In order for dS to be an exact differential, the condition expressed in (12.64) must hold. A calculation confirms Hilbert's observation that (12.64) is equivalent to the validity of the Euler equation at (x, y) for the extremal passing through this point.

We now rewrite dS in the form

$$dS = \left(\frac{\partial f}{\partial y'} \frac{dy}{dx} + f - \frac{\partial f}{\partial y'} p \right) dx \ , \tag{12.68}$$

or

$$dS = \left(f(x, y, p) + \frac{\partial f}{\partial y'}(x, y, p)(y' - p) \right) dx \ . \tag{12.69}$$

Here y' denotes the slope at (x, y) of the comparison arc $y = y(x)$. The variational integral evaluated from a to b is therefore given by the formula

$$S\big(b, y_0(b)\big) = \int_a^b \left(f(x, y, p) + \frac{\partial f}{\partial y'}(x, y, p)(y' - p) \right) dx \ . \tag{12.70}$$

The left side of (12.70) is equal to I, and the right side is equal to I^*. Hence we have established that I^* is invariant.

12.10.2. Some modern perspectives. In deriving the necessary condition (12.49), Weierstrass took the arc (0 3) (Fig. 12.4) to be an extremal, i.e., a solution to the Euler equation. Since the point 3 must be distinct from 2, Weierstrass stipulated that there can be no point conjugate to 0 along the arc (0 1). Thus in order to carry out the given construction, Jacobi's condition must hold.

Later authors beginning with Goursat removed the requirement that the neighbouring arc (0 3) be an extremal. All that is needed is that it be given in some definite manner. This is how Weierstrass's necessary condition is derived in modern textbooks. Weierstrass's condition is now known as the second necessary condition, the first being Euler's and the third and fourth being Legendre's and Jacobi's, respectively. The reason for this reordering of the original historical presentation of the necessary conditions is that Legendre's condition may be derived from Weierstrass's, given certain continuity assumptions.

It is nevertheless important to note that Weierstrass himself required the comparison arc to be an extremal and that consequently Jacobi's condition is logically prior to (12.49) in his development. Thus in Weierstrass's original theory the field concept was already implicit in the derivation of the necessary condition involving the E-function. There was a unity to his treatment of this condition and the more involved analysis required to prove sufficiency. This unity is not present today. In the latter, Weierstrass's necessary condition is derived without using comparison extremals and his sufficient condition is obtained in a different way (described above) by means of Hilbert's invariant integral.

Finally we note a peculiarity of modern exposition of the theory of sufficiency in the calculus of variations. In modern treatment one will search in vain for an account of the traditional case, $\int_a^b f(x, y, y', y'')\, dx \to \max / \min$, studied in such detail by Legendre, Jacobi, Spitzer and Hesse. This seems rather surprising, since this problem arises in mechanics (for example, the determination of the elastica) and is well suited to illustrate the various issues that arise in a general theory of sufficiency. Instead, such cases are reduced today to a problem of constrained optimization following Clebsch's procedure set forth above (12.7); the investigation of sufficiency (using either transformation or field methods) is then subsumed under the general and somewhat abstruse theory of the Lagrange problem. The practice of formulating the general variational problem as a Lagrange problem (or alternatively, as a Mayer or a Bolza problem) is widely followed in modern literature on the subject of sufficiency. An unfortunate result of this situation is that there is a distinct contrast between the elementary case $n = 1$, on the one hand, and the elaborate theory of the fully general Lagrange problem on the other.

12.11. Variational methods in mechanics

Throughout its history the calculus of variations has been closely associated with research in theoretical mechanics. Problems in statics and dynamics have provided examples and a sphere of application for the mathematical theory. The latter has been deployed in turn in the invention of new methods in physics. To understand something of the historical interaction of the two subjects, we shall briefly examine the dynamical research of Lagrange, William Rowan Hamilton and Jacobi.

