12. Edmund Husserl's Contributions to the Theory of the Second Variation in the Calculus of Variations

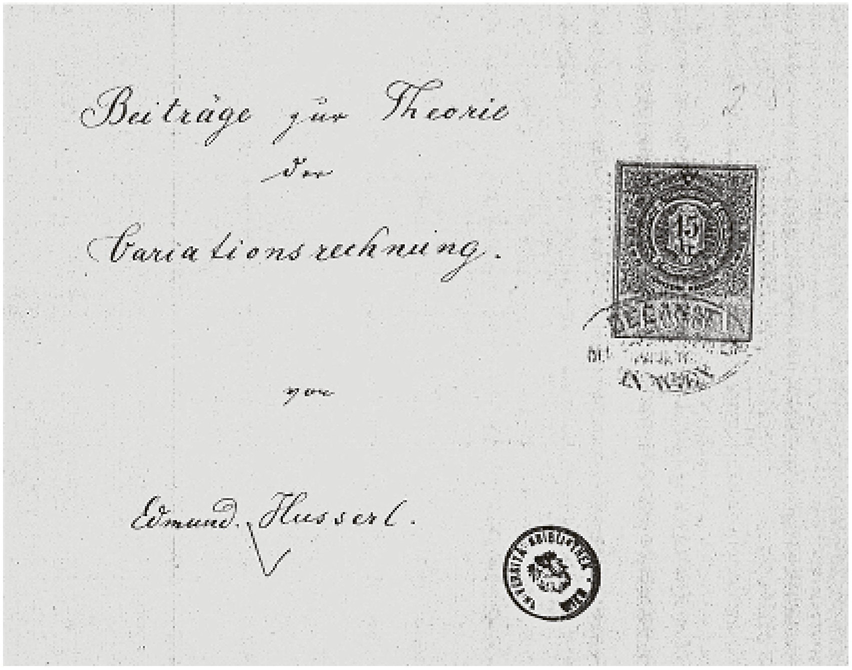
*Craig Fraser**

1. Introduction

Edmund Husserl (1859-1938) was a native of Moravia (today in the Czech Republic and then part of the Austrian Empire), and studied mathematics at the Universities of Leipzig (1876-1878), Berlin (1878-1881) and Vienna (1881-1883). In Berlin he attended and made notes on the lectures of Karl Weierstrass in the calculus of variations. In November of 1882 he completed his dissertation at the University of Vienna under the direction of Leo Koenigsberger. The title of his handwritten thesis was *Beiträge zur Theorie von Variationsrechnung* ("Contribution to the Theory of the Calculus of Variations", see Figure 1). As he embarked on the revision and preparation for publication of this work he became interested in philosophical and religious subjects, soon abandoning the dissertation project and the field of technical mathematics forever. He went on to achieve fame as the founder of the phenomenological movement in philosophy. His dissertation in mathematics was eventually published in 1983 in French translation.

The subject of Husserl's dissertation was the theory of the second variation, a prominent part of the calculus of variations from the 1830s to the end of the century. In a famous paper of 1837 Carl Gustav Jacobi outlined the basis of a new theory of the conditions required to ensure a maximum or minimum in the calculus of variations. His paper, in which proofs were omitted and little explanatory detail was provided, became the basis for a vigorous program of mathematical research. Textbooks on

* Institute for the History and Philosophy of Science and Technology, Room 316, Victoria College, University of Toronto, 91 Charles St. West, Toronto, Ontario M5S 1K7, Canada, craig.fraser@utoronto.ca. December 30, 2017.


Beiträge zur Theorie der Variationsrechnung.

von

Edmund Husserl.

Figure 1

calculus of variations (both in the nineteenth century and today) focus on the case where there is a single dependent variable and only the first derivative of this variable appears in the variational integrand. However, substantial difficulties in the theory of the second variation arise in problems that are more general than this, where there are derivatives of higher order in the integrand or where there is more than one dependent variable. The considerable body of work stimulated by Jacobi's paper was devoted largely to an investigation of the transformation of the second variation in this more general setting. The transformation – assuming it was possible to carry out – permitted one to infer Adrien-Marie Legendre's condition for the existence of a maximum or minimum. Among notable contributors here were Charles-Eugène Delaunay (1841), Gaspare Mainardi (1852), Simon Spitzer (1854-1855), Otto Hesse (1857), Alfred Clebsch (1858a, b) and Adolph Mayer (1866, 1868).

