The Culture of Research Mathematics in 1860s Prussia: Adolph Mayer and the Theory of the Second Variation in the Calculus of Variations

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Abstract The paper examines the intellectual culture of higher mathematics in Prussia and more broadly in Germany in the middle of the nineteenth century. There was at this time a strong ethos of pure mathematics in which the subject was pursued more for its intrinsic interest than for its utility or practical applications. This outlook was reflective of the prominence of neohumanism in German culture of the period. In the case of mathematics, it led to a higher degree of logical sophistication in the elaboration of theories. The work of Adolph Mayer in the calculus of variations at Prussia's University of Königsberg is presented as a case study that illustrates the outlook and underlying values of German higher mathematics in the second half of the century. The self-consciously theoretical character of this mathematics distinguished it in a qualitative way from the style and mentality of the Enlightenment masters of analysis a century earlier. Our study provides evidence for the basic historicity of the development of analysis from 1750 to 1870.

1 Introduction

The first half of the nineteenth century witnessed several developments which pointed to the emergence of a new mentality across large parts of mathematics. Among the different strands, one could mention the foundational transformation that mathematics underwent. The invention of non-Euclidean geometry, the development of rigor in analysis, the construction of novel algebraic structures, and the emergence of mathematical logic all pointed to the arrival of new epistemic vistas, characterized among other things by a greater self-consciousness concerning the fundamental character of mathematics. There was a shift away from the notion that

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M. Zack, D. Schlimm (eds.), *Research in History and Philosophy of Mathematics*, Proceedings of the Canadian Society for History and Philosophy of Mathematics/ Société canadienne d'histoire et de philosophie des mathématiques,
https://doi.org/10.1007/978-3-319-90983-7_8

mathematics is about a subject that is given, to a view of mathematics as something that is created from within, a self-made intellectual construction governed by internal precepts and laws.

In 1868, Adolph Mayer (1839–1908) published an important article in the *Journal für die reine und angewandte Mathematik* on the calculus of variations, a branch of analysis that was intensively studied in the nineteenth century. Mayer investigated the conditions that must be satisfied to ensure that a given function that satisfies the Euler differential equation is a genuine maximum or minimum. The impetus for this article originated with research on the second variation published by the Königsberg mathematician Carl Jacobi (1804–1851) in the 1830s. Mayer's result was only one among a large range of new findings in analysis, not to mention the whole of mathematics. It nonetheless provides evidence for the basic historicity of the development of analysis from 1750 to 1870. One would not have encountered the kind of theoretical concerns that occupied Mayer and his contemporaries in the work of the Enlightenment masters of analysis. At a fairly concrete level, in a central part of mathematics, the intellectual orientation of researchers had moved away significantly from the point of view that had traditionally prevailed.[1]

2 Prussia and the Discipline of Mathematics 1825–1865

Mayer carried out his investigation of the second variation in the early 1860s at the University of Königsberg. In order to understand the background to this work and the broader intellectual milieu in which he worked, it is necessary to examine the state of Prussian university education in the first half of the nineteenth century. Mayer was the beneficiary of a new culture of scientific research that had emerged in Prussian universities at this time.[2] Although Prussia was the leader in educational reform, there was some movement of faculty and much movement of students between the different German universities. Universities such as Heidelberg, Göttingen, and Leipzig experienced similar changes to their Prussian counterparts. A cultural outlook that was most clearly present in Prussia developed throughout the German states.

[1]This essay originated in response to Josipa Petrunic's call to explore how epistemic cultures in mathematical practice and theory are identified, and how they interact. A workshop on this subject was held at the University of Toronto in the summer of 2012. I am grateful to Hardy Grant for comments on an earlier draft of the paper, and to the referees for their comments.

[2]In the present essay, the word "culture" is used not as a formal theoretical concept of sociology, but rather in its everyday sense as the common system of beliefs and values that members of society bring to projects of literary, artistic, and intellectual achievement. Within history of science, our understanding of this word is consistent with the way it is used in an article such as Paul Forman's (1971) "Weimar culture, causality and quantum theory, 1918–1927: adaptation by German physicists and mathematicians to a hostile intellectual environment." For a study that articulates the notion of a culture in science on a more explicit theoretical level, see Karin Knorr Cetina's *Epistemic Cultures: How Sciences Make Knowledge* (1999).

The reform of Prussian universities in the first half of the nineteenth century had its origins in the government ministry's response to military setbacks experienced by Prussian forces in the Napoleonic Wars.[3] The institutions which were prominent in this development were the University of Königsberg, the University of Halle-Wittenberg, the University of Breslau, the University of Berlin upon its establishment in 1810, and the University of Bonn on its establishment in 1818.

The reform of Prussian secondary and university education has been studied extensively by historians.[4] The expansion of education was accompanied by the emergence of universities as the place in which advanced scientific research was carried out. Prussia led the way in the shift from the academies of the Enlightenment to the universities of the nineteenth century and today.[5] Insofar as physical science is concerned, historians identify the University of Königsberg as a pivotal institution in this development. During the 1830s, Jacobi and Franz Neumann (1798–1895) established a seminar at Königsberg in which original scientific research was presented and discussed. The participants in the seminar were typically future gymnasium teachers and university professors.

Prominent themes in histories of Prussia education in the nineteenth century include the professionalization of science and mathematics, the role of the Prussian ministry in the reform of education, the importance of competition and decentralization, and the growth of specialization. From the perspective of the present study, the most significant development was the emergence of a research-oriented ethos in science. Fundamental to the new vision was the cultural outlook of neohumanism, what historians such as Stephen Turner have called *Wissenschaftsideologie*. The ideal of critical and original research had developed in the subjects of philology and history, and this ideal was extended by men such as Jacobi and Neumann to mathematical science. Jacobi himself had been a student in Berlin of the great classical philologist August Böckh (1785–1867). (Of course, unlike the case of philology or history, mathematicians already had a high standard to emulate in the mathematics being done in France.)