Lagrange's *Mécanique analytique* of 1788 was a comprehensive textbook on statics and dynamics based on a general statement of the principle of virtual work. This principle was formulated and applied using the δ-formalism of the calculus of variations. Lagrange's central technical achievement was to derive the "Lagrangian" form of the differential equations of motion

$$\frac{\partial T}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \frac{\partial V}{\partial q_i} \tag{12.71}$$

for a system with n degrees of freedom and generalized coordinates q_i $(i = 1, \ldots, n)$. The quantities T and V are scalar functions denoting what in later physics would be called the kinetic and potential energies of the system. The advantages of these equations are well known: their applicability to a wide range of physical systems, the freedom to choose whatever coordinates are suitable to describe the system, the elimination of forces of constraint, and their simplicity and elegance.

In addition to presenting powerful new methods of mechanical investigation, Lagrange also provided a discussion of the various principles of the subject. The *Mécanique analytique* would be an important source of inspiration for such nineteenth-century researchers as Hamilton and Jacobi. In investigating problems in particle dynamics in the early 1830s, Hamilton hit upon the idea of taking a certain integral and regarding it as a function of the initial and final coordinate values. He was able to show that the given integral regarded in this way—the so-called principal function—satisfies two partial differential equations of the first order. Although Hamilton employed variational ideas and techniques, his analysis was developed within the established theory of analytical dynamics.

Hamilton's theory was a very original and seminal contribution to the formal development of dynamics. He himself said in 1834 in a letter to his friend William Whewell that he had "made a revolution in mechanics." His findings were published in English in the Philosophical Transactions of the Royal Society. Hamilton was very fortunate to have in Jacobi a reader who immediately appreciated the significance of his work and who was also an exceptional mathematician in his own right. Jacobi took what he referred to as Hamilton's "beautiful discovery" and developed an improved and revised theory. Whereas Hamilton had stipulated that the conservation of mechanical energy (live forces) should hold, Jacobi observed that this equation can be derived without such an assumption. Furthermore, Jacobi emphasized the integration problem and used the theory of partial differential equations to obtain a solution to the dynamical ordinary differential equations in terms of the solution of the corresponding Hamilton-Jacobi equation.

Jacobi confined his investigation to the primary problem in analytical mechanics. In 1858 Clebsch used some of the ideas of the Hamilton-Jacobi theory in his mathematical investigation of the second variation. In the course of doing so, he provided a simple and general exposition of Jacobi's derivation of the Hamilton-Jacobi equation. Mayer, in his study several years later of the second variation (see 12.6), also summarized some of the essential ideas of the Hamilton-Jacobi theory. Clebsch and Mayer developed this theory as a mathematical subject, largely independent of any connection to mechanics.

An historical exposition of the Hamilton-Jacobi theory is beyond the scope of the present paper. We can however make one observation which is germane to our study. It is of interest to note the way in which the later concept of a field of extremals is implicit in the Hamilton-Jacobi development. In Clebsch's derivation of the Hamilton-Jacobi partial differential equation, it is assumed that the given region of the x-y plane is covered with a family of curves that are solutions to the Euler differential equation; it is also assumed implicitly that there is a unique such solution joining the initial point and any subsequent point in the region. The slope of the extremal passing through each point gives rise to a well-defined field function defined over the region.

The germ of this idea can be traced to Hamilton's original derivation of his principal function in his paper of 1834 (and even earlier, to his draft memoirs). Hamilton was working within a dynamical framework and did not conceptualize his result in terms of the calculus of variations. For example, a key step in his derivation of the Hamilton-Jacobi equation required the assumption that the trajectory followed by the system was described by the dynamical equations of motion (expressed in terms of canonical coordinates); viewed as a problem in the calculus of variations, what was being assumed was that the Euler variational equation holds, i.e., that the given trajectory is an extremal.

Interest in the Hamilton-Jacobi theory in the later nineteenth century seems to have been largely based on its role in integrating the variational differential equations. A particular integration of the Euler equations in terms of canonical constants was employed by Clebsch and Mayer in their study of the second variation. In mechanical investigations efforts were concentrated on the question of transforming the coordinates of a system in order to obtain a set of coordinates that yielded a tractable solution to the integration problem.