Husserl's dissertation was not a polished work. He was rather selective in his approach and did not always provide a full discussion of the particular point under consideration. His dissertation nonetheless contained some original and significant contributions. These are:

1. A discussion of Legendre's transformation of the second variation for the simplest variational problem, with an analysis of Jacobi's key insight concerning the second variation.
2. An estimation of Mayer's contribution to the study of the second variation. Husserl identified Mayer as having made a decisive theoretical breakthrough in the study of the second variation. He also significantly improved on Mayer's theory, and provided a general and simple procedure for selecting the constants required in Mayer's transformation of the second variation.
3. A proof of the necessity of Jacobi's condition in the general problem of the Lagrange. This is the only part of the dissertation that seems to have been influenced by Weierstrass.

The subject of the present essay is the first two parts of Husserl's dissertation, which make up three quarters of his work.

Husserl's dissertation is of historical interest as an indication of the mathematical preoccupations of the period, which included a significant place for the calculus of variations. Furthermore, although the thesis was not published and remained unknown for one hundred years, it nonetheless contributed in substantial ways to the subject and remains of interest today for this reason. Husserl possessed a remarkable ability to look into the theory and identify the essence of a mathematical result, to bring out in the formulation of the result the core features of the theoretical situation. Finally, whatever light Husserl's doctoral dissertation sheds on his thought at this early stage will be of some interest in understanding his general frame of mind as he embarked on his investigations in the philosophy of mathematics.

2. Theory of the Second Variation[1]

2.1. Jacobi, Delaunay and Hesse

In his 1837 paper *Zur Theorie der Variations-Rechnung und der Differential-Gleichungen* Jacobi succeeded in establishing a consistent theory of the conditions required to ensure a maximum or minimum in the calculus of variations. The path followed today in the calculus of variations is to derive a set of necessary conditions and then to show that these conditions taken jointly are sufficient to produce a minimum or maximum. Before the latter part of the nineteenth century there was little explicit awareness of the familiar modern distinction between necessary and sufficient conditions, terms which in fact did not appear in the writings of researchers. In showing that the nature of the problem leads to a given condition, they assumed that a function satisfying this condition yielded the solution of the problem. Thus researchers typically believed that they had arrived at a satisfactory resolution of the problem. If this was shown not to be the case they attempted to find further conditions. The Euler differential equation was deduced from the requirement that the variation of the variational integral is zero. Legendre observed however that a given solution of this equation need not yield a maximum or a minimum. In later parlance the Euler equation is a necessary but not a sufficient condition. Legendre therefore derived an additional condition concerning the sign of the second partial derivative of the variational integrand. Jacobi showed in turn that this condition may be satisfied by functions which are not minimal or maximal; hence he imposed further requirements on the solution.

In a short passage in his 1837 paper Jacobi had drawn attention to another aspect of the variational problem, what in later mathematics would be called the theory of the conjugate point. The essential point is the following. Consider the problem of maximizing or minimizing the integral $I=\int_{x_0}^{x_1} f(x,y,y')\mathrm{d}x$ on the interval $[x_0,x_1]$. Among all arcs $y=y(x)$ joining the endpoints, the optimal arc $y=y_0(x)$ will be a solution to the Euler equation $V=0$. The general solution of the Euler equation

[1] For surveys of the history of the calculus of variations see (Todhunter 1861), (Goldstine 1981) and (Fraser 2003).

for curves that are required to go through the initial point will contain an arbitrary constant α, $y_{,} = y_0(x, \alpha)$. We suppose that $\frac{\partial y_0(x_0,0)}{\partial\alpha} = 0$. Consider the variation $\delta y = \frac{\partial y_0(x,\alpha)}{\partial\alpha}\alpha$ and comparison arcs of the form $y(x) + \frac{\partial y_0(x,\alpha)}{\partial\alpha}\alpha$. If the condition $\frac{\partial y_0(x_1,0)}{\partial\alpha} = 0$ also holds then $\delta y = \frac{\partial y_0(x,\alpha)}{\partial\alpha}\alpha$ is an admissible variation on the interval $[x_0, x_1]$. The first variation of I is $\delta I = \int_{x_0}^{x_1} V\partial y \mathrm{d}x$ and so the second variation of I is $\delta^2 I = \int_{x_0}^{x_1} \delta V\partial y \mathrm{d}x$. Now for $y = y_0(x)$ we have $V = 0$. Hence $\delta V = 0$ and so $\delta^2 I = 0$. Thus for $y = y_0(x)$ and the admissible variation $\delta y = \frac{\partial y_0(x,\alpha)}{\partial\alpha}\alpha$ we have both $\delta I = 0$ and $\delta^2 I = 0$. The first and second variations of I are both zero. It is clear in this situation that there can be no maximum or minimum, because the sign of the third variation can (in general) be made either positive or negative[2].