Prussian university students in the philosophical faculty participated in seminars in which the professor expounded on the subject of his research. The students were similar to today's graduate students and were expected to carry out original research. The cultural outlook of neohumanism involved an emphasis of the pursuit of knowledge for its own sake without any concern for utility or practical application. Emil du Bois-Reymond (1818–1896), a leading scientific figure at the middle of the century, pronounced on the benefit students received in such training, a benefit "which accrues even to the mediocre mind, that, at least once in his life, he has been compelled to one step over the threshold of pure learning and has felt the breath of its spirit; that at least once he has seen the truth sought, found and cherished for its own sake." (Quoted by Turner 1971, p. 153)

[3]Gerstell (1975) provides an account of mathematics education in Prussia in the first part of the nineteenth century.

[4]See Turner (1971) and Schubring (1981, 2005) and the references contained therein.

[5]For a standard account of this development, see Joseph Ben-David's "German scientific hegemony and the emergence of organized science," (Ben-David 1971, Chapter 7).

The professionalization of the professoriate also contributed to an emphasis on the disciplinary subject as an intrinsic object of value. It was necessary to establish the autonomy of members of the discipline with respect to external authority (government, industry, and commerce). There arose among mathematical professionals an emphasis on the subject-specific character of mathematics and on the pursuit of mathematics for its own sake. Gert Schubring (1981, p. 118) in his essay "The conception of pure mathematics as an instrument in the professionalization of mathematics," calls attention to Jacobi's "rejection of the externally defined value of usefulness, and on his corresponding emphasis on the internal values of the discipline."

The rise of neohumanism in university circles was accompanied by an increased emphasis on scientific research of a very theoretical nature. Advanced mathematics as it was understood by Jacobi and his contemporary August Crelle (1780–1855) encompassed subjects such as theoretical mechanics that would today be regarded as part of applied mathematics or physics. Crelle observed in 1828:

> Mathematics by itself, or so-called *pure mathematics*, does not depend on its applications. It is completely idealist; its objects, *number*, *space*, and *force*, are not taken from the external world, but are primitive ideas. They pursue their own independent development from deductions drawn on their basic concepts [...]. Every addition of and ties to applications, on which it does not depend, is therefore disadvantageous to the science itself. (Quoted in Schubring 2005, p. 484)

The distinctive feature of this conception of mathematics was not its independence from empirical concepts such as force and space, but rather its non-utilitarian, idealist, and purely theoretical character.[6]

In the eighteenth century, investigators at the academies engaged in scholarly projects that included what the later neohumanists would likely have called pure research. However, there was during this earlier period no particular impulse to cultivate the subject as something that stood separate from the various utilitarian purposes to which it could be put. What was notable about *Wissenschaftsideologie* was the self-conscious goal of studying mathematics as a body of knowledge with its own concepts, precepts, and internal dynamics that stood apart from practical or utilitarian concerns. It was something that was to be contemplated and appreciated in and for itself. In so doing the mathematician was engaged in an intellectual project

[6]For a wide-ranging discussion of historical and philosophical notions of pure and applied mathematics, see Ian Hacking's "Applications," Chapter 5 of *Why Is There a Philosophy of Mathematics at All* (2014). Hacking (p. 153) distinguishes the philosophical meaning of pure as that which does not involve any experiential input from the more common meaning of pure as that which is independent of practical purposes. Authors such as du Bois-Reymond and Crelle seem to have understood pure mathematics in the second sense. What Jakob Fries called applied mathematics would certainly count as pure in the second sense. The theory of the second variation in the calculus of variations belongs to pure mathematics in either sense, while a subject such as Hamilton–Jacobi theory in dynamics—cultivated as an abstract theory—would evidently belong to pure mathematics in the second sense.

of the same cultural prestige as that of the humanist.[7] This conception rings out in the following comments of Crelle on the training of teachers of mathematics:

> It is therefore initially very important *that pure mathematics remain strictly separated from its applications,* particularly in an institute training mathematics teachers while simultaneously helping to attain an important goal of mathematics: to be a means of education. The applications of mathematics are noble and useful fruits of the same; mathematics alone can ripen these fruits and do this fully without disadvantage to itself only if it has previously been allowed to develop purely from within itself without hindrance. (Quoted in Schubring 2005, p. 485)

The concept of pure mathematics is also found in some of the philosophical work of the period, in particular the writings of Jakob Fries (1773–1843). Fries held professorships at the University of Heidelberg and the University of Jena, and authored several influential books on Kantian philosophy. He is known primarily for having developed a psychological interpretation of Kant's critical philosophy. In contrast to contemporary philosophical thinkers such as Friedrich Schelling (1775–1854) and Georg Hegel (1770–1831), he was very interested in mathematics and natural science. He was the rare figure during the period whose work in philosophy took account of contemporary scientific advances. In his *Mathematical Philosophy of Nature* (1822), Fries divided mathematics into pure and what he called applied mathematics. Both parts of mathematics dealt with *a priori* knowledge. Pure mathematics was combinatorial and symbolic and had as its base arithmetic and algebra; applied mathematics was focused on the pure doctrine of motion as the foundation of analytical dynamics. For Fries, mathematical statements were apodictic or necessarily true; they were, however, not pure products of intuition—as Kant had believed—but rather were discursive and philosophical in character. According to Pulte (2006, p. 118), Fries "takes into account the general development of mathematics in his time, which is characterized by an increasing abstraction and self-reference of its laws and by the complexity of its structures."