Weierstrass seems to have carried out his research in the calculus of variations, including his development of field methods, largely independently of an interest in the Hamilton-Jacobi theory. Although some of Beltrami's investigations were associated directly with this theory, he seems not to have had much influence on the main development of the subject, which unfolded in Germany. Kneser's *Lehrbuch der variationsrechnung* of 1900 was the first major treatise containing an exposition of both field methods and the Hamilton-Jacobi theory. Subsequently, Carathéodory (Carathéodory 1935) explored systematically the relationship between the calculus of variations, the Hamilton-Jacobi theory and the theory of partial differential equations.[9]

12.12. Existence questions

Although we have concentrated on single-integral variational problems, the theory extends in a natural way to problems with more than one independent variable. Let $u = (x, y)$ and define $p = \partial u/\partial x$, $q = \partial u/\partial y$. Consider the problem of maximizing or minimizing the integral

$$\iint_R f(x, y, p, q)\, dx\, dy \tag{12.72}$$

where R is some region in the x-y plane, and u is assumed to take on a specified value on the boundary C of R. The solution u must satisfy the Euler differential equation

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial p} - \frac{\partial}{\partial y}\frac{\partial f}{\partial q} = 0. \tag{12.73}$$

As an example, suppose that

$$f = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = p^2 + q^2\ . \tag{12.74}$$

In this case the Euler equation reduces to what is known as the potential equation or Laplace's equation for the function u:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ . \tag{12.75}$$

Nineteenth-century researchers in potential theory and complex analysis were led to the problem of finding a function that satisfies Laplace's partial differential equation on a given region and takes on specified values on the boundary. George Green, William Thomson and Peter Lejeune Dirichlet in potential theory and Bernhard Riemann in complex analysis inferred the existence of such a function from the fact that it would be a solution to a well-defined problem in the calculus of variations. Thus the calculus of variations became the guarantor for the existence of a function needed in another part of analysis. This process of reasoning was given the name "Dirichlet's principle" by Riemann.

In 1870 Weierstrass observed that it is not the case that a problem in the calculus of variations will always have a solution. He considered the example

$$\int_{-1}^{1} x^2 \left(\frac{dy}{dx}\right)^2 dx\ , \tag{12.76}$$

where the values of y at $x = -1$ and $x = +1$ are different. He showed that although it is possible to find admissible functions $y = y(x)$ which make the integral arbitrarily small, the minimum value of zero is never actually attained.

Weierstrass's result called into question the validity of Dirichlet's principle, as an a priori method of analysis, and for a period the principle fell into disrepute. In his Paris address of 1900, Hilbert called in Problem 20 for the further investigation of existence questions in the calculus of variations. In articles published during the

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[9] Thiele (Thiele 1997) discusses the historical connections between Hamilton-Jacobi theory and field theory.

period 1901–1906 (for references see (Monna 1975, 132)), he resuscitated Dirichlet's principle by showing that under certain specified conditions the variational problem will always have a solution. To secure this result, he employed a so-called "direct" method: Instead of deriving the Euler differential equation and attempting to obtain an integral of this equation, he used a suitable limiting process to show that a solution to the original variational problem existed. Hilbert's research initiated an active programme of twentieth-century variational research in which questions of existence have played a prominent part.

Bibliography

Bernoulli, Jakob 1744a. *Solutio problematum fraternorum, peculiari programmate cal. January 1697 Groningae, nec non Actorum Lips. mens June & December 1696, & February 1697 propositorum: una cum propositione reciproca aliorum*, Published in Jakob's Opera (1744), pp. 768-778. Reproduced in (Bernoulli, Jakob; Bernoulli Johann 1991), pp. 271-282., Acta eruditorum (1697), 211-217.

