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HAMILTON-JACOBI METHODS AND WEIERSTRASSIAN FIELD THEORY IN THE CALCULUS OF VARIATIONS: A STUDY IN THE INTERACTION OF MATHEMATICS AND PHYSICS

The relationship between mathematics and empirical science has evolved and developed historically, and is continuing to evolve and develop today. The twentieth century has seen many new and different ways in which mathematics is used, in non-linear analysis, computer science, operations research, industrial engineering, economics, and so forth. The present paper discusses some aspects of the interaction of mathematical analysis and theoretical mechanics during the period 1700-1900, with attention to the relatively classical example of the calculus of variations on the one hand, and Hamilton-Jacobi theory on the other.

MATHEMATICS AND PHYSICS 1700-1900

The rejection of geometric conceptions that occurred in the work on analysis of Euler and other eighteenth-century researchers was not accompanied by a realization that the calculus could be developed in full logical isolation as part of pure analysis. In Euler's analytical writings the relationship between foundation, theoretical development and problem generation is not worked out. Euler's entire project in the calculus of variations consisted of the derivation of differential equations for general problems, each of which embodied characteristics found in a given set of examples from geometry or mechanics. As Euler's work evolved and developed the separation of analysis from geometry was made more explicit at a theoretical level. His variational investigations however remained centered on the derivation of general differential equational forms. He provided no account of how the problems in question might originate or be generated within this or any other branch of pure analysis.

Euler sometimes wrote as if problems are things that are external to analysis that guarantee its meaning and validity. In a memoir published in 1758 he investigated singular solutions to ordinary differential equations, that is, solutions which are not included in the general integral containing arbitrary constants. He took a differential equation and exhibited a particular function $y = f(x)$ that satisfied the equation but was not in the general solution. He wrote: "Concerning the example that I have just set forth, as it is drawn from fantasy, one could doubt whether this case is ever encountered in a real problem. But the same examples that I adduced in order to clarify the first paradox, will serve also to clarify this one."[1] (The examples in question concerned curves in the plane that satisfied certain tangent conditions.)

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The point here is connected to a larger difference of outlook between eighteenth-century and modern mathematics. That the problems of geometry and mechanics should conform to treatment by pure analysis was something that Euler implicitly accepted as a point of philosophical principle. The term philosophy (or metaphysics) is here being used in the sense identified by Daston (1991, 522): "the presuppositions (often unexamined) that inform a scientist's work, which may be of either epistemological or ontological import...metaphysics is what is left over once the mathematical and empirical content have been subtracted...." In the writings of such post-positivist intellectual historians as E. A. Burtt the term metaphysics in this sense referred to very broad assumptions, such as a general Platonic belief among early modern thinkers in the mathematical character of physical reality.[2] We suggest that it is also useful at a more concrete level in explaining certain tacit but definite attitudes displayed by Euler in his research in geometry and analysis.

Demidov (1982, 37) writing of the failure of Euler and d'Alembert to understand each other's point of view in the debate over the wave equation observes:

> A cause no less important of this incomprehension rests, in our opinion, on the understanding of the notion of a solution of a mathematical problem. For d'Alembert as for Euler the notion of such a solution does not depend on the way in which it is defined...rather the solution represents a certain reality endowed with properties that are independent of the method of defining the solution. To reveal these properties diverse methods are acceptable, including the physical reasonings employed by d'Alembert and Euler.

A biographer (Grimsley 1963, 248) of d'Alembert has noted his insistence on "the elementary truth that the scientist must always accept the essential 'giveness' of the situation in which he finds himself." The sense of logical freedom that is inherent in modern mathematics was notably absent in the eighteenth century.

During the nineteenth century mathematical analysis experienced a profound logical transformation as it underwent successive phases of a process known as "arithmetization." This change was accompanied by a larger shift in the way mathematics was understood, a shift that was manifested most prominently in such subjects as the foundations of geometry, algebraic number theory and mathematical logic. The deep intellectual transformation that took place in mathematical outlook after 1800 is expressed succinctly in Dedekind's famous statement of 1888 that "Numbers are free creations of the human mind," a statement which presupposes a sense of logical freedom that would have been foreign to eighteenth-century masters of the subject. This transformation was also accompanied by a new sense of the nature of mathematics and its possibilities, expressed for example in George Boole's remark of 1854 that "It is not of the essence of mathematics to be conversant with the ideas of number and quantity."