It is difficult to connect philosophical positions directly to the activities of working mathematicians. We know that Gauss supported Fries's work and asserted that Fries was the only philosopher he could trust (Schubring 1981, p. 120).[8] Certainly, Fries's valuation of mathematics and his articulation of its *a priori* character would have been consistent with the paradigm of pure theory that infused German research of the period. A given branch of mathematics was to be explored for the sake of its intellectual interest. Rather than view mathematics as something

[7] Pyenson (1983) in his book *Neohumanism and the Persistence of Pure Mathematics in Wilhelmian Germany* maintains that this outlook hindered developments in physics education at the end of the century, although his thesis has been challenged by Rowe (1985).

[8] Schubring cites Leonard Nelson (ed.), "Vier Briefe von Gauss und Weber an Fries," *Abhandlungen der Fries'schen Schule* V. 1 (1906), 431–440, p. 437. Concerning Fries 1822 book, Gauss provided the following advice to a skeptical student: "Young man, if after three years of intense study you have progressed to where you understand and appreciate this book, you can leave the university with the conviction that you have made use of your time better by far than the majority of your fellow students." Quoted in Gregory (1983, p. 186).

that unfolds following immutable laws of development, or to view it primarily for its utilitarian purposes, there was a natural tendency to reflect on the inner connections of the theory and to explore the relations between its parts. Scharlau (1981) has observed that the concept of pure mathematics that developed in nineteenth-century Germany was manifested in the identification of unexpected connections between different areas of the subject.

3 Mayer and His Mathematical Circle

In the entry on Jacobi in the *Dictionary of Scientific Biography*, Christoph Scriba (1973, p. 51) writes "Such were Jacobi's forceful personality and sweeping enthusiasm that none of his gifted students could escape his spell; they were drawn into his sphere of thought, working along the manifold lines he suggested, and soon represented a "school."" [9] The term "Königsberg school" as used by historians of mathematics refers not simply to researchers at the University of Königsberg but also to graduates of this institution who became established professors elsewhere, to investigators who spent some time in the early part of their career at Königsberg, and more broadly to figures that were influenced by these groups of researchers. The first generation of mathematicians who were educated in the reformed system of Prussian education was born in the period from about 1795 to 1810. Mathematicians connected to Königsberg included Franz Neumann (1798–1895), Jacobi (1804–1851), Ludwig Sohncke (1807–1853), Friedrich Richelot (1808–1875), Otto Hesse (1811–1874), and a few years later Johann Rosenhain (1816–1887), Karl Borchardt (1817–1880), Philipp von Seidel (1821–1896), and Eduard Heine (1821–1881).

Adolph Mayer belonged to a second generation of mathematicians born in the 1830s that included Carl Neumann (1832–1925), Rudolf Lipschitz (1832–1903), and Alfred Clebsch (1833–1872). Younger researchers who overlapped with Mayer in Königsberg were Karl VonderMühl (1841–1912), Heinrich Weber (1842–1913), and Albert Wangerin (1844–1933). Mayer appeared on the mathematical scene as the German states were becoming the dominant force in European mathematics in terms of the number of researchers, volume of research, and level of creative achievement.[10]

Particular note should be made of Richelot, who was a student of Jacobi and is regarded as a prominent member of the Königsberg school.[11] Richelot worked in the field of elliptic functions but had wide-ranging interests in analysis. He lectured to a generation of students, welcoming them to his home and going on walks with them. Among his doctoral students were Heinrich Schröter (1829–1892), who pursued a career at the University of Breslau, and Carl Neumann (1832–1925), who secured a chair in mathematics at the University of Leipzig. Richelot was very interested

[9] For more information on the Jacobi school (see Klein 1926, pp. 112–115).

[10] For biographical information on Mayer (see Thiele 1999).

[11] For biographical information on Richelot (see Cantor 1889).

in mathematics education and made recommendations to the Prussian ministry that were included in the 1866 teaching regulations, particularly the stipulation that a teaching candidate be required to carry out original work in some area of mathematics (Rowe 1985, p. 128).

Mayer was the son of a prosperous Leipzig merchant and was able to pursue an academic career in the highly competitive German system, something that involved years of independent and unremunerated research beyond the doctorate. He carried out university studies at Heidelberg and Göttingen, studying at the former under Otto Hesse (1811–1874). Hesse took an interest in Mayer and encouraged his mathematical studies, enlisting his help in producing the corrections to his *Analytischen Geometrie des Raumes* of 1861. Mayer obtained his doctorate from Heidelberg in 1861. From 1862 to 1865, he studied at Königsberg where he attended Franz Neumann's famous seminar on mathematical physics. Neumann's son Carl, who founded the *Mathematischen Annalen* with Alfred Clebsch (1833–1872) in 1868, would later become his colleague at Leipzig. Mayer's interest in the calculus of variations was stimulated by the lectures in 1864–1865 of Richelot, who suggested the topic of his Habilitationsschrift. The latter was dedicated to Richelot and successfully defended by Mayer at the University of Leipzig in 1866 and published in the same year. In 1867 Mayer began to teach at this university, where he remained for the rest of his life. At Leipzig he joined with Carl Neumann to give a seminar on mathematics, modeled after the seminar in Königsberg. Mayer is also known in the history of mathematics for his role in bringing the results of the Norwegian researcher Sophus Lie to the attention of the mathematical community.

Although Mayer took as the starting point of his Habilitationsschrift some work of Hesse and Clebsch on the second variation, he was more focused on the detailed investigation of the deductive structure of the theory. Unlike Hesse and Clebsch, he remained active in the calculus of variations throughout his career and made important contributions to the subject up to his death in 1908. Particularly notable were Mayer's (1886) proof of the general form of Lagrange's multiplier rule in the calculus of variations, and his papers of 1904 and 1906 on the new field methods.