Bernoulli, Jakob 1744b. *Analysis magni problematis isoperimetrici*, Published in Jakob's Opera (1744), pp. 895-920, and in Johann's Opera V. 2 (1742), pp. 219-234. Reproduced in (Bernoulli, Jakob; Bernoulli Johann 1991), Acta eruditorum (1701), 213-228.

Bernoulli, Jakob 1744c. *Jacobi Bernoulli Basileensis opera* (Gabriel Cramer, ed.), 2 volumes, Geneva, 1744.

Bernoulli, Johann 1718. *Remarques sur ce qu'on a donné jusqu'ici de solutions des Problêmes sur les Isoperimetres, avec une nouvelle methode courte & facile de les resoudre sans calcul, laquelle s'étend aussi à d'autres problêmes qui ont rapport à ceux-là*, Reproduced in (Bernoulli, Jakob; Bernoulli Johann 1991), pp. 527-568, Mémoires de l'Académie Royale des Sciences (1718), 100-138.

Bernoulli, Johann 1742. *Jacobi Bernoulli opera omnia* (Gabriel Cramer, ed.), 4 volumes, Lausanne and Geneva, 1742.

Bernoulli, Jakob; Bernoulli, Johann 1991. *Die Streitschriften von Jacob und Johann Bernoulli Variationsrechnung* (H.H. Goldstine with the assistance of P. Radelet-de-Grave, eds.), Introduction by Goldstine, pp. 4-113, Naturforschenden Gesellschaft, 1991; , Birkhäuser Verlag, Basel, Boston and Berlin.

Bliss, G.A. 1925. *Calculus of variations*, Published for Mathematical Association of America by Open Court Publishing, 1925.

Bliss, G.A. 1946. *Lectures on the calculus of variations*, University of Chicago Press, 1946.

Bolza, O. 1904. *Lectures on the calculus of variations*, University of Chicago Press, 1904.

Bolza, O. 1906. *Weierstrass' theorem and Kneser's theorem on transversals for the most general case of an extremum of a simple definite integral*, Transactions of the American Mathematical Society **7** (1906), 459-488.

Bolza, O. 1909. *Vorlesungen über Variationsrechnung*, Teubner, Leipzig and Berlin, 1909.

Bos, H.J.M. 1974. *Differentials, higher-order differentials and the derivative in the Leibnizian calculus*, Archive for History of Exact Sciences **4** (1974), 1-90.

Bottazzini, U. 1986. *The higher calculus: A history of real and complex analysis from Euler to Weierstrass*, Springer-Verlag, New York, Berlin, etc., 1986.

Carathéodory, C. 1952. *Einführung in Eulers Arbeiten über Variationsrechnung, in Euler's Opera Omnia*, Ser. 1, vol. 24, 1952, pp. viii-li.

Carathéodory, C. 1935. *Variationsrechung und Partielle Differentialgleichungen erster Ordnung*, B.G. Teubner, Leipzing and Berlin, 1935.

Clebsch, R.F.A. 1858. *Ueber die Reduktion der zweiten Variation auf ihre einfachste Form*, Journal für die reine und angewandte Mathematik **60** (1858), 254-273.

Clebsch, R.F.A. 1858b. *Ueber diejenigen Probleme der Variationsrechnung, welche nur eine unäbhangige Variable enthalten*, Journal für die reine und angewandte Mathematik **60** (1858b), 335-355.

Delaunay, C. 1841. *Thèse sur la distinction des maxima et des minima dans les questions qui dépendent de la méthode des variations*, Journal de mathématiques pures et appliqées **6** (1841), 209-237.

du Bois-Reymond, P. 1879. *Erläuterungen zu Anfangsgründen der Variationsrechnung*, Mathematische Annalen **15** (1879), 283-314.

Erdmann, G. 1877. *Ueber unstetige Lösungen in der Variationsrechnung*, Journal für die reine und ungewandte Mathematik **82** (1877), 21-30.

Euler, L. 1952. *Problematis isoperimetrici in latissimo sensu accepti solutio generalis*, in Commentarii academiae scientiarum Petropolitanae **6** (1732/3), 123-155; Reprinted in Euler's Opera Omnia, Ser. 1, vol. 25, 1952, pp. 13-40.

