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THE CONCEPT OF ELASTIC STRESS IN EIGHTEENTH-CENTURY MECHANICS: SOME EXAMPLES FROM EULER*

CRAIG G. FRASER†

Introduction. In the historical evolution of any highly mathematized physical theory there occurs an interesting interplay between the development of physical concepts and ideas, on the one hand, and the application and extension of mathematical techniques on the other. The subject of continuum mechanics offers in this respect a very clear example. Combining mathematical elements taken from calculus, analytic geometry and linear algebra, it provides a general and powerful approach to the investigation of physical phenomena, one based nevertheless on a specific and highly idealized conception of the way in which matter interacts.

The subject as it emerged in the writings of Navier and Cauchy in the 1820s had roots in several different branches of pure and applied science.[1] Considerable knowledge was available about engineering design and the strength of material structures, a field of research that had been investigated in the eighteenth-century and was familiar to French-trained engineers in the post-Revolutionary period. Theoretical mechanics, the science that became known in nineteenth-century France as "rational" mechanics, had reached a very high level of development in the classic treatises of Euler, Lagrange and Laplace. Finally, the advent of the wave theory of light led to an interest in analyzing wave phenomena based on an elastic-solid model of the ether.

In surveying the eighteenth-century background to Cauchy's work one it struck by how little of the modern theory was in place in the period before Coulomb. A wide range of problems was considered, numerous special results were reached, and sophisticated mathematical techniques were frequently employed; overall, however, the subject remained in a very rudimentary state of development. Even the most basic results concerning the distribution of shear stress in beams, for example, were beyond the scope of the theory, such as it was then.

The present paper seeks to examine aspects of the early history of the concept of stress, focusing on examples from the work of Euler, and to indicate some of the difficulties that were involved in the original formulation of the stress principle. It is intended as a case study in concept formation

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[1] Belhoste [1991, Chapter 6] provides a discussion of Cauchy's early work on elasticity.

within the domain of mathematical physics. In reference to physical science in the eighteenth-century the historian Thomas Hankins [1985, 30] has observed: ". . . then as now physical concepts developed much more slowly than did new mathematical techniques. Physical concepts are extraordinarily difficult to create. Mathematical methods may not be any simpler, but when they are needed they seem to be found much more quickly."[2]

The stress principle. The stress principle is used to analyze the static and dynamic behaviour of a continuous body when it is acted upon by applied forces. There are two distinct ideas involved in this principle. The first, which is logically independent of the concept of stress *per se*, is that a body may for the purposes of analysis be imagined to be divided into two parts: one of these parts is isolated for study and the effect of the second part upon the first is replaced by a specified set of forces acting at the boundary. The second idea is that the given separation into two parts may be regarded as being accomplished by a plane of division, with the effect of the second part on the first being understood in terms of forces per unit area or stresses distributed over this plane and acting across it.

It is this second idea that is taken up and developed in the full theory of stress. One distinguishes between normal and shear stress and analyzes the mechanical behaviour of the body in terms of the relations between the various stress components acting on an infinitesimal cube of material contained in the body.

Background to Euler's early research. In the early eighteenth century there was considerable work on determining the shape assumed by a perfectly flexible body under various loadings. The basic idea, developed by the brothers Johann and Jakob Bernoulli and published by Jakob Hermann, was to analyze the tensions acting on a segment of a hanging cable and to obtain a differential equation that described the resulting curve. Since the cable was regarded as inextensible and perfectly flexible no assumptions were necessary concerning the elastic behaviour of the material comprising the cable.

Assumptions about elasticity did enter into research on two distinct and important problems, the problem of fracture and the problem of the elastica. In the study of fracture one attempted to determine the maximum load that a beam of given material and dimensions can sustain without breaking. In the problem of bending one was concerned with determining the shape assumed by a rod or lamina in equilibrium when subject to given

[2] Hankins is referring to the history of physical science where the mathematical techniques in question were already largely in place. It should be noted that within mathematics itself the process of concept formation is far from straightforward. The establishment of such subjects as Galois theory, differential geometry or partial differential equations illustrates the slowness and difficulty with which the relevant techniques and concepts were identified and developed. On a more philosophical level one could argue that the absence of an empirical referent in pure mathematics places a special premium on the formation and development of concepts there.

external forces. In the case of the elastica these forces were assumed to act at the ends of the rod causing it to bend into a curve.