4 Mayer's Result

Some branches of modern analysis originated at particular historical times and developed into well-defined and substantial areas of research. One example is the theory of integral equations, which basically emerged *ex nihilo* during the 1890s. The calculus of variations is very different, having been an active area of investigation from the invention of calculus in the 1680s. Single-integral variational problems were the subject of extensive research in the eighteenth, nineteenth, and early twentieth centuries, and are subsumed today within the wider part of mathematics devoted to problems of optimization. Throughout its history the calculus of variations has been engaged with problems from mathematical mechanics, and variational principles in mathematical physics continued to play a role in what was an increasingly abstract part of analysis.

Key works in the development of the calculus of variation prior to the 1830s include Leonhard Euler's (1708–1783) *Methodus inveniendi lineas curvas* of 1744, Joseph Lagrange's (1736–1813) paper of 1762 on the δ-algorithm, and Lagrange's textbooks on analysis around 1800.[12] In a famous paper of 1837, Jacobi (1837) outlined the basis of a new theory of the conditions required to ensure a maximum or minimum in the calculus of variations. His paper, in which proofs and justifications were omitted, would become the basis for a vigorous program of mathematical research.[13] Textbooks on calculus of variations (both in the nineteenth century and today) focus on the case where there is a single dependent variable and only the first derivative of this variable appears in the variational integrand. However, the essential difficulties in the theory of the second variation arise in problems that are more general than this, where there are derivatives of higher order in the integrand or where there is more than one dependent variable. The considerable body of work stimulated by Jacobi's paper was devoted largely to an investigation of the transformation of the second variation in this more general setting. The transformation—assuming it was possible to carry out—permitted one to infer Legendre's necessary condition for the existence of a maximum or minimum. Among notable contributors here were Charles Delaunay (1816–1872) (1841), Gaspare Mainardi (1800–1879) (1852), Simon Spitzer (1826–1887) (1854), Hesse (1857), and Clebsch (1858a, b).

In a short passage in his 1837 paper, Jacobi had drawn attention to another aspect of the variational problem, what in later mathematics would be called the theory of the conjugate point. The essential point is the following. Consider the problem of maximizing or minimizing the integral $I = \int_{x_0}^{x_1} f\left(x, y, y'\right) dx$ on the interval $[x_0,x_1]$. Among all arcs $y=y(x)$ joining the endpoints, the optimal arc $y=y_0(x)$ will give a solution to the Euler equation $V=0$. The general solution of the Euler equation for curves that are required to go through the initial point will contain an arbitrary constant α, $y=y_0(x,\alpha)$. We suppose that $\frac{\partial y_0(x_0,0)}{\partial \alpha} = 0$. Consider the variation $\delta y = \frac{\partial y_0(x,\alpha)}{\partial \alpha}\alpha$ and comparison arcs of the form $y(x) + \frac{\partial y_0(x,\alpha)}{\partial \alpha}\alpha$. If the condition $\frac{\partial y_0(x_1,0)}{\partial \alpha} = 0$ also holds, then $\delta y = \frac{\partial y_0(x,\alpha)}{\partial \alpha}\alpha$ is an admissible variation on the interval $[x_0,x_1]$. The first variation of I is $\delta I = \int_{x_0}^{x_1} V \delta y dx$ and so the second variation of I is $\delta^2 I = \int_{x_0}^{x_1} \delta V \delta y dx$. Now for $y=y_0(x)$ we have $V=0$. Hence $\delta V=0$ and so $\delta^2 I=0$. Thus, for $y=y_0(x)$ and the admissible variation $\delta y = \frac{\partial y_0(x,\alpha)}{\partial \alpha}\alpha$ we have both $\delta I=0$ and $\delta^2 I=0$. The first and second variations of I are both zero. It is clear in this situation that there can be no maximum or minimum, because the sign of the third variation can (in general) be made either positive or negative.[14]

If we start at the initial point (x_0,y_0) on the curve $y=y_0(x,\alpha)$, we may eventually arrive at a second point (x_1,y_1) with $\frac{\partial y_0(x_1,0)}{\partial \alpha} = 0$. In this situation, there will

[12] See Goldstine (1980) and Fraser (1994).

[13] For accounts of these developments. See Goldstine (1980) and Fraser (1996, 2003).

[14] Noteworthy is the way in which the operational character of the variational process δ enters into the reasoning. See Fraser (1994).

be an admissible variation for which $\delta^2 I=0$, and it will then not be the case for the given variational problem that y_0 is an optimal curve on the interval $[x_0,x_1]$. This second limiting point, the value of which cannot be reached or exceeded if a minimum or maximum is to obtain, would become known in later mathematics as a conjugate point (a term introduced by Karl Weierstrass (1815–1897)). The condition that there be no such second point on the interval is known in the modern subject as Jacobi's condition. Jacobi illustrated this restriction using the example of the elliptical motion of a particle moving about a force center, in which the trajectory is deduced from the principle of least action. In lectures delivered in the early 1840s and published posthumously in his *Vorlesungen* (1866, p. 46), he introduced an even simpler example, the case of a single particle constrained to move on the surface of sphere but otherwise subject to no force. The principle of least action leads here to the conclusion that the trajectory must be a geodesic or path of shortest distance. Hence, the particle moves on a great circle, i.e., the intersection of the surface of sphere and a plane through its center. If we begin at a given point *A* and traverse an angular distance of 180°, we reach a point *C* conjugate to *A*. If the second point *B* coincides with or is beyond *C* then it is not difficult to see that there are comparison paths of equal or shorter distance.[15]

During the period from 1837 to 1865, the efforts of researchers were devoted almost exclusively to the transformation problem. While the idea of a conjugate point had a natural analytical basis, in practice the existence of such points was revealed in geometrical examples. There was a pronounced formal orientation to the research stimulated by Jacobi's paper. From a theoretical viewpoint, it is not at all obvious how one would in the general case connect the transformation problem to the theory of the conjugate point.