If we start at the initial point (x_1, y_1) on the curve $y = y_0(x, \alpha)$ we may eventually arrive at a second point (x_1, y_1) with $\frac{\partial y_0(x_1,0)}{\partial\alpha} = 0$. In this situation there will be an admissible variation for which $\delta^2 I = 0$ and it will then not be the case for the given variational problem that y_0 is an optimal curve on the interval $[x_0, x_1]$. This second limiting point, the value of which cannot be reached or exceeded if a minimum or maximum is to obtain, would become known in later mathematics as a conjugate point (a term introduced by Karl Weierstrass (1815-1897)). The condition that there be no such second point on the interval is known as Jacobi's condition.

Jacobi illustrated this restriction using the example of the elliptical motion of a particle moving about a force centre, in which the trajectory is deduced from the principle of least action. In lectures delivered in the early 1840s and published posthumously in his *Vorlesungen* (1866, 46) he introduced an even simpler example, the case of a single particle constrained to move on the surface of sphere but otherwise subject to no force. The principle of least action leads here to the conclusion that the trajectory must be a geodesic or path of shortest distance. Hence the particle moves on a great circle, i.e., the intersection of the surface of

[2] Noteworthy is the way in which the operational character of the variational process δ enters into the reasoning. See (Fraser 1994).

sphere and a plane through its centre. If we begin at a given point A and traverse an angular distance of *180* degrees we reach a point C conjugate to A. If the second point B coincides with or is beyond C then it is not difficult to see that there are comparison paths of equal or shorter distance[3].

During the period from 1837 to 1865 the efforts of researchers were devoted almost exclusively to the transformation problem. While the idea of a conjugate point had a natural analytical basis, in practice the existence of such points was revealed in geometrical examples. There was a pronounced formal orientation to the research stimulated by Jacobi's paper. From a theoretical viewpoint it is not at all obvious how one would in the general case connect the transformation problem to the theory of the conjugate point.

The body of research on the transformation problem divided into two main lines. The first, stemming from Jacobi's original memoir, focused on a result in the theory of ordinary differential equations that Jacobi used to transform the second variation for the general problem where the variational integrand contains derivatives of y with respect to x of arbitrary order. Jacobi also introduced a crucial insight involving the use of variations given as partial derivatives of the Euler solutions with respect to the arbitrary constants appearing in them to effect this transformation. Although several researchers continued this line of research, the most important figures were Delaunay (1841) and Hesse (1857)[4]. Delaunay followed Jacobi's outline most closely; indeed his memoir is an indispensable aid in understanding and making sense of the 1837 paper, which was very compressed and even somewhat cryptic in places. Hesse brought this whole line of research to its culmination, with an article in 1857 in Crelle's journal. Hesse also provided a complete analysis of the simplest case of the variational problem, where the integrand has the form $f(x,y,y')(y'=dy/dx)$. In fact he arrived at the first sufficiency proof, showing in this case that if a given function $y=y(x)$

[3] For a history of Jacobi's work in mathematical dynamics see (Nakane, Fraser 2002).

[4] Lebesgue (1841) and Jellett (1850) were other figures who worked on the transformation problem, using the Jacobi transformation. An overview of relevant researches of the period up to 1861 is provided by (Todhunter 1861, Chapters 9 and 10, 229-310).

satisfied Euler's equation $\partial f/\partial y - d/dx(\partial f/\partial y') = 0$, Jacobi's condition was satisfied (no conjugate point on the interval), and Legendre's condition held ($\partial^2 f/\partial y'^2 > 0$), then $y = y(x)$ produced a minimum.

The Jacobi-Delaunay line of research to all intents and purposes ended with Hesse's 1857 article in Crelle. In many respects the whole development was idiosyncratic. It was not at all a natural approach to the problem of sufficiency, and it seems surprising that it ever originated at all. Despite the singular character of Jacobi's new transformation, there were two fundamental and enduring aspects of his work: the notion of the conjugate point, and the recognition that any transformation of the second variation would involve variations obtained as partial derivatives of the Euler solutions for the problem.

2.2. Spitzer and Clebsch

A second line of attack was launched by Gaspare Mainardi in 1852 and continued by Simon Spitzer two years later.[5] Spitzer's development of the theory was more systematic and detailed than Mainardi's and seems to have had a greater impact on subsequent research. Instead of using Jacobi's new transformation, Spitzer followed Legendre's original idea and began with a general identity between the second variation in its standard form and an expression for the second variation in positive definite form. Such an approach was natural, and avoided the *ad hoc* nature of Jacobi's transformation. Spitzer (1854,1025) himself stated that he had developed a much simpler method "by means of which Jacobi's complicated and difficult transformation is avoided". However, Spitzer did adopt Jacobi's key insight of working with variations obtained as partial derivatives of functional solutions to the Euler equation to effect the transformation. The framework erected by Spitzer became the basis for the theory of the second variation in the modern subject.