Euler, L. 1736. *Curvarum maximi minimive proprietate gaudentium inventio nova et facilis*, in Commentarii academiae scientiarum Petropolitanae **8** (1736), 159-190; Reprinted in Opera Omnia, Ser. 1, vol. 25, 1952, pp. 54-80.

Euler, L. 1744. *Methodus inveniendi curvas lineas maximi minimive proprietate gaudentes sive solution problematis isoperimetrici lattisimo sensu accepti*, Lausanne, 1744; Reprinted in Opera Omnia, Ser. 1, vol. 24, 1952.

Euler, L. 1771. *Methodus nova et facilis calculum variationum tractandi*, Nova commentarii academiae scientiarum Petropolitanae **16** (1771), 35-70; , Reprinted in Opera Omnia, Ser. 1, vol. 25, 1952, pp. 208-235.

Feigenbaum, L. 1985. *Brook Taylor and the method of increments*, Archive for history of exact sciences **34** (1985), 1-140.

Fraser, C. 1985. *J.L. Lagrange's changing approach to the foundations of the calculus of variations*, Archive for history of exact sciences **32** (1985), 151-191.

Fraser, C. 1992. *Isoperimetric problems in the variational calculus of Euler and Lagrange*, Historia mathematica **19** (1992), 4-23.

Fraser, C. 1994. *The origins of Euler's variational calculus*, Archive for history of exact sciences **47** (1994), 103-141.

Fraser, C. 1996. *The background to and early emergence of Euler's analysis*, Analysis and synthesis in mathematics. History and philosophy. Boston studies in the philosophy of Science (M. Otte and M. Panza, eds.), Kluwer, Dordrecht-Boston-London, 1996.

Funk, P. 1962. *Variationsrechnung und ihre Andwendung in Physik und Technik*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1962.

Galileo, Galilei 1639. *Discorsi e dimostrazioni matematichi inforno à due nuove scienze*, 1639; In Le Opere de Galileo Galilei (1890-1909), vol. 8, pp. 190-313.

Goldstine, H.H. 1980. *A history of the calculus of variations from the 17th through the 19th century*, Springer-Verlag, 1980.

Goursat, E. 1905. *Cours d'analyse mathémtique*, vol. 2, Gauthier-Villars, Paris, 1905, Chapter 23, pp. 590-632, is devoted to the calculus of variations.

Hadamard, J. 1910. *Leçons sur le calcul des variations*, A. Hermann et Fils, Paris, 1910.

Hamilton, William Rowan 1834. *On a general method employed in Dynamics, by which the study of the motions of all free Systems of attracting or repelling points is reduced to the search and differentiation of one central solution or characteristic function*, Philosophical transactions of the Royal Society of London **124** (1834), 247-308.

Hamilton, William Rowan 1835. *Second essay on a general method in Dynamics*, Philosophical transactions of the Royal Society of London **125** (1835), 95-144.

Hamilton, William Rowan 1940. *Geometrical optics*, The mathematical papers of Sir William Rowan Hamilton, vol. 1, 1931; *Dynamics*, ibid., vol. 2, University Press, Cambridge, 1940.

Hankins, T.L. 1980. *Sir William Rowan Hamilton*, Johns Hopkins Press, Baltimore and London, 1980.

Hesse, L.O. 1857. *Ueber die Criterien des Maximums und Minimums der einfachen Integrale*, Journal für die reine unde angewandte Mathematik **54** (1857), 227-273.

Heath, Thomas H. 1921. *A history of Greek mathematics*, vol. II, Clarendon Press, Oxford, 1921, Reprinted by Dover Publications, Inc. 1981.

Hilbert, D. 1900. *Mathematische Probleme. Vortag, gehalten auf dem inernationalen Mathematiker-Kongress zu Paris 1900*, Göttingen Nachrichten (1900), 253-297; Archiv der Mathematik und Physik **1** (3) (1901), 44-63, 213-237; An English translation appears in, Bulletin of the American Mathematical Society **7** (2) (1902), 437-479.