The body of research on the transformation problem divided into two main lines. The first, stemming from Jacobi's original memoir, focused on a result in the theory of ordinary differential equations that Jacobi used to transform the second variation for the general problem where the variational integrand contains derivatives of y with respect to x of arbitrary order. Jacobi also introduced a crucial insight involving the use of variations given as partial derivatives of the Euler solutions with respect to the arbitrary constants appearing in them to effect this transformation. Although several researchers continued this line of research, the most important figures were Delaunay (1841) and Hesse (1857). Delaunay followed Jacobi's outline most closely; indeed his memoir is an indispensable aid in understanding and making sense of the 1837 paper, which was very compressed and even somewhat cryptic in places. Hesse brought this whole line of research to its culmination, with an article in 1857 in Crelle's journal. Hesse also provided a complete analysis of the simplest case of the variational problem, where the integrand has the form $f(x,y.y')$ $(y'=dy/dx)$. In fact he arrived at the first sufficiency proof, showing in this case that if a given function $y=y(x)$ satisfied Euler's equation $\partial f/\partial y\text{-}d/dx(\partial f/\partial y')=0$, Jacobi's condition was satisfied (no conjugate point on the interval), and Legendre's condition held $(\partial^2 f/\partial y'^2 > 0)$, then $y=y(x)$ produced a minimum.

15 For a history of Jacobi's work in mathematical dynamics (see Nakane and Fraser 2002).

The Jacobi–Delaunay line of research to all intents and purposes ended with Hesse's 1857 article in Crelle. In many respects, the whole development was idiosyncratic. It was not at all a natural approach to the problem of sufficiency, and it seems surprising that it ever originated at all. Despite the singular character of Jacobi's new transformation, there were two fundamental and enduring aspects of his work: the notion of the conjugate point, and the recognition that any transformation of the second variation would involve variations obtained as partial derivatives of the Euler solutions for the problem.

A second line of attack was launched by Gaspare Mainardi in 1852 and continued by Simon Spitzer two years later.[16] Spitzer's development of the theory was more systematic and detailed than Mainardi's and seems to have had a greater impact on subsequent research. Instead of using Jacobi's new transformation, Spitzer followed Legendre's original idea and began with a general identity between the second variation in its standard form and an expression for the second variation in positive definite form. Such an approach was natural, and avoided the ad hoc nature of Jacobi's transformation. Spitzer (1854, p. 1025) himself stated that he had developed a much simpler method "by means of which Jacobi's complicated and difficult transformation is avoided." However, Spitzer did adopt Jacobi's key insight of working with variations obtained as partial derivatives of functional solutions to the Euler equation to effect the transformation. The framework erected by Spitzer became the basis for the theory of the second variation in the modern subject.

Both Mainardi and Spitzer also investigated the problem where there is more than one dependent variable, and made some important progress in its formulation and solution. The particular issues that arise in extending the theory to multiple dependent variables resemble those that arise in the traditional problem where there is only one dependent variable and where higher-order derivatives appear in the variational integrand.

Clebsch was a product of the Königsberg school, having studied there in the early 1850s under Richelot, Neumann, and Hesse. His investigation of the second variation was carried out early in his career, when he was in his middle twenties, at a time when he was also working on problems in applied mathematics. He would subsequently leave these fields and for the remaining years of his somewhat short life achieved distinction for his research in algebra and the theory of invariants. His aptitude and tastes in mathematics were already evident in his research in the calculus of variations, which was distinguished by its combinatorial skill and formal brilliance.

In two memoirs published in Crelle's journal in 1858, Clebsch developed the theory in a systematic way for the general variational problem involving multiple dependent variables. Clebsch's paper followed Hesse's paper in Crelle by only a

[16]Spitzer came from a Jewish family in Mikulov in Moravia. The second variation in the calculus of variations was his first area of research in mathematics. He is perhaps best known for his writings on the Laplace transform, where he championed the priority of Laplace and became involved in a dispute with his Vienna contemporary Joseph Petzval (see Deakin 1981). Spitzer's career was at the Vienna Handelsschule and he also taught at the Polytechnische Institut in Vienna.

year, and Hesse is the only researcher other than Jacobi cited by Clebsch. In the literature on the calculus of variations, Clebsch's research is viewed as a fairly direct continuation of Hesse's work. Nevertheless, the conception at the base of Clebsch's approach is very similar to the method of Spitzer (and Mainardi before him). Clebsch formulated the theory for arbitrarily many dependent variables. (Of the earlier researchers, only Mainardi and Spitzer had attempted a detailed treatment of problems with more than one dependent variable.) Clebsch also made the basis for the whole theory an identity between the second variation and an expression in positive definite form. The prominent line of research involving Jacobi and Hesse's special transformation is completely absent in Clebsch's memoirs. By contrast, some of the leading ideas of his investigation are echoed in the work of Spitzer, although Clebsch succeeded brilliantly at the difficult task of a completely general transformation of the second variation.

Clebsch developed the theory from the outset at a high level of generality, with none of the careful consideration of particular problems and cases that had occupied earlier researchers. This approach brought with it a heavy reliance on notation in which results were derived and expressed in detailed formulas and equations and in which the entire subject was experienced intellectually at a rather abstract level. Given the formal difficulty of the project Clebsch set for himself in his 1858 articles, it is possible that he was only successful because he possessed such a cast of mind.