[5] The second variation in the calculus of variations was Spitzer's first area of research in mathematics. He is perhaps best known for his writings on the Laplace transform, where he championed the priority of Laplace and became involved in a dispute with his Vienna contemporary Joseph Petzval. (See (Deakin 1981).) Spitzer's career was at the Vienna Handelsschule and he also taught at the Polytechnische Institut in Vienna. Both Spitzer and Husserl were Jews from Moravia, Husserl from Prostějov and Spitzer from Mikulov.

Both Mainardi and Spitzer also investigated the problem where there is more than one dependent variable, and made some important progress in its formulation and solution. The particular issues that arise in extending the theory to multiple dependent variables resemble those that arise in the traditional problem where there is only one dependent variable and where higher order derivatives appear in the variational integrand.

In two memoirs published in Crelle's journal in 1858 Clebsch developed the theory in a systematic way for the general variational problem involving multiple dependent variables. Clebsch's paper followed Hesse's paper in Crelle by only a year, and Hesse is the only researcher other than Jacobi cited by Clebsch. In the literature on the calculus of variations Clebsch's research is viewed as a fairly direct continuation of Hesse's work. Nevertheless, the conception at the base of Clebsch's approach is very similar to the method of Spitzer (and Mainardi before him). Clebsch formulated the theory for arbitrarily many dependent variables. (Of the earlier researchers, only Mainardi and Spitzer had attempted a detailed treatment of problems with more than one dependent variable.) Clebsch also made the basis for the whole theory an identity between the second variation and an expression in positive definite form. The prominent line of research involving Jacobi and Hesse's special transformation is completely absent in Clebsch's memoirs. By contrast, some of the leading ideas of his investigation are echoed in the work of Spitzer, although Clebsch succeeded brilliantly at the difficult task of developing at a formal level a completely general theory of the second variation.

There are two approaches to the presentation of any mathematical theory. One can select the simplest problems that illustrate the main features present in the general theory and provide a complete account of these problems. This account may then be followed by a presentation of the general theory itself. The advantage of this approach is that it is focused on concrete examples that capture the complexities or refinements present in the general case. This is the approach that was followed by pretty much all of the researchers on the second variation prior to Clebsch. However, it was not Clebsch's way. He developed the theory from the outset at a very high level of generality, with none of the careful consideration of particular problems and cases that had occupied earlier researchers. His philosophy was part of a trend within the calculus of variations and possibly within mathematics as a whole that seemed to value generality for the sake of generality. This approach brought with it a heavy reliance on notation in which results were derived and expressed in detailed formulas and equations and in which the entire subject was experienced intellectually at a rather abstract level. Given the formal difficulty of the project Clebsch set for himself in his 1858 articles it is possible that he was only successful because he possessed such a cast of mind. An expository account of Clebsch's theory was given by Mayer in his 1866 *Habilitationsschrift*, a work that is very useful indeed in understanding Clebsch's work. (That Mayer should have felt it necessary to devote such effort to an exposition of this work is revealing in itself.)

In a departure from tradition Clebsch took as primary the general problem in which there are side conditions in the form of differential equations, the so-called problem of Lagrange. In 1797 and 1801 Lagrange had shown how the variational equations may be obtained in this situation by means of a multiplier rule. Suppose for example that there are two dependent variables $y_1 = y_1(x)$ and $y_2 = y_2(x)$ and the variational integral has the form

$$\int_{x_0}^{x_1} f(x, y_1, y_2, y_1', y_2')\mathrm{d}x. \tag{1}$$

Consider now the traditional case in which there is one dependent variable y and the variational integrand contains first and second-order derivatives of y:

$$\int_{x_0}^{x_1} f(x, y, y', y'')\mathrm{d}x. \tag{2}$$

Clebsch showed that the latter problem can be reduced using the multiplier rule to the case where the variational integral is of the form (1). We introduce the auxiliary condition

$$y_2 - y_1' = 0 \tag{3}$$

and form the function $f + \lambda(y_2 - y_1')$, where λ is a multiplier function. Applying the multiplier rule to $\int_{x_0}^{x_1}(f + \lambda(y_2 - y_1'))\mathrm{d}x$ we have the Euler equations

$$\frac{\partial(f + \lambda(y_2 - y_1'))}{\partial y_1} - \frac{d}{dx}\frac{\partial(f + \lambda(y_2 - y_1'))}{\partial y_1'} = 0, \tag{4a}$$

$$\frac{\partial(f+\lambda(y_2-y_1'))}{\partial y_2}-\frac{d}{dx}\frac{\partial(f+\lambda(y_2-y_1'))}{\partial y_2'}=0. \tag{4b}$$

Let us suppose that the function f does not contain y_2 so that $f=f(x,y_1,y_1',y_2')$. Then (4) reduce to

$$\frac{\partial f}{\partial y_1}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}-\lambda\right)=0, \tag{5a}$$

$$\lambda-\frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right)=0. \tag{5b}$$

Equations (5) in turn becomes

$$\frac{\partial f}{\partial y_1}-\frac{d}{dx}\frac{\partial f}{\partial y_1'}+\frac{d^2}{dx^2}\frac{\partial f}{\partial y_1''}=0, \tag{6}$$

which is the Euler equation corresponding to integral (2).