In the problem of fracture researchers such as Leibniz, Varignon and Parent obtained results that can be readily interpreted in terms of modern formulas and theory. Typically they assumed that the beam was joined transversally to a wall and that rupture occurred at the joining with the wall. Here the physical situation directly concentrated attention on the plane of fracture. The conception then current of the loaded beam as comprised of longitudinal fibres in tension is readily understood today in terms of stresses acting across this plane.

In the problem of elastic bending by contrast researchers were much slower to develop an analysis that connected the phenomenon in question to the internal structure of the beam. Here there was nothing in the physical situation that identified for immediate study any particular cross-sectional plane. In all of Jakob Bernoulli's seminal writings on the elastica the central idea of stress fails to receive clear identification and development.

Two papers by Euler. The formative period in the development of Euler's early research in mechanics occurred during the ten years following his move at age twenty to St. Petersburg in 1728. A colleague of Hermann's (until 1731) and Daniel Bernoulli's, he benefited from the favourable intellectual atmosphere that existed at this time for research in the Russian academy. If one surveys his early work on mechanics it shows a familiarity with Johann and Jakob Bernoulli's writings in the *Acta Eruditorum*, Newton's *Principia Mathematica*, Hermann's *Phoronomia*, Taylor's *Methodus incrementorum* and Varignon's various papers on particle dynamics in the Paris Academy's memoirs.

(a) Euler's "De oscillationibus annulorum elasticorum"

An example illustrating how Euler understood elastic deformation during the period is provided by an unpublished paper, dating (we believe) from sometime the 1730s.[3] Euler considered an "annulus" (washer-like ring) that is disturbed from its equilibrium position and set as a consequence into motion. Figure 1 shows a part $abBA$ of the ring in its normal and stressed configuration. The segment $AaeE$ is regarded as being composed of concentric filaments. The inner line ae remains constant under deformation while the outer line AE is stretched to $A\varepsilon$. The triangle $eE\varepsilon$ shows the stretching of the filaments as one proceeds outward from a to A. Euler sets $c = Aa$, $ds = ab$ and $dt = E\varepsilon$.

[3] The essay was eventually published in 1862. Dating of the paper is discussed by Fraser [1992, p. 244, n.6], who suggests the middle 1730s as the most plausible time of composition.

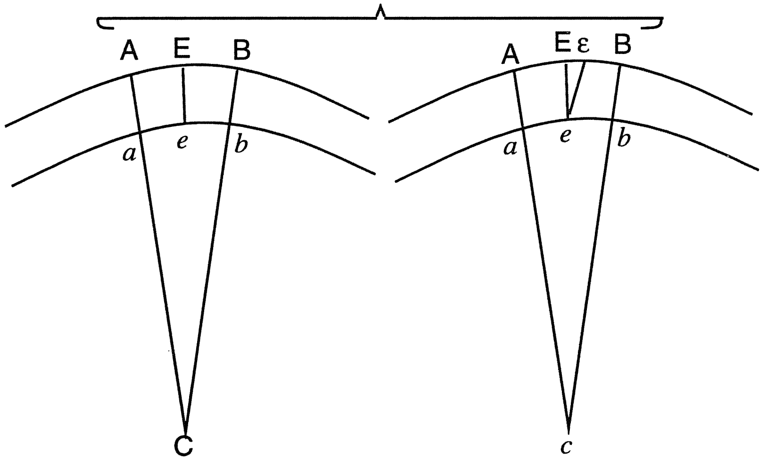

Figure 1 (from Euler [1862, Fig. 3])

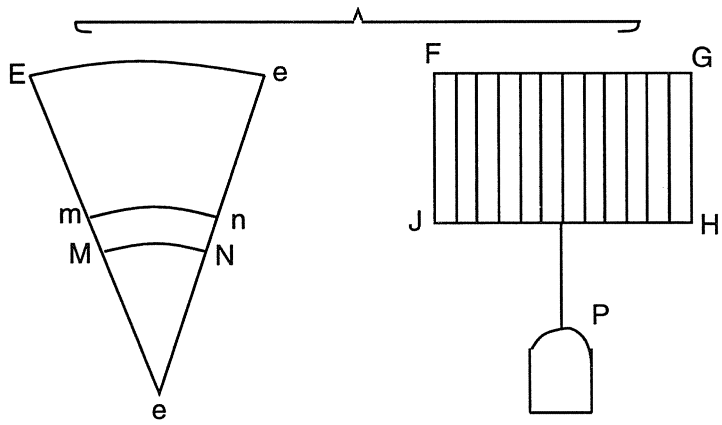

Figure 2 (from Euler [1862, Fig. 4])