In a departure from tradition, Clebsch and later Mayer took as primary the general problem in which there are side conditions present in the form of differential equations, the so-called problem of Lagrange. In 1797 and 1801, Lagrange had shown how the variational equations may be obtained in this situation by means of a multiplier rule. Suppose, for example, that there are two dependent variables $y_1=y_1(x)$ and $y_2=y_2(x)$ and the variational integral has the form

$$\int_{x_0}^{x_1} f\left(x, y_1, y_2, y_1', {y_2}'\right) dx. \tag{1}$$

Consider now the traditional case in which there is one dependent variable y and the variational integrand contains first- and second-order derivatives of y:

$$\int_{x_0}^{x_1} f\left(x, y, y', y''\right) dx. \tag{2}$$

Clebsch showed that the latter problem can be reduced using the multiplier rule to the case where the variational integral is of the form (1). In the latter case, introduce the auxiliary condition

$$y_2 - {y_1}' = 0, \tag{3}$$

and form the function $f+\lambda(y_2-y_1')$, where λ is a multiplier function. Applying the multiplier rule to $\int_{x_0}^{x_1}\left(f+\lambda\left(y_2-y_1'\right)\right)dx$, we have the Euler equations

$$\frac{\partial\left(f+\lambda\left(y_2-y_1'\right)\right)}{\partial y_1}-\frac{d}{dx}\frac{\partial\left(f+\lambda\left(y_2-y_1'\right)\right)}{\partial y_1{}'}=0, \tag{4a}$$

$$\frac{\partial\left(f+\lambda\left(y_2-y_1'\right)\right)}{\partial y_2}-\frac{d}{dx}\frac{\partial\left(f+\lambda\left(y_2-y_1'\right)\right)}{\partial y_2{}'}=0. \tag{4b}$$

Let us suppose that the function f does not contain y_2 so that $f=f(x,y_1,y_1{}',y_2{}')$. Then (4a) and (4b) reduce to

$$\frac{\partial f}{\partial y_1}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_1{}'}-\lambda\right)=0, \tag{5a}$$

$$\lambda-\frac{d}{dx}\left(\frac{\partial f}{\partial y_2{}'}\right)=0, \tag{5b}$$

Equations (5a) and (5b) in turn become

$$\frac{\partial f}{\partial y_1}-\frac{d}{dx}\frac{\partial f}{\partial y_1{}'}+\frac{d^2}{dx^2}\frac{\partial f}{\partial y_1''}=0, \tag{6}$$

which is the Euler equation corresponding to integral (2).

By applying the multiplier rule in the case where there are additional dependent variables, we may repeat this argument to obtain the Euler equation when f contains derivatives of arbitrary order of y with respect to x. Clebsch was thereby able to formulate a general variational problem using only the first derivatives of the dependent variables. His approach may have been motivated by analytical dynamics, where the Hamiltonian function (as it later became known) was a function of position and momentum variables involving only first derivatives. Employing methods from Hamilton–Jacobi theory, Clebsch went on to write the variational equations in canonical form and posited a solution of these equations containing canonical constants of integration.

Taking off from Clebsch Mayer focused on single-integral variational problems and formulated the latter for the case in which the variational integrand contains arbitrarily many dependent variables and where only first derivatives of these variables appear in the integrand. The analysis was also developed for the general problem of Lagrange. Mayer undertook a detailed investigation of the variations required in the transformation of the second variation. The coefficients that appear in these variations will be subject to certain relations that arise from auxiliary differential equations that appear in the transformation. Serious study of this subject had begun with Delaunay in 1841. The investigation of these relations was a crucial thread in the historical development of the theory from Jacobi to Mayer. Not only must suitable variations be found, but it must also be shown that it is possible to find among such variations ones for which the transformed variational integrand remains finite on the given interval. In special cases (where only two or three dependent variables are present), it was a matter of some difficulty to establish the existence of the requisite variations.

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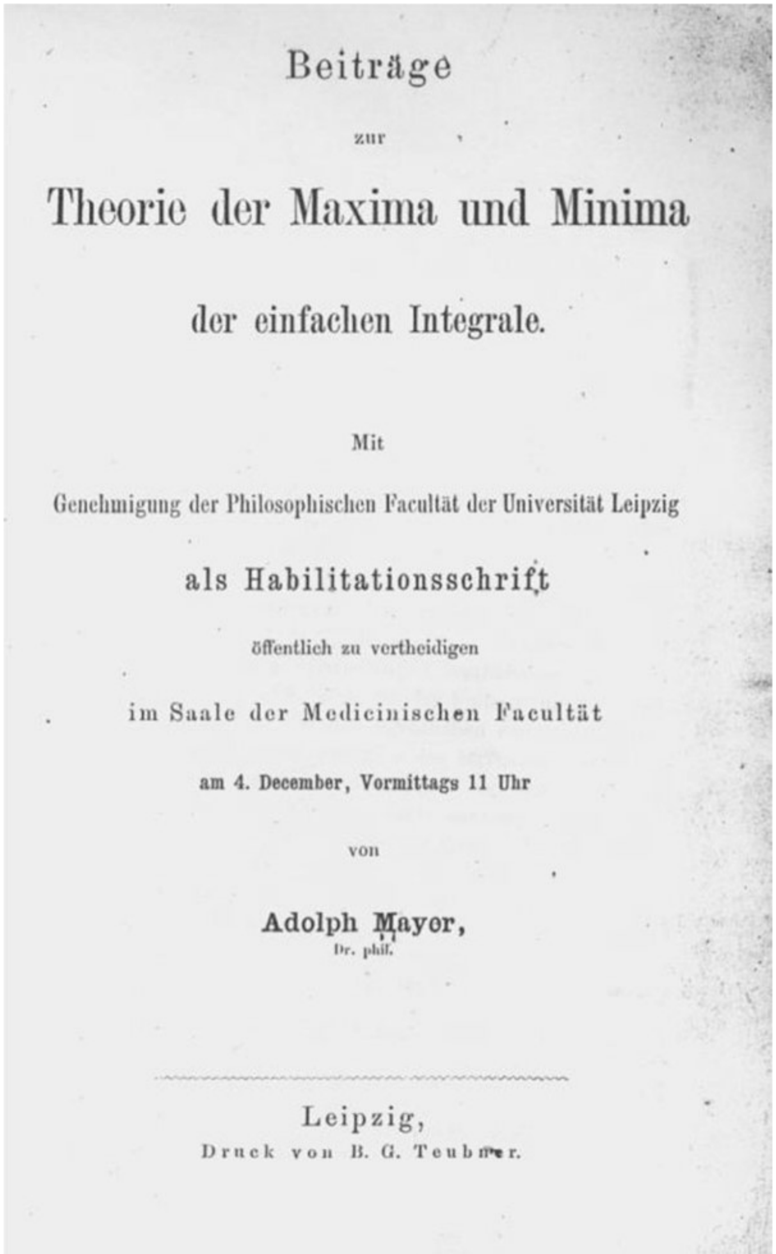
A textbook published by Lorenz Lindelöf and Abbé Moigno in 1861 indicated the point of view of advanced research of the period. These authors recognized that it is necessary to ensure that there is no admissible variation which makes the second variation equal to zero. This aspect of the analysis was seen as straightforward, a preliminary condition that must be verified before a more detailed study is possible. Their emphasis was decidedly on the problem of transforming the second variation in order to obtain Legendre's condition. They drew attention to the need to find a system of functions in the transformation for which a certain determinant in the denominator of the integrand of the second variation is nonzero throughout the given interval. The coefficient constants appearing in these functions must satisfy certain auxiliary relations. Lindelöf and Moigno (1861, p. 192) observed: "The question of recognizing if such a system of constants does or does not exist is in reality the most delicate part of Jacobi's theory, and it awaits yet a general solution."