By applying the multiplier rule in the case where there are additional dependent variables we may repeat this argument to obtain the Euler equation when *f* contains derivatives of arbitrary order of *y* with respect to *x*. Clebsch was thereby able to formulate a general variational problem using only the first derivatives of the dependent variables. His approach may have been motivated by analytical dynamics, where the Hamiltonian function (as it later became known) was a function of position and momentum variables involving only first derivatives[6].

Clebsch presented the theory for the Lagrange problem where there are *n* dependent variables y_i $(i = 1, \ldots, n)$ and *m* constraint equations. Here the variational integral is

$$\int_{x_0}^{x_1} f(x,y_1,\ldots,y_n,y_1',\ldots,y_n')\mathrm{d}x. \tag{7}$$

and the constraint equations are of the form

$$g_k(x,y_1,\ldots,y_n,y_1',\ldots,y_n')=0 \qquad (k=1,\ldots,m). \tag{8}$$

[6] In 1858-1859 Clebsch gave lectures at the Karlsruhe Polytechnische Hochschule on mechanics along Lagrangian lines.

In this setting Clebsch constructed a transformation of the second variation that allowed one to infer a general form of Legendre's condition. Assume that the general solution of the variational equations is $y_i = y_i(c_1, \ldots, c_{2n})$ $(i = 1, \ldots, n)$ where the c_k are constants of integration. Following Jacobi's original idea Clebsch constructed variations by taking partial derivatives of the solution functions y_i with respect to the *2n* constants of integration c_k. The general form for the variations z_i obtained in this way is

$$z_i = a_{i1}\frac{\partial y_i}{\partial c_1} + \ldots + a_{i2n}\frac{\partial y_i}{\partial c_{2n}} \qquad (i = 1, \ldots, n), \tag{9}$$

where a_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, 2n)$ are coefficient numbers.

In his second paper of 1858 Clebsch drew on some ideas from Hamilton-Jacobi theory in dynamics and formulated the variational equations in canonical form. He considered solutions to these equations containing so-called canonical constants of integration[7]. The traditional equations for the Lagrange problem consisted of the Euler equations on the one hand, and the equations of constraint on the other. There was a basic asymmetry between the Euler equations and the constraint equations. A significant formal advantage of expressing the equations of the problem in canonical form is that this asymmetry is absent.

3. Mayer

Following the completion of his doctorate at the University of Heidelberg in 1861, Adolph Mayer spent three years studying at the University of Königsberg. His interest in the calculus of variations was stimulated by the lectures in 1864-65 of Jules Richelot, who suggested the topic of his *Habilitationsschrift*. The latter was dedicated to Richelot and successfully defended by Mayer at the University of Leipzig in 1866 and published in the same year. Two years later Mayer published a much shorter account of the main result in an article in Crelle[8].

[7] Clebsch was not the first to employ Hamilton-Jacobi methods and canonical equations in the calculus of variations. Mikhail Ostrogradsky had already done so in a memoir published in 1850, although Clebsch does not seem to have been aware of Ostrogradsky's research.

[8] For a discussion of the wider intellectual context of German theoretical mathematics of the period see (Fraser 2018).

Taking off from Clebsch Mayer focused on single-integral variational problems and formulated the latter for the case in which the variational integrand contains arbitrarily many dependent variables and where only first derivatives of these variables appear in the integrand. The analysis was also developed for the general problem of Lagrange. Mayer undertook a detailed investigation of the variations (9) required in the transformation of the second variation. The coefficients that appear in (9) will be subject to certain relations that arise from auxiliary differential equations that appear in the course of transformation. Serious study of this subject had begun with Delaunay in 1841. The investigation of these relations was a crucial thread in the historical development of the theory from Jacobi to Mayer. Not only must suitable variations of the form (9) be found, but it must also be shown that it is possible to find among such variations ones for which the transformed variational integrand remains finite on the given interval. In special cases (where only two or three dependent variables are present) it was a matter of some difficulty to establish the existence of the requisite variations.