With the emergence in the nineteenth century of an internalized and logically self-contained conception of mathematics, there also developed a corresponding notion of theoretical physics, in which a highly mathematical approach was combined with a clear disciplinary separation of physics from mathematics. Jungnickel and McCormmach, in their social study of theoretical physics from Ohm to Einstein, subtitle volume one, "The torch of mathematics 1800-1870", and volume two, "The now mighty theoretical physics

1870-1925". They call attention to the disappearance in the second half of the century of the figure of the mathematician-physicist, of whom Gauss and Riemann had been outstanding examples. They write:

> Mathematicians continued to be of help to physicists as the needs of physical theory came to include bodies of mathematical knowledge not contained in Riemann's original manual on the partial differential equations of physics. But the position of intermediary between mathematics and physics, as Riemann was seen to hold, was increasingly taken over by a new kind of specialist, the theoretical physicist. The theoretical physicist might consult or even collaborate with a mathematician, but he always worked as a physicist rather than a mathematician. As a physicist, he was knowledgable in mathematics, and although he did not do original research in mathematics, he was capable of adapting new mathematics to physical uses and, in the process, of offering mathematicians new mathematical opportunities. (1986, V. 1, 185).

This new disciplinary alignment was apparent in the growing distinction between mathematical physics, a subject practised by mathematicians on the one hand, and theoretical physics, a subject of concern to physicists on the other (Jungnickel and McCormmach 1986 V. 2, 346-7). The physicist Wilhelm Wien (1915) publicly discussed this dichotomy in an article titled "Ziele und Methoden der theoretischen Physik."

WEIERSTRASSIAN FIELD THEORY IN THE CALCULUS OF VARIATIONS

In the 1870s the calculus of variations entered a new phase as German researchers began to investigate the subject in a rigorous way from the standpoint of the theory of a function of a real variable. In 1877, G. Erdmann published a paper giving conditions under which broken extremals, functions whose derivatives are discontinuous at a finite number of points, are solutions to a variational problem. Two years later Paul Du Bois-Reymond carried out a detailed study of the basic variational processes in terms of real-variable analysis. In the middle 1880s Ludwig Scheeffer published researches in which he subjected the traditional conditions of Euler, Legendre and Jacobi to very close critical scrutiny.

The leading figure in the new calculus of variations was Karl Weierstrass. More than any other researcher Weierstrass established the logical outlook of this branch of analysis as a modern mathematical subject. In his lectures the distinction between necessary and sufficient conditions appears clearly for the first time. He carefully specified the continuity properties that must be satisfied by functions and their variations. In problems of constrained optimization he used theorems on implicit functions to ensure that the optimizing arc was embedded in a suitable family of comparison curves.

Before the 1860s researchers did not identify at the outset of their investigation the precise class of comparison arcs in a given variational problem. There was no prior logical conception concerning the nature of this class. The δ-process introduced by Lagrange required however that both the comparison arc and its slope at each point differ by only a small amount from the actual curve. This condition was imposed implicitly by the nature of the variational process.

Weierstrass broadened the notion of a solution to include a much larger class of comparison arcs. At a general level his approach to the calculus of variations involved a very basic logical re-orientation of the subject. In earlier variational research the nature of the mathematical objects was determined implicitly by the methods employed. Weierstrass by contrast began with objects defined constructively in terms of an explicit theory of a function of a real variable.

In his lectures of the 1870s and early 1880s Weierstrass pioneered a new method which provided conditions sufficient to ensure the existence of a maximum or minimum in single-integral variational problems. His basic idea, involving the so-called excess function, allowed one to develop the theory in reference to an extensive class of comparison variations. To apply his method one must show that the hypothetical minimizing or maximizing arc $y_0 = y_0(x)$ may be embedded in a strip or region ("Flächenstreife") of the plane containing $y_0(x)$ and covered by a family of solutions to the Euler equation. This family satisfies the property that there is a unique member joining the initial point 0 and any subsequent point in the region; at each such point there will therefore be a well-defined slope function, given as the slope of the extremal passing through the point. In his 1899 *Lehrbuch derVariationsrechnung* Adolf Kneser introduced the formal term "field of extremals" to designate such a family of curves.

An important simplification of Weierstrass's technique was presented by Hilbert in his famous Paris address of 1900. Writings based on Weierstrass's field methods were published during the period 1895-1905 by Ernst Zermelo, Kneser, E. R. Hedrick, Oscar Bolza and E. J. B. Goursat. Major textbooks of Bolza (1909) and Jacques Hadamard (1910) provided a masterful synthesis of contemporary achievements in the subject.

