Bulletin CSHPM/SCHPM November/Novembre 2025 Number/le numéro 77, pp. 11-13.
Review by Craig Fraser,

Book Review: Real Functions of One Variable

Geschichte der reellen Funktionen einer Veränderlichen: ein quellenorientierter Abriss der Entwicklung vom Beginn des 17. bis zur Mitte des 20. Jahrhunderts (History of Real Functions of One Variable: A Source-oriented Outline of Developments from the Beginning of the 17th to the Middle of the 20th Century), by Rüdiger Thiele. Münster: WTM – Verlag für wissenschaftliche Texte und Medien, 2022, 530 pp. €74,90.

Rüdiger Thiele has written books on Leonhard Euler and the history of the calculus of variations. He was the invited speaker at the 1997 meeting of the CSHPM in St. John's, where he spoke on Euler, and the 2000 meeting in Hamilton, where he presented the paper "Early Calculus of Variations and the Concept of Function." Both lectures appeared in Glen Van Brummelen and Michael Kinyon, eds., *Mathematics and the Historian's Craft: The Kenneth O. May Lectures* (Springer, 2005), pp. 81—140, 243–295. Thiele also participated in the joint meeting of the CSHPM and BSHM in 1999 in Toronto.

The 530-page book under review is a history of real analysis from the seventeenth century to the early twentieth century. The frontispiece (Figure 1) is a portrait of Carl Friedrich Gauss with the inscription (in translation)

> One should never forget that functions, like all mathematical concepts, are only our own inventions, and that when the definition one started with ceases to make sense, one should not really ask "what is?" but rather assume what is convenient, so that one may remain consistent.

The function concept has been a *Leitmotif* in the development of mathematics since the eighteenth century. It is the essential thread in Thiele's book, connecting the work of Euler, Lagrange, Cauchy, Dirichlet, Weierstrass, Riemann, and others. The function becomes a symbol of the *Weltanschauung* of modern analysis.

The second and third chapters provide a detailed account of developments in seventeenth and early eighteenth-century mathematics. Subjects of interest include logarithms, analytic geometry, and the invention of calculus. The narrative really takes off with Euler in chapter three. Drawing on ideas in the writings of Johann and Jacob Bernoulli, Euler's achievement was to formally introduce the function concept and to organize various parts of analysis around it. His achievement comes into focus when one compares what he did in the 1740s and 1750s with contemporary writings on calculus. For example, Colin Maclaurin's 1742 *Treatise of Fluxions* was a major work of great foundational interest, but the function concept was absent in his development of the subject. Nor does it figure in Maria Gaetana Agnesi's celebrated books of 1748 on calculus.

Man sollte überhaupt nie vergessen, dass die Funktionen, wie alle mathematischen Begriffszusammensetzungen [Konstruktionen], nur unsere eigenen Geschöpfe sind, und dass, wo die Definition von der man zeuging aufhört einen Sinn zu haben, man eigentlich nicht fragen soll was ist? sondern was *convenirt* [passend ist]? annehmen, damit ich immer consequent bleiben kann.

Carl Friedrich Gauß über Funktionen

In einem Brief an Bessel vom 21. November 1811

Figure 6: Thiele's frontispiece.

Morris Kline has observed that in the nineteenth century, "Partial differential equations became and remain the heart of mathematics" (*Mathematical Thought from Ancient to Modern Times* (1972, p. 671)). A substantial section of Thiele's book is devoted to examining how the function concept was deployed in the investigations of these equations, with the development of Fourier analysis being a major area of concern. The investigation of trigonometric series and the representation of functions led to the identification of "pathological" functions. One example consisted of a function that was everywhere continuous but nowhere differentiable. Weierstrass in 1872 produced a famous example of such a function. Interesting work in this vein that is less well-known was carried out by his Swiss contemporary Charles Cellérier. Kline (ibid, p. 954) observes that the discovery of pathological functions "made mathematicians all the more fearful of trusting intuition or geometrical thinking."

In Euler's time there was no distinction between real and complex analysis. The emergence of complex analysis was a major event in the nineteenth century. Thiele's focus on real functions arises for pragmatic reasons and in acknowledgement of the fact that the

Figure 7: Rüdiger Thiele

consideration of complex analysis would require another book. On the relation of the two branches of analysis he (p. 13) defers to Gauss, who wrote in an 1811 letter to Friedrich Bessel:

> First of all I would ask someone who wants to introduce a new function into analysis to explain whether he wants it to be applied strictly to real quantities (real values of the function's arguments), and considers the imaginary values of the arguments as a kind of superfluous element, or whether he subscribes to my principle that in the realm of quantities, the imaginary $a+b\sqrt{-1}=a+bi$ must be viewed as enjoying equal rights with the real ones... Analysis is to me an independent science, which would lose a great deal of beauty and completeness by the elimination of those fictitious quantities, and would be forced at every moment to add extremely burdensome restrictions to truths that are otherwise generally valid.

Included in Thiele's book are several Russian mathematicians not as well-known as their European contemporaries. Although we tend to associate Nikolai Lobachevsky with geometry, he also contributed to analysis, writing in Russian and providing a source and vocabulary for that subject in his country. (Lobachevsky is introduced with some lines from Mikhail Lermontov's poem *The Sail*: A lonely sail is flashing white/In the blue mist of the sea! .../What does it seek in foreign lands?/What did it leave behind at home? ...) Lobachevsky considered a more general concept of function involving an arbitrary relation among variables, a conception that was also found in Dirichlet. Of note here is historian F. A. Medvedev's 1991 book *Scenes from the History of Real Functions* (Birkhäuser). (Medvedev also makes central the concept of a function and its role in the development of real analysis.) Other Russian analysts encountered in Thiele's book are Mikhail Ostrogradsdky, Pafnuty Chebyshev, Viktor Bunyakovsky, and Dimitri Egorov.

Thiele extends his account of functions to logicians and philosophers of mathematics, most notably Gottlob Frege, who in 1891 published the article "Function and concept." Frege's general view was to regard number as something that is defined by the context in which it appears, this context being given by propositions and language. Frege adopted this view to avoid psychological or formal conceptions of number, which he considered to be untenable. His investigations of the function notion involved both the philosophy of mathematics and the philosophy of language, and it is the latter that is sometimes seen as his most original contribution.

The book contains attractive reproductions of text and images, as well as a range of portraits of historical personages. Mathematical exercises at the end of each chapter connect the subject matter to problems of concrete interest. Each chapter ends with a valuable bibliography of relevant literature.

Craig Fraser

AGM of CSHPM/SCHPM

The Annual General Meeting of the Canadian Society for History and Philosophy of Mathematics took place at George Brown College, Toronto, ON, on June 1, 2025. The meeting was called to order at 12:30 pm EDT by Robert Bradley, President, with 22 members present.

AGM Agenda

1. Approval of agenda