A textbook published by Lorenz Lindelöf and Abbé Moigno in 1861 indicated the point of view of advanced research of the period. These authors recognized that it is necessary to ensure that there is no admissible variation which makes the second variation equal to zero. This aspect of the analysis was seen as straightforward, a preliminary condition that must be verified before a more detailed study is possible. Their emphasis was decidedly on the problem of transforming the second variation in order to obtain Legendre's condition. They drew attention to the need to find a system of functions in the transformation for which a certain determinant in the denominator of the integrand of the second variation is non-zero throughout the given interval. The coefficient constants appearing in these functions must satisfy certain auxiliary relations. Lindelöf and Moigno (1861, 192) observed: "The question of recognizing if such a system of constants does or does not exist is in reality the most delicate part of Jacobi's theory, and it awaits yet a general solution".

Let us assume that in the given problem under consideration there is no conjugate point on the interval $[x_0, x_1]$ In his Habilitation work *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale* Mayer was able to show that if this condition holds then it is possible using canonical methods to find variations that allow one to carry out the transformation of the second variation and infer Legendre's condition. His derivation was devoted to a detailed analysis of the various possible systems of integration constants and coefficients and to establishing the existence of suitable set of constants that could be used in the transformation.

Assume that Jacobi's condition holds, so that there is no point c on the interval $[x_0, x_1]$ that is conjugate to x_0 It is necessary to show that it is possible to select variations in Clebsch's transformation of the second variation for which the transformation remains finite on the interval $[x_0, x_1]$. Analytically this problem reduces to showing that a certain determinant $\Delta(x)$ in the denominator of the analytical expression appearing in the transformation is non-zero on $[x_0, x_1]$. Following Clebsch Mayer considered the Euler equations in canonical form, so that the constants c_i in (9) are so-called canonical constants of integration. He showed that it is possible to select the coefficients a_{ij} in (9) in terms of these constants in such a way that the given determinant $\Delta(x)$ is non-zero on $[x_0, x_1]$. The article in Crelle that appeared in 1868 laid out an abbreviated and refined version of his result[9]. With this work Mayer had resolved the central theoretical question in the classical theory of the second variation.

4. Husserl

4.1. Husserl's Discussion of the Second Variation in the Simplest Case

Husserl began his dissertation with a discussion of the theory of the second variation in the simplest case of one dependent variable and where only a derivative of order one appears in the variational integrand. His intention seems to have been to articulate the logical basis of Jacobi's theory upon which a more general framework could be erected.

[9] Mayer's 1866 *Beiträge zur Theorie der Maxima und Minima der einfachen Integrale* seems to have had rather limited distribution, and is not widely cited in the literature on the calculus of variations. There are copies at the major German university libraries. There is a copy in the British Library, but no copy in the Bibliothèque nationale de France. In the United States, no copies exist in the libraries of Harvard and the Library of Congress. While the University of Chicago was only founded in 1890, it acquired many older books in the calculus of variations (an important field of mathematical research at Chicago into the middle of the twentieth century), although Mayer's book was not among them. There are copies at Yale and Princeton although it is not indicated in their catalogues when the book was acquired. Of course, every major university possesses Crelle's journal, in which Mayer's 1868 article appeared. Today Mayer's *Beiträge* is available online at Google books.

There are two essential ideas in Jacobi's theory of the second variation. The first is the idea of the conjugate point. The second and related idea is that a certain type of variation may be used to transform the second variation to positive definite form. The requisite variation is obtained by taking partial derivatives of the Euler solution functions with respect to the constants of integration appearing in these solutions. Husserl (1883, 12) referred to these discoveries by Jacobi as something "whose sudden appearance almost borders on the miraculous".

A fundamental question in the whole theory is how one is led to the second insight. In Jacobi's original paper it is simply asserted that the variations in question provide solutions to the differential equations required to transform the second variation. In modern textbooks the variations are introduced as if somehow given and it is verified that they provide the requisite solution to the equations of transformation. It was this central point that Husserl focussed on in the opening part of his dissertation. He considered the simplest case of maximizing or minimizing the integral $\int_{x_0}^{x_1} f(x,y,y')\mathrm{d}x$. In many respects his treatment was similar to Spitzer's development of the theory, although he seems not to have been aware of Spitzer's writings and his own work was at a higher level of logical sophistication. What are noteworthy in Husserl's treatment were his recognition of the fundamental point in question, and his presentation of the theory in a way that led to the right sort of variation as a natural development of the theory.

The sources cited by Husserl were (Clebsch 1857) and (Mayer 1868)[10]. It seems clear that he had not read Jacobi. Nowhere does he discuss or show any awareness of the Jacobi-Delaunay-Hesse transformation of the second variation. It is as if this robust line of mathematical inquiry had been excised from history. There is also no indication that he had read Mayer's *Habilitationsschrift*.