Let us assume that in the given problem under consideration there is no conjugate point. In his habilitation work *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale* (1866) (Figure 1), Mayer was able to show that if this condition holds then it is possible to find suitable auxiliary functions involving the canonical constants that allow one to carry out the transformation of the second variation and infer Legendre's criterion.[17] In essence, his derivation was devoted to a detailed analysis of the various possible systems of arbitrary constants and to establishing the existence of a suitable set of functions involving these constants that could be used in the transformation. Mayer published an abbreviated and refined version of his result two years later in an article in Crelle's journal.[18] With this result, he had resolved the central theoretical question in the classical theory of the second variation.

The theoretical sensibility that Mayer brought to the subject was also evident in some later work of the young researcher Edmund Husserl (1859–1938). Husserl came from a Jewish family in Moravia (today in the Czech Republic and then part

[17]One gets a sense for the theoretical character of mathematics in Mayer's time in the theses that he included as part of the public defense of his Habilitation. Such theses were often only distantly connected to the subject of the Habilitation. Including among the eight theses presented by Mayer were the following: in purely analytic problems, geometric considerations are insufficient; infinitesimals may lead to new mathematical theorems but cannot be part of their proof; Lagrange's multiplier method in the calculus of variations is in need of a rigorous and scientific foundation; in optics the ether must be taken to be incompressible; and the law of inertia is a pure truth of experience.

[18]Mayer's (1866) *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale* seems to have had rather limited distribution, and is not widely cited in the literature on the calculus of variations. There are copies at the major German university libraries. There is a copy in the British Library, but no copy in the Bibliothèque nationale de France. In the United States, no copies exist in the libraries of Harvard and the Library of Congress. While the University of Chicago was only founded in 1890, it acquired many older books in the calculus of variations (an important field of mathematical research at Chicago into the middle of the twentieth century), although Mayer's book was not among them. There are copies at Yale and Princeton, although it is not indicated in their catalogues when the book was acquired. Of course, every major university possesses Crelle's journal, in which Mayer's (1868) article appeared. Today Mayer's *Beiträge* is available online at Google Books.



Beiträge

zur

Theorie der Maxima und Minima

der einfachen Integrale.

Mit

Genehmigung der Philosophischen Facultät der Universität Leipzig

als Habilitationsschrift

öffentlich zu vertheidigen

im Saale der Medicinischen Facultät

am 4. December, Vormittags 11 Uhr

von

Adolph Mayer,

Dr. phil.

Leipzig,

Druck von B. G. Teubner.

Fig. 1 Title page of Mayer's 1866 habilitation of the Austrian Empire), and studied mathematics at the Universities of Leipzig (1876–1878), Berlin (1878–1881), and Vienna (1881–1883). In Berlin, he attended and made notes on the lectures of Karl Weierstrass in the calculus of variations. In November 1882, he completed his dissertation at the University of Vienna under the direction of Leo Koenigsberger. The title of his handwritten thesis was *Beiträge zur Theorie von Variationsrechnung* ("Contribution to the Theory of the Calculus of Variations"). As he embarked on the revision and preparation for publication of this work, he became interested in philosophical and religious subjects, soon abandoning the dissertation project and the field of technical mathematics forever. He went on to achieve fame as the founder of the phenomenological movement in philosophy. His dissertation in mathematics was eventually published in 1983 in French translation.

Husserl's dissertation contained original insights into the methods of Jacobi and Mayer, and displayed an unusual sensitivity to the theoretical side of the subject. His most important contribution was to show that the existence of a given system of constants required in the Clebsch–Mayer transformation was a consequence of quite general considerations that make no reference to any particular process of selection. As Husserl (1882, p. 38) put it,

> Although it [Mayer's method] leads to the results in a rigorous way, such a method possesses certain disadvantages. To make a completely special determination of the constants involves necessarily an element of chance, of something arbitrary, and does not make clear the basis of things. Although there is then nothing to add to these results, it would not perhaps be without interest to find a general and natural procedure, free of all secondary calculations, in order to arrive directly at the criteria, starting out from the transformation of Clebsch and Jacobi.