To obtain a measure of the elastic force Euler considered the material membrane $FGHJ$ (Figure 2) consisting of the extended part of a series of stretched filaments. $FJ = g$ is the magnitude of the extension and $FG = f$ is the width of the membrane. He supposed that the weight P is sufficient to sustain this stretching so that P/fg is a measure for the given material of the elastic force per unit of extension and per unit of width.

The stretched part of the ring segment $AaeE$ is comprised of the triangle $eE\varepsilon$ (Figure 2). Consider the portion $MNnm$ of $eE\varepsilon$ located a radial distance $eN = x$ from e. $MNnm$ is composed of a series of concentric extended filaments bounded by MN and mn. Since $MN = (x/c)\ (E\varepsilon) = (x/c)dt$ the area of $MNnm$ is $(xdxdt)/c$. The elastic force that gives rise to

$MNnm$ is therefore equal to $(Pxdxdt/cfg)$. Euler calculated the moment of this force about the point e to be $(Px^2dxdt)/c^2fg$. (Rather curiously he took x/c instead of x as a measure of the moment arm eM.) By integrating this expression from 0 to c he obtained a value of the total "force of cohesion", $(Pcdt)/(3fg)$.[4] By relating this formula to the radii of curvature of ab in its normal and deformed states he arrived at an expression that he was able to use to investigate the vibratory motion of the ring.[5]

What is striking in Euler's treatment of this problem is the absence of anything that could be interpreted from a later perspective as an application of the stress principle. The elastic forces that arise are regarded as being distributed over the plane in which they act, not over a transverse cross-section. This plane is not regarded logically as something that divides the body into two parts; rather it is viewed as a component element (rather like a membrane) of the body upon which it is a necessary to calculate the forces acting. These forces are themselves viewed as an absolute function of the displacement dt. Thus Euler lacked the concept of elastic strain. The formula $(Pcdt)/(3fg)$ itself fails to relate in a satisfactory manner the bending moment to the cross-sectional structure of the ring.[6]

(b) Euler's "Solutio problematis de inveniendi curva quam format lamina utcunque elastica in singulus punctis a potentiis quibuscunque sollicitata"

Although unsuccessful Euler's paper "De oscillationibus" is of interest because of the detailed picture it presents of his understanding at this time of elastic phenomena. In this published writings he would abandon any attempt at analyzing the elastic behaviour of a body in terms of its internal constitution. The approach that he would follow publicly throughout his career was established in the 1732 paper "Solution of the problem of finding the curve formed by an elastic lamina acted upon at each point by arbitrary soliciting powers". Its purpose was to provide a coordinated treatment of the various results then available concerning mechanical lines, the catenary, parabola, elastica, velarium, lintearia and so on.

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[4] The text that is reprinted in the *Opera* has $Pc^2dt/3fg$ rather than $Pcdt/3fg$. (The latter is what actually follows from the preceding step of the derivation.) That this is a misprint is evident from the fact that Euler immediately sets $dt = (a-b)cds/ab$ and obtains $Pcc(a-b)ds/3abfg$, the formula he actually works with in the paper. (That the c^2 is a typographical error is also clear from the fact that the Euler always denotes the square of a single-letter variable a by aa rather than a^2.)

[5] A detailed critical account of Euler's analysis is presented by Cannon and Dostrovsky [1981, 37–43].

[6] Through a series of "corrections" it is possible to obtain the modern formula for the bending moment of a prismatic beam from Euler's procedure. In calculating the moment of the elastic force we use x rather than x/c as the moment arm; we replace dt by the strain dt/ds; we incorporate the thickness h of the ring into the final formula; finally, we interpret $E = P/fg$ as "Young's modulus". With these changes Euler's formula becomes $E(c^2/3)(dt/ds)$, which is the flexure formula for a prismatic beam in which the neutral axis is assumed to lie on an outer surface. (It is on the basis of an argument something like this that Truesdell [1960, 145] arrives at a high evaluation of Euler's paper.)