The variational integral I is given as

$$I = \int_{x_0}^{x_1} f(x,y,y'). \tag{10}$$

[10] Husserl refers to (Hesse 1857), but only the part on pp. 255-260 where the conjugate point and the transformation of the second variation for the case $n = 1$ are discussed. If he was acquainted with the other 90% of Hesse's memoir it is not apparent in his dissertation. For a study of Hesse's memoir, see (Fraser 1996).

The variation δy of y is denoted by z, $z = \delta y$. The second variation is given in standard form as

$$\delta^2 I = \int_{x_0}^{x_1} 2\Omega_2 \mathrm{d}x, \tag{11}$$

where

$$\Omega_2 = \frac{1}{2}\frac{\partial^2 f}{\partial^2 y^2} z^2 + \frac{\partial^2 f}{\partial y' \partial y} z z' + \frac{1}{2}\frac{\partial^2 f}{\partial^2 y'^2} z'^2. \tag{12}$$

We introduce the function $\nu = \nu(x)$ and subtract and add the term $(\nu z^2)_{x_0}^{x_1}$ to $\int_{x_0}^{x_1} \Omega_2 \mathrm{d}x$:

$$\int_{x_0}^{x_1} \Omega_2 \mathrm{d}x = \int_{x_0}^{x_1} \left(\frac{1}{2}\frac{\partial^2 f}{\partial y^2} z^2 + \frac{\partial^2 f}{\partial y \partial y'} z z' + \frac{1}{2}\frac{\partial^2 f}{\partial y'^2} z'^2 \right) \mathrm{d}x - (\nu z^2)_{x_0}^{x_1} + (\nu z^2)_{x_0}^{x_1}. \tag{13}$$

It will be convenient in what follows to introduce some standard abbreviations:

$$P = \frac{\partial^2 f}{\partial y^2}, \quad Q = \frac{\partial^2 f}{\partial y' \partial y}, \quad R = \frac{\partial^2 f}{\partial y'^2}. \tag{14}$$

We write, $(\nu z^2)_{x_0}^{x_1} = \int_{x_0}^{x_1} (\nu z^2)' \mathrm{d}x$, according to which (13) may be expressed as

$$\int_{x_0}^{x_1} \Omega_2 \mathrm{d}x = \int_{x_0}^{x_1} \left(\left(\frac{1}{2} P - \nu' \right) z^2 + (Q - 2\nu) z z' + \frac{1}{2} R z'^2 \right) \mathrm{d}x + (\nu z^2)_{x_0}^{x_1}. \tag{15}$$

The integrand in (6) is a quadratic form in z and z' and so (15) may be expressed as

$$\int_{x_0}^{x_1} \Omega_2 \mathrm{d}x = \int_{x_0}^{x_1} \left(\frac{R}{2} \left(z' + \frac{Q - 2\nu}{R} z \right)^2 + z^2 \left(\frac{1}{2} P - \nu' - \frac{(Q - 2\nu)^2}{2R} \right) \right) \mathrm{d}x + (\nu z^2)_{x_0}^{x_1}. \tag{16}$$

Expressed in terms of the second variation $\delta^2 I$ we find

$$\delta^2 I = \int_{x_0}^{x_1} \left(R \left(z' + \frac{Q - 2\nu}{R} z \right)^2 + 2 z^2 \left(\frac{1}{2} P - \nu' - \frac{(Q - 2\nu)^2}{2R} \right) \right) \mathrm{d}x + (2\nu z^2)_{x_0}^{x_1}. \tag{17}$$

Suppose the discriminant in (17) is zero:

$$\frac{1}{2} P - \nu' - \frac{(Q - 2\nu)^2}{2R} = 0. \tag{18}$$

Then (17) reduces to

$$\delta^2 I = \int_{x_0}^{x_1} \left(R \left(z' + \frac{Q - 2\nu}{R} z \right)^2 \right) \mathrm{d}x + (2\nu z^2)_{x_0}^{x_1}. \tag{19}$$

For variations that vanish at the endpoints we arrive at Legendre's condition for a maximum or a minimum. This conclusion is based on the assumption that we can find a suitable function ν on the interval $[x_0, x_1]$ Husserl attributed the theory up to this point to Lagrange, although the method of transformation originated with Legendre.