Hilbert, D. 1906. *Zur Variationsrechnung*, Mathematische Annalen **62** (1906), 351-371.

Hildebrandt, S. 1989. *The calculus of variations today*, Mathematical Intelligencer **11** (1989), 50-60.

Jacobi, C.G.J. 1837. *Zur Theorie der Variations-Rechnung und der Differential-Gleichungen*, Journal für die reine und angewandte Mathematik **17** (1837), 68-82.

Jacobi, C.G.J. 1838. *Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systemes gewöhnlicher Differentialgleichungen*, Journal für die reine und angewandte Mathematik **17** (1838), 97-162.

Jacobi, C.G.J. 1866. *Volesungen über Dynamik* (A. Clebsch, eds.), Verlag Georg Reimer, Berlin, 1866.

Jacobi, C.G.J. 1886. *C.G.J. Jacobi's Gesammelte Werke*, vol. 4, K. Weierstrasse. Georg Reimer, Berlin, 1886.

Klein, F. 1926. *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert Teil 1*, Berlin, 1926.

Kneser, A. 1900. *Lehrbuch der Variationsrechnung*, Second edition 1925, Braunschweig, 1900.

Kneser, A. 1904. *Variationsrechnung*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Part 8, vol. IIA, 1904, pp. 571-625.

Lacroix, S.F. 1806. *Traité élémentaire de calcul différentiel et de calcul intégral.* Second edition. Courcier, Paris.

Lagrange, J.L. 1760-1761. *Essai sur une nouvelle méthode pour déterminer les maxima et les mimima des formules intégrales indéfinies*, Miscellanea physico-mathematica societatis Taurinensia, vol. ii, 1760-61, pp. 407-418; In *Oeuvres*, vol. 1, pp. 333-362.

Lagrange, J.L. 1788. *Méchanique analitique*, Second edition appeared in two volumes in 1811 and 1815, 1788; Reprinted in *Oeuvres*, vol. 11, 12.

Lagrange, J.L. 1797. *Théorie des fonctions analytiques*, Paris, 1797; The second edition appeared in 1813 and is reprinted as *Oeuvres* **9**.

Lagrange, J.L. 1806. *Leçons sur le calcul des fonctions. Nouvelle edition*, Includes *un traité complèt du calcul des variation*, Paris, 1806; Reprinted as *Oeuvres* **10**.

Lagrange, J.L. 1867-1892. *Oeuvres de Lagrange*, 14 volumes, Gauthier-Villars, Paris, 1867-1892.

Lecat, M. 1913. *Bibliographie du calcul des variations 1850-1913*, Ghent and Paris, 1913.

Lecat, M. 1916. *Bibliographie du calcul des variations, depuis les origines jusqu'à 1850 comprenant la liste des travaux qui ont préparé ce calcul*, Ghent and Paris, 1916.

Legendre, A.M. 1786. *Sur la manière de distinguer les maxima des minima dans le calcul des variations*, Mémoires de l'Académie Royale des Sciences (1786), 7-37.

Mach, E. 1893. *The science of mechanics: A critical and historical account of its development*, was first published in 1893, 1883, English translation by Thomas McCormack.

Mayer, A. 1866. *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale*, Habilitationsschrift, Leipzig, 1866.

Mayer, A. 1868. *Ueber die Kriterien des Maximums und Minimums der einfachen Integrale*, Journal für die reine und angewandte Mathematik **69** (1868), 238-263.

Mayer, A. 1886. *Begründung der Lagrange'sche Multiplicatorenmethode in der Variationsrechnung*, Mathematische Annalen, vol. 26, 1886, pp. 74-82.

Mayer, A. 1904. *Ueber den Hilber'schen Unabhängigkeitssatz in der methode in der Variationsrechnung*, Mathematische Annalen **58** (1904), 235-248.