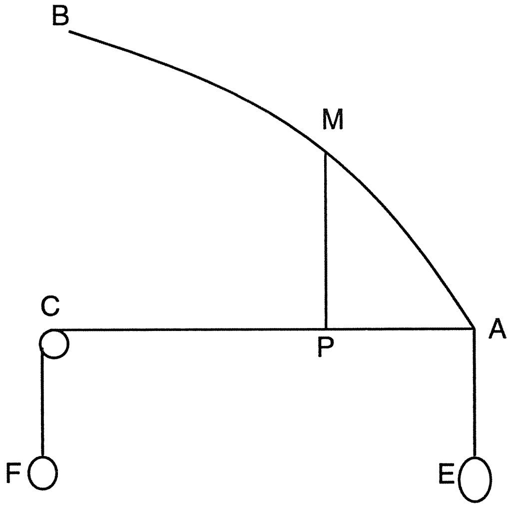

Figure 3 (from Euler [1732, Fig. 4])

In the "general problem" considered by Euler we are given a heavy lamina BMA that is subjected at its end to vertical and horizontal forces E and F (Figure 3). It is necessary to determine the shape of the curve subject to this loading. Consider any point M on the lamina. Euler took the resisting moment at M to be Av/r, where v is an elastic constant, A is a constant of proportionality and r is the radius of curvature of the curve at M. The resisting moment will balance the moment of all forces acting on the section of the system to the right of M. Let A be the origin of a Cartesian coordinate system, $AP = x$, $PM = y$. The moment about M of the forces E and F is $Ex + Fy$. Euler calculated the moment of the body forces acting on the lamina MA to be

$$\int_0^x P dx + \int_0^y Q dy,$$

where P and Q are equal to

$$P = \int_0^x F_V dx, \qquad Q = \int_0^y F_H dy,$$

F_v being the vertical force per unit distance along the x-axis and F_H being the horizontal force per unit distance along the y-axis. (Euler obtained these last expressions in a preliminary lemma. The result is demonstrated geometrically in terms of curves and graphs. The argument reduces to the following. If $M(x)$ is the moment due to the vertically-acting body forces

then we have $dM = \int_0^x (F_V(t)dx)dt$. Hence it is clear that $P = \int_0^x F_V(t)dt$.)
The total moment of all forces must balance the resisting moment, leading to the general equation

$$\frac{Av}{r} = Ex + Fy + \int_0^x Pdx + \int_0^y Qdy. \tag{1}$$

In the case of perfectly flexible bodies the elastic force $v = 0$ and (1) reduces to

$$Ex + Fy + \int_0^x Pdx + \int_0^y Qdy = 0. \tag{2}$$

If furthermore we suppose $Q = 0$ we have the case of the hanging cable given by

$$Ex + Fy + \int_0^x Pdx = 0. \tag{3}$$

Differentiation yields this equation in the form

$$Edx + Fdy + Pdx = 0. \tag{4}$$

Euler further differentiated (4) with respect to s, where s is the distance along the curve:

$$dP + \frac{Fdxddy - Fdyddx}{dx^2} = 0. \tag{5}$$

Because ds is constant we have $d(dx^2 + dy^2) = 0$ or $dxddx + dyddy = 0$. (5) may therefore re-expressed in the form

$$dPdx^2dy = Fds^2ddx. \tag{6}$$

An alternate form of (6) was obtained by Euler by setting the radius of curvature r equal to $dsdy/ddx$:

$$rdPdx^2 = Fds^3. \tag{6'}$$

Euler observed that all the various types of the catenary are expressed by (6′).

Euler illustrated (6) with several examples, one of which will be described here. Suppose the curve BMA is a segment of a light flexible hanging cable fixed at B from which a uniform heavy horizontal load is suspended (Figure 4). We have $E = 0$ and $dP = adx$ where a is a constant. F may here be regarded as the tension supporting the cable at A. Equation (4) becomes $adx^3dy = Fds^2ddx$, which integrated yields $ay = -(Fds^2)/(2dx^2)$. Because the applied force at A acts to the right we replace F by $-F$ and (using $ds^2 = dx^2 + dy^2$) arrive at $dx = (dy\sqrt{F})/\sqrt{(2ay - F)}$. This is the equation, Euler noted, of the Apollonian parabola.