HAMILTON-JACOBI THEORY IN ANALYTICAL DYNAMICS

Lagrange's *Mécanique Analytique* of 1788 was a comprehensive textbook on statics and dynamics based on a general statement of the principle of virtual work. The central technical achievement of this treatise was to introduce the "Lagrangian" form of the differential equations of motion, $\partial T/\partial q_i - d(\partial T/\partial \dot{q}_i)/dt = \partial V/\partial q_i$, for a system with n degrees of freedom and generalized coordinates q_i ($i=1,..., n$). The quantities T and V are scalar functions denoting what in later physics would be called the kinetic and potential energies of the system. The advantages of these equations are well known: their applicability to a wide range of physical systems; the freedom to choose whatever coordinates are suitable to describe the system; the elimination of forces of constraint; and their simplicity and elegance.

In addition to presenting powerful new methods of mechanical investigation Lagrange also provided a discussion of the various principles of the subject. The *Mécanique Analytique* would be an important source of inspiration for such nineteenth-century researchers as Hamilton and Jacobi. In investigating problems in particle dynamics in the early 1830s Hamilton hit upon the idea of taking a certain integral and regarding it as a function of the initial and final coordinate values. He was able to show that the given integral regarded in this way – the so-called principal function — satisfies two partial differential equations of the first order. Although Hamilton employed variational ideas and techniques his analysis was developed within the established theory of analytical dynamics.

Hamilton's theory was a very original and seminal contribution to the formal development of dynamics. He himself reported in 1834 in a letter to his friend William Whewell that he had "made a revolution in mechanics."[3] His findings were published in English in *Philosophical Transactions* of the Royal Society. Hamilton was fortunate to have in Jacobi a reader who immediately appreciated the significance of his work and was also an exceptional mathematician in his own right. Jacobi took what he referred to as Hamilton's "beautiful idea" and developed an improved and revised theory. Whereas Hamilton had stipulated that the conservation of mechanical energy (live forces) holds, Jacobi observed that this equation can be derived without such an assumption. Furthermore, Jacobi emphasized the integration problem and used the theory of partial differential equations to obtain a solution to the dynamical ordinary differential equations in terms of the solution of the corresponding Hamilton-Jacobi equation.

Jacobi confined his investigation to the primary problem in analytical mechanics. In 1858 Clebsch used some of ideas of the Hamilton-Jacobi theory in his mathematical investigation of the second variation. In the course of doing so he provided a simple and general exposition of Jacobi's derivation of the Hamilton-Jacobi equation. Mayer, in his study several years later of the second variation, also summarized some of the essential ideas of the Hamilton-Jacobi theory.

A detailed exposition of the Hamilton-Jacobi theory is beyond the scope of the present paper. There is however one observation which we can make that is germane to our study. It is of interest to note the way in which the later mathematical concept of a field of extremals is implicit in the Hamilton-Jacobi development. In Clebsch's derivation of the Hamilton-Jacobi partial differential equation it is assumed that the given region of the x-y plane is covered with a family of curves that are solutions to the Euler differential equation; it is also assumed implicitly that there is a unique such solution joining the initial point and any subsequent point in the region. The slope of the extremal passing through each point gives rise to a well-defined field function defined over the region.

The germ of this idea can be traced to Hamilton's original derivation of his principal function in his paper of 1834 (and even earlier, to his draft memoirs). Hamilton was working within a dynamical framework and did not conceptualize his result in terms of the calculus of variations. For example, a key step in his derivation of the Hamilton-Jacobi equation required assuming that the trajectory followed by the system is described by the dynamical equations of motion (expressed in terms of canonical coordinates); viewed as a problem in the calculus of variations what was being assumed in this step was that the Euler variational equation holds, i.e. that the given trajectory is an extremal. Although it is not within the scope of the present paper it would be of some interest to provide a detailed analysis of Hamilton's original theory and to present a step-by-step comparison of his derivation with that which would obtain using later variational theory.

Interest in the Hamilton-Jacobi theory in the second half of the nineteenth century seems to have been largely based on its role in integrating the variational differential equations. A particular integration of the Euler equations in terms of canonical constants was employed by Clebsch and Mayer in their study of the second variation. In order to transform the second variation to positive definite form it was necessary to introduce functions containing certain constants, and Clebsch and Mayer were able to obtain a solution in terms of the constants of integration for the variational equations given in canonical form. In mechanical investigations efforts were concentrated on the question of transforming the coordinates of a system in order to obtain a set of coordinates that yielded a tractable solution to the integration problem.