Monna, A.F. 1975. *Dirichlet's principle: A mathematical comedy of errors and its influence on the development of analysis*, Oosthoek, Scheltema and Holkema, Utrecht, 1975.

Nakane, M. 1991. *The role of the three-body problem in the construction of the characteristic function for mechanics by W.R. Hamilton*, Historia scientiarum **11** (1991), 27-38.

Nakane, M. 1995. *From dynamics to mathematics: The construction of the Hamilton-Jacobi formalism in the calculus of variations*, Paper presented at the 5th Tsuda Conference of the History of Mathematics for Mathematicians, Japan, 1995.

Newton, Isaac 1687. *Philosophiae naturalis principia mathematica*, University of Chicago Press, 1687, English translation of the third edition by Motte (1729), revised by Florian Cajori, is available in a two-volume paperback.

Newton, Isaac 1974. *The mathematical papers of Isaac Newton, Volume VI 1684-1691* (D.T. Whiteside, ed.), Cambridge University Press, 1974.

Woodhouse, R. 1810. *A treatise on isoperimetrical problems and the calculus of variations*, Cambridge, 1810.

Scheeffer, L. 1885. *Die Maxima und Minima der einfachen Integrale zwischen festen Grenzen*, Mathematischen Annalen **25** (1885), 522-595.

Scriba, C.J. 1973. *Jacobi, Carl Gustav Jacob*, Dictionary of scientific biography (Charles C. Gillispie, ed.), vol. 7, Charles Scribner's Sons, New York, 1973, pp. 50-55.

Spitzer, S. 1854. *Über die Kriterien des Grössten and Kleinsten bei den Problemen der Variationsrechnung*, Sitzungsberichte der Mathematisch-Naturwissenschaften Classe der Kaiserlichen Akademie der Wissenschaften, Part One appears in Volume 13 (1854), pp. 1014-1071. Part Two appears in Volume 14 (1855), pp. 41-120, Vienna, 1854-1855.

Stäckel, P. 1762. *Abhandlungen über Variations-Rechnung Zweitzer Theil: Abhandlungen von Lagrange (1762, 1770), Legendre (1786) und Jacobi (1837)*, Number 47 of Wilhelm Ostwald's series Die Lkassiker der exakten Wissenschaften, 1894.

Stäckel, P.; Taylor, B. 1715. *Methodus incrementorum dirrectra and inversa*, London, 1715.

Thiele, R. 1982. *Leonhard Euler*, Volume 56 of the series *Biographien hervorragender Naturwissenschaftler, Techniker und Mediziner*, BSB B.G. Teubner, Leipzing, 1982.

Thiele, R. 1997. *Some contributions to the field theory in the calculus of variations*, Historia mathematica **24** (1997), 281-300.

Todhunter, I. 1861. *A history of the progress of the calculus of variations during the nineteenth century*, 1861; Reprinted by Chelsea Inc. in 1962 under the title "A history of the calculus of variations during the nineteenth century".

Todhunter, I. 1871. *Researchers in the calculus of variations, principally on the theory of discontinuous solutions*, Macmillan and Co., London and Cambridge, 1871.

Todhunter, I.; Turner, R.S. 1971. *The growth of professorial research in Russia, 1818 to 1848–causes and context*, Historical studies in the physical sciences, vol. 3, 1971, pp. 137-182.

Weierstrass, K. 1927. *Vorlesungen über Variationsrechnung* (Rudolf Rothe, ed.), Akademische Verlagsgesellschaft, Leipzig, 1927.

Wisan, W.L. 1974. *The new science of motion: A study of Galileo's De motu locali*, Archive for history of exact sciences, vol. 13, 1974, pp. 103-306.

Zermelo, E.; Hahn, H. 1895. *Untersuchungen zur Variationsrechnung*, Berlin, 1895.

Zermelo, E.; Hahn, H. 1904. *Weiterentwicklung der Variationsrechnung in der Letzten Jahren*, Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Part 8a, vol. IIA, 1904, pp. 626-641.