Essentially, Husserl's underlying insight involved properties of linear algebraic equations that belong to what is known today as linear algebra.

In retrospect, one could say that the particular approach adopted by Mayer was essentially contingent and arose from his tacit understanding that the coefficients appearing in the transformation must be given in terms of the constants of integration, and more particularly in terms of those constants involved in the integration of the variational equations expressed in canonical form. Clebsch's second paper (1858b) would have reinforced this point of view, providing as it did a general characterization of the transformation functions in terms of the canonical constants of integration.

5 Conclusion

Euler's primary achievement in the calculus of variations consisted of synthesizing a large number of examples and problems in terms of standard differential equational forms. Although he did so with great skill and inventiveness, he did not otherwise advance the development of the subject as a mathematical theory. Lagrange in his books of 1797 and 1806 displayed a more systematic theoretical understanding of the subject, in the sense of recognizing questions of deductive organization and linkage between different parts of the subject. Nevertheless, his primary orientation remained centered on the development of techniques and methods that arose more or less directly in the solutions to definite problems. While there was certainly an underlying rational basis to the wider analysis of Euler and Lagrange, it was implied by their way of doing analysis rather than consciously formulated as a set of principles.[19]

Mayer's paper by contrast concerned a purely theoretical question, the problem of identifying and relating the different conditions which are implicit in a general analysis of the second variation. Although this question is a fundamental one, it is remarkable how much work was done in the period 1837–1870 without any recognition of the point at issue. The advanced character of Mayer's investigation is apparent when one compares it to contemporary accounts in the textbook literature of the period. Focused on the exposition of particular methods and examples, these accounts failed to connect the different conditions involved in Jacobi's theory. A substantial part of Mayer's achievement was simply to clearly identify the fundamental theoretical features of the problem and to recognize the need to establish from first principles the connections between the given conditions.

Mayer's result represented mathematical work of a qualitatively different sort from the kind found traditionally in the subject. He focused on a detailed analysis of the conditions needed to ensure an extremum, and established the existence of a transformation required to infer the validity of these conditions. His investigation embodied a theoretical awareness and a consciousness of the logical interdependence of the parts of the theory that was not present in the eighteenth-century paradigm of analysis, a paradigm it should be noted that was still influential in the decades leading up to the 1860s.

The work of Clebsch and Mayer was at a higher level of sophistication than the textbooks and research memoirs of contemporary French and English mathematicians. Although Mayer's result was a key contribution to the study of the second variation, it would become part of the specialist literature in the calculus of variations. Clebsch and Mayer seem to have conceived their audience in terms of a rather small group of advanced researchers. In a manner that is sometimes typical of mathematical investigators, they made a limited effort to reach out to a broader readership. They were participants in an ivory tower of cutting-edge research, players in a *Glasperlenspiel* of pure mathematics. Mayer chose to develop the analysis at a very general level within a framework involving the multiplier rule and Hamilton–Jacobi methods of integration. These powerful mathematical tools, as well as the rather complicated way in which they were deployed in the proof, served to make his achievement more difficult and abstruse than in fact it was. It did not help that Mayer's 1868 paper in Crelle's journal was a highly refined and very economical work that eliminated much of the explanatory and motivational discussion of his Habilitationsschrift. One result of the rather abstruse character of

[19]For a critical account of the outlook and mentality that d'Alembert, Euler, Lagrange, and other eighteenth-century figures brought to their work in analysis (see Ferraro 2008).

Mayer's paper is that the full significance of his result was not always appreciated by later researchers.[20]

While Mayer's theory may not have achieved widespread recognition, there were several later authorities who appreciated the value of his work and provided expositions of his main result. Accounts of the Clebsch–Mayer development were presented by Camille Jordan (1838–1922) (1896), and Gustav Ritter von Escherich (1849–1935) (1899). While Oscar Bolza (1857–1942) (1909) does not seem to have understood fully all aspects of Mayer's result, he provided an account of it and gave references to the literature, and his textbook *Vorlesungen über Variationsrechnung* was a widely read work on the calculus of variations in its modern form.

6 Epilogue

The theory of Mayer and Clebsch and indeed all work on the problem of sufficiency up to that point may be characterized generally in terms of what are known as expansionist methods. The differential or increment of the variational integral is expanded as a Taylor series, and the question of sufficiency is investigated by examining the second variation. In the 1880s, Karl Weierstrass (1927) gave lectures at the University of Berlin on the calculus of variations, copies of which circulated in the last years of the century and which were eventually published in 1927. Working at a high theoretical and critical level, Weierstrass set out a new approach to the problem of sufficiency. This approach became the subject of a substantial research program in the period 1890–1920. Stimulated by some contributions of David Hilbert (1862–1943), it led to the creation of what is known as field theory in the calculus of variations.[21] The success of the Weierstrassian program tended to overshadow the earlier contributions of Clebsch and Mayer and may have contributed to the relative neglect of their work. There were certainly prominent researchers such as von Escherich (1899) who continued to uphold the older expansionist methods and found the Weierstrassian field theoretic approach to be somewhat artificial. Nevertheless, proponents of field theory dominated the study of sufficiency in the first half of the twentieth century, and the earlier work of Mayer receded into history.

[20] The biographies of Mayer in the *Dictionary of Scientific Biography*, the St. Andrews *MacTutor* website, and *Wikipedia* are based on the obituaries by VonderMühll (1908) and Liebmann (1908). Both of these authors mention Mayer's Habilitation without providing any details concerning its contents or significance.

[21] For a detailed history of this development (see Thiele 2007); Fraser (2009) provides an English-language essay review of Thiele's book.

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