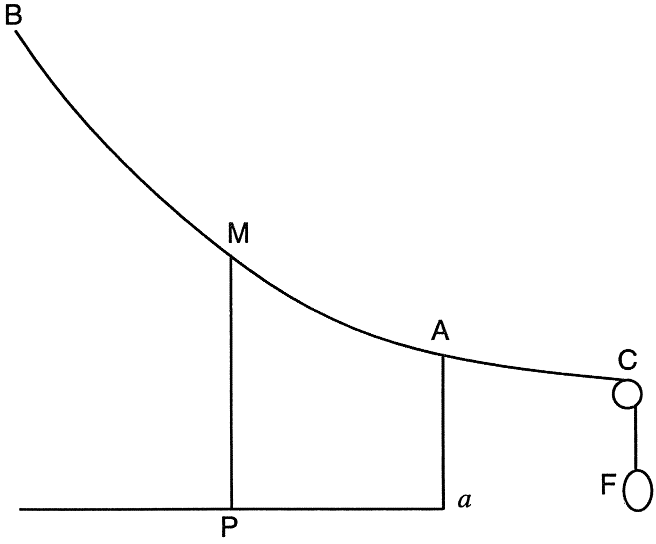

Figure 4 (from Euler [1732, Fig. 5])

Returning later in the paper to the more general equation (1) Euler considered examples in which the elastic constant v is non-zero, deriving differential equations to describe the resulting curve. His research here was preparatory to his famous essay of 1744 on the elastica in which he provided a full and systematic analysis of the different solutions obtained from (1) when F_H and F_V are zero.[7]

Let us turn now to a critical evaluation of Euler's 1732 paper. Earlier researchers in mechanics – Jacob Bernoulli and Hermann foremost among them – had analyzed the catenary and elastica separately. The equation of the catenary was obtained by examining tensions along a segment of the hanging cable following the approach that is more or less customary today. The equation of the elastica by contrast was obtained by examining the elastic behaviour of the fibres that make up the lamina and arriving at an estimate for the resisting moment at each point. Using the fact that the resisting moment of a perfectly flexible body is everywhere zero, Euler was

[7] Euler's essay of 1744 is the subject of [Fraser 1991].

able by means of equation (1) to provide a single method that covered both flexible and elastic lines.

Euler's condition for equilibrium is that the moment of all forces acting on the body to the right of a given point M must balance the resisting moment at M. In effect he is dividing the line into two parts, isolating for study the part MA, and examining at the boundary point M the effect of the second part BM on MA. This principle had been used implicitly by Jakob Bernoulli and Hermann when they equated the tension acting at a given point of a hanging cable to the sum of all forces acting on the cable to the right (or to the left) of this point. Euler himself came to the realization by 1750 that the principle could be used in balancing either the force or the moment acting at a point.[8] In a memoir of 1771, "Genuine principles of the doctrine of the equilibrium and motion of flexible and elastic bodies", he provided a more explicit statement of this condition, supplementing it with a derivation of the general differential equation relating the tension, normal shear and moment acting at an arbitrary point of a line. This paper, as well as one published five years later in 1776, contained Euler's finished theory of the static equilibrium of mechanical lines.

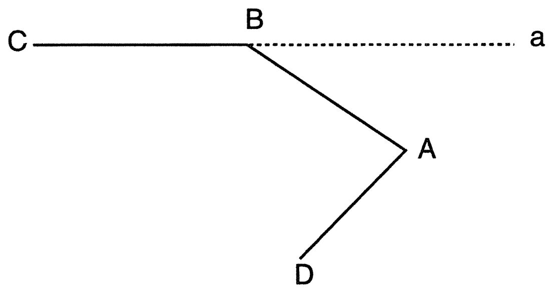

Figure 5 (from Euler [1732, Fig. 1])

Central to Euler's conception of an elastic line is the assertion that the resisting moment at a given point is proportional to the radius of curvature of the line there. This proposition is a consequence of the "hypothesis" presented at the beginning of the 1732 paper. Euler considered two rods CB and Ba joined initially at B in a straight line (Figure 5). The action of the force or power AD at A causes the segment Ba to assume the position BA. Euler stated that the moment of this power at B will be proportional conjointly to the "elastic force" at B and the angle aBA. This hypothesis, he wrote [1732, 71], "is commonly assumed and could probably be demonstrate physically if the angle aBA is extremely small."