Although the volumes of Jacobi's collected works published in the 1880s were edited by Weierstrass, there is no record that Weierstrass took much interest in the papers in the fourth volume on mechanics. If he did he never integrated this interest into his study of the calculus of variations. His development of field methods seems to have been a work of pure analysis, carried out largely independently of any interest in theoretical mechanics.

At the end of the century mathematicians such as Adolf Kneser involved in the development of field methods did become very interested in the Hamilton-Jacobi theory. My own study of the technical sources has led me to hypothesize that some familiarity with this theory may have contributed to Hilbert's original development in 1900 of the invariant integral. I believe that there are technical grounds for supposing that Hilbert arrived at the idea for this integral by taking such standard results as the variable-endpoint formula, and developing these results using a Hamilton-style conception of the variational integral.[4] It should be noted that Beltrami's independent discovery in 1868 of the invariant integral [as discussed in (Thiele 1997a)] was associated with his interest in the Hamilton-Jacobi theory. It should also be noted that in his discussion of the invariant integral in his published Paris address, Hilbert called attention to Kneser's related researches and referred to the Hamilton-Jacobi equations.

I should emphasize that although there would seem to be intellectual reasons for believing that Hilbert's work on the invariant integral was influenced by the Hamilton-Jacobi theory, we have no actual documentary evidence that this was the case. In an examination of Hilbert's lectures from around 1900, Thiele [personal communication to the author; see also his (1997a)] has found no evidence of an interest on Hilbert's part in the Hamilton-Jacobi theory.

In any case, the Hamilton-Jacobi theory was a central concern of Kneser's and received detailed coverage in both his *Lehrbuch* of 1899 as well as in Bolza's major textbook a decade later. Subsequently, Carathéodory (1935) would investigate in a systematic way the relationship between the calculus of variations, the Hamilton-Jacobi theory and the theory of partial differential equations.

DISCUSSION

We would probably classify Hamilton as a mathematician-physicist, of the sort represented by Gauss and Riemann, rather than a theoretical physicist, of the sort represented by Helmholtz or Einstein. Nevertheless, Hamilton's dynamical researches of the 1830s were clearly part of physics rather than mathematics. His leading conceptions were carried over from his earlier optical researches, and the intellectual process leading to his major innovations was conceptualized in dynamical rather than analytical terms.

Theoretical mechanics and analysis provide clear examples of what Emily Grosholz (1999) has called autonomous but rationally related domains. Both the autonomy and rational relatedness have been manifested in different ways in different historical periods. In the eighteenth century mechanics was conceived in much the same way as geometry was, as part of mathematics. The physical objects of study in mechanics were quasi-Archimedean entities, hybrids to use a term introduced by Grosholz, capable of mathematical-deductive study by the tools of advanced analysis. In mathematical analysis itself however researchers such as Euler emphasized the logical independence of this subject from geometrical and mechanical conceptions. Euler's viewpoint was very different from that of the early pioneers, who conceived of the foundation of the calculus in terms of geometric conceptions, or that of the nineteenth-century researchers, for whom the numerical continuum provided a fundamental structure of interpretation.

As the nineteenth century progressed, researchers — particularly in Germany — increasingly emphasized the autonomous, physical, empirical, *anschaulich* character of mechanics *vis à vis* mathematics. Meanwhile analysis itself was re-conceptualized as a logical subject independent of physical science. By the end of the century, when analysis in general and the calculus of variations in particular had achieved complete technical and philosophical separation from empirical science, Hamilton-Jacobi theory provided an external source of new and potent mathematical ideas. The rational relatedness of the two subjects was manifested is a deep, unexpected and highly fruitful way.

NOTES

1. "Pour l'example que je viens d'alléguer ici, comme il est formé à fantaisie, on pourrait aussi douter, si ce cas se recontre jamais dans la solution d'un problème réel. Mais les mêmes exemples, que j'ai rapportés pour éclaircir le premier paradoxe, serviront aussi à éclaircir celui-ci."
2. Daston is identifying the sense in which the term metaphysics is used by Burtt and others. She is somewhat critical of this usage because it does not take into account the various actual historical systems of metaphysics which prevailed in the early modern period. To the extent however that the term serves to designate certain extra-scientific or extra-mathematical attitudes in past research it remains a useful concept of historical analysis.
3. See (Hankins 1980, xviii).
4. This is explained in more detail in (Fraser, forthcoming) in a collection of essays on the history of analysis (Jahnke and Knoche, forthcoming).

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