[8] See Euler's unpublished notebook EH 5, pp. 268–269, cited in [Truesdell 1961, p. 395, note 1, example 3].

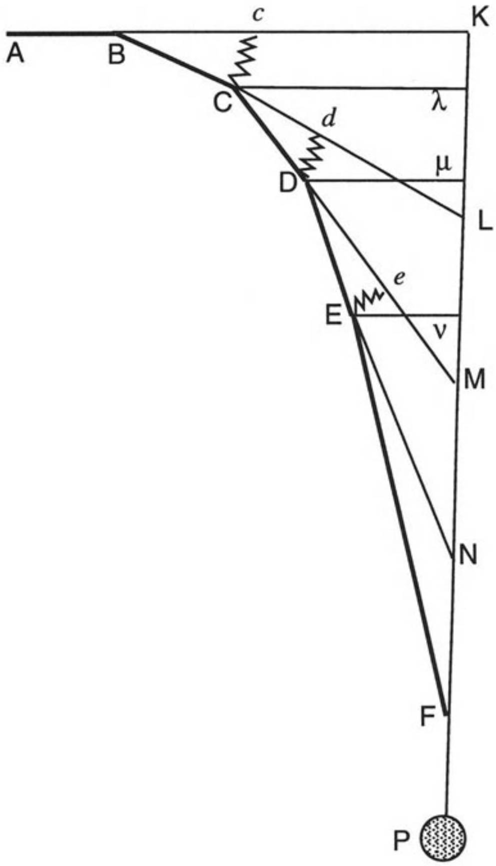

Figure 6 (from Lagrange [1771, Fig. 1])

Note that in this conception physical objects such as rods and beams are idealized as one-dimensional lines, possessing a certain degree of elasticity but lacking any internal constitution that can be further analyzed. This point of view seems to have become standard in eighteenth-century theoretical mechanics. A paper published by Lagrange in 1771 opened with a discussion of the elastic curve. The author wished to show that the moment of an applied weight P acting at the end of the curve would at each point equal P times the perpendicular distance from the point to the vertical through P. The validity of this result for rigid bodies was known and it was necessary to demonstrate it for elastic bodies. To do so Lagrange considered a curve made up of polygonal segments; at each vertex a force represented as sort of an elastic hinge was assumed to exert a resisting moment (Figure 6). He devised a certain argument on the basis of this model to establish the result, which was then extended to continuous curves by assuming that the number of sides of the polygonal line increases indefinitely.

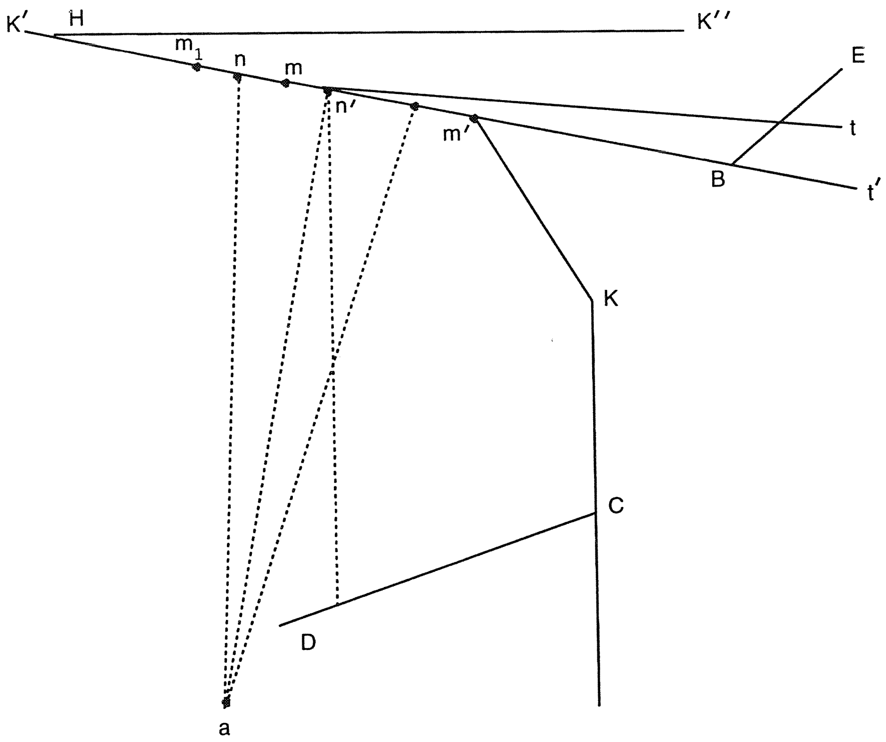

Figure 7 (from Poisson [1811, Fig. 40])

In 1811 in the first volume of his *Traite de Mécanique* Siméon Poisson adopted Euler's conception of the elastic line. He justified treating the elastic lamina as a geometrical line by noting that its thickness would be assumed constant along its length. In an extended discussion (pp. 212–230) he analyzed an elastic line as an infinite-sided polygonal line in which an elastic moment is exerted at each vertex (Figure 7). He assumed as hypothesis the proportionality of the moment to the angle between two successive polygonal segments.

In the second edition of the *Traité*, published in 1833, Poisson discarded this model, replacing it by an analysis in terms of stresses acting across cross-sectional planes.[9] He provided no discussion of this change, although

[9] Poisson's analysis of the static equilibrium of an elastic lamina in the 1833 edition of the *Traité* is presented on pp. 600–607 of volume 1. On pp. 617–620 he presents an analysis of elastic bending for various assumptions concerning the shape of the transverse cross-section of the lamina.

it is clear that the appearance of stress analysis in the treatises of Navier and Cauchy influenced his treatment. By the 1830s the older Eulerian conception of an elastic line had become something of an historical curiosity.

The employment of the elastic line as a conceptual entity was indicative of the distinctive style of theoretical mechanics as it was practiced in the eighteenth-century. It was at this period not simply a case of applying mathematics to physical reality. In the characteristic processes of abstraction and conceptualization the entities under study emerged as objects possessing both a mathematical and physical identity. The elastic line was a kind of mathematico-physico hybrid. Although treated formally as a geometrical object in a quasi-Archimedian sense, its ultimate ontological status (paradoxically) was that of a physical object. There was an understanding of the relationship of mathematics to empirical reality in which the autonomous mathematical character of conceptual entities was acknowledged to a greater degree than it would be in later physics.[10]

Concluding remarks. If one surveys eighteenth-century work for anticipations of the concept of stress they are found in researches that focus on concrete physical problem. Problems such as the fracture of beams or the buckling of columns led researchers to analyze the internal forces acting across a given cross-sectional plane, be it the plane of fracture or the plane perpendicular to the long axis of the column. In each case the object under study was a definite engineering structure possessing a well defined physical identity.

When scientists attempted to develop a general mathematical theory of elasticity they turned to models that were physically very restrictive. Euler's treatment of the statics of elastic lines is a case in point. His investigation was limited at the outset by the adoption of an idealization of physical structures that excluded any analysis of their properties in terms of the concept of internal stress.[11] He was himself conscious of limitations of his approach, writing [1771, 381] at the beginning of his 1771 paper that "we are still far removed from a complete theory which is capable of determining the figure of elastic surfaces as well as bodies", in consequence of which he would restrict his study to "simple strings whether perfectly flexible or elastic, as they have been treated till now by geometers."

It was Cauchy's achievement to develop a theory that was mathemat-

[10] Further discussion of the general philosophical point at issue here is provided by Grosholz [1990].

[11] The papers of 1771 and 1772 are excellent representatives of exact science as it was practiced by Euler. A given point of view is worked out in a systematic and methodical manner with the careful derivation and study of the requisite differential equations. What results is an orderly and satisfactory analysis whose overall character and scope are substantially restricted by the limitation of the initial point of view. (Miller's [1916, 238] general comments are relevant here: 'The great volume of Euler's writings is partly due to the fact that he went into great details, and presented even the simpler matters with considerable completeness.")

ically sophisticated and at the same time grounded in a physical analysis generally applicable to continuous structures and media. Consideration of the eighteenth-century background enables us to appreciate better his originality by indicating some of the difficulties of mathematical technique, physical conception and underlying methodology that were involved in the early evolution of the subject.

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