It is natural to consider variations for which the second variation equals zero, since it is necessary to ensure that there are no such variations which also vanish at the endpoints. Husserl supposed that ν is any function with no conditions imposed on it. Consider the differential equation obtained by setting the integrand of (17) equal to zero:

$$R \left(z' + \frac{Q - 2\nu}{R} z \right)^2 + 2z^2 \left(\frac{1}{2} P - \nu' - \frac{(Q - 2\nu)^2}{2R} \right) = 0. \tag{20}$$

The goal is to find a solution $z = \pi(x)$ to this non-linear equation in z. To do so let us suppose that the first term in (20) is equal to zero:

$$R \left(\pi' + \frac{Q - 2\nu}{R} \pi \right)^2 = 0. \tag{21}$$

(20) and (21) imply that

$$\frac{1}{2} P - \nu' - \frac{(Q - 2\nu)^2}{2R} = 0. \tag{22}$$

Using (21) and (22) we eliminate ν to obtain

$$R\pi'' + R'\pi' + (Q' - P)\pi = 0, \tag{23}$$

which is linear in π. ((23) is sometimes known as "Jacobi's differential equation" in the modern subject.) Husserl observed that if $\pi(x)$ is a solution to (23) then the second variation given in (19) reduces to its limit value:

$$\delta^2 I = (2\nu\pi^2)_{x_0}^{x_1}. \tag{24}$$

Husserl proceeded to investigate further conditions – independent of the question of transformation – under which the second variation is zero. To do so he derived from (11) another expression for $\delta^2 I$:

$$\delta^2 I = \int_{x_0}^{x_1} z\left(\frac{\partial \Omega_2}{\partial z} - \frac{d}{dx}\frac{\partial \Omega_2}{\partial z'}\right)dx + \left(z\frac{\partial \Omega_2}{\partial z'}\right)_{x_0}^{x_1}. \qquad (25)$$

(25) will reduce to its limit value if we have

$$\frac{\partial \Omega_2}{\partial z} - \frac{d}{dx}\frac{\partial \Omega_2}{\partial z'} = 0. \qquad (26)$$

Expanded out we obtain from (26) the linear differential equation (23).

Husserl next turned to a closer examination of the Euler equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \qquad (27)$$

Husserl[11] observed that:

$$\frac{\partial 2\Omega_2}{\partial z} - \frac{d}{dx}\frac{\partial 2\Omega_2}{\partial z'} = \delta\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right). \qquad (28)$$

Let $y = \Psi(x, c_1, c_2)$ be the general solution of (27), where c_1 and c_2 are constants of integration. Consider the variation $z = \delta y = \varepsilon \frac{\partial \psi}{\partial c_1}$ Then for the given y and z we have

$$\frac{\partial 2\Omega_2}{\partial z} - \frac{d}{dx}\frac{\partial 2\Omega_2}{\partial z'} = \delta\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) = 0. \qquad (29)$$

A similar conclusion follows for $z = \delta y = \varepsilon \frac{\partial \psi}{\partial c_2}$. Hence the general solution of (26) is

$$z = \varepsilon a_1 \frac{\partial \psi}{\partial c_1} + \varepsilon a_2 \frac{\partial \psi}{\partial c_2}, \qquad (30)$$

[11] Husserl does not have the factor of *2* on the left side of equation (28). On p. 21 he writes $\Omega_2 = \delta(\Omega_1)$, where $\Omega_1 = \frac{\partial f}{\partial y}z + \frac{\partial f}{\partial y'}z'$. This is evidently a slip, given his original definition of Ω_2 in (12).

where a_1 and a_2 are arbitrary constants. We now conclude that the integral of the linear differential equation (23) arising in the transformation of the second variation is given by (30). The function ν may then be obtained from (21). The functions π and ν needed in the transformation are derived from the solution of the Euler equation, Jacobi's original insight. Husserl observed that the result has been reached without any "transcendental operation", apparently referring to the integration of (23).

Husserl noted that the presentation of the result given in (Clebsch 1857) and (Mayer 1868) was simply a verification that did not bring out the source of the result. In making this statement he was emphasizing that the functions required in the transformation were not derived by these authors within the analysis itself but were simply introduced and then verified[12].

4.1.2. An Alternative Derivation

We provide an alternative analysis to Husserl's derivation that is more streamlined, and possibly provides a simpler account of the point in question.

Let us assume that (18) holds so that we have equation (19) for the second variation:

$$\delta^2 I = \int_{x_0}^{x_1} R\left(z' + \frac{Q-2\nu}{R}\right)^2 \mathrm{d}x + (2\nu z^2)_{x_0}^{x_1}. \tag{19}$$

Suppose now that z is given by (30). In this case equation (26) holds and equation (25) reduces to

$$\delta^2 I = \left(z\frac{\partial \Omega_2}{\partial z'}\right)_{x_0}^{x_1}. \tag{31}$$

Equating (19) and (31) we find

$$\left(z\frac{\partial \Omega_2}{\partial z'}\right)_{x_0}^{x_1} = \int_{x_0}^{x_1} R\left(z' + \frac{Q-2\nu}{R}\right)^2 \mathrm{d}x + (2\nu z^2)_{x_0}^{x_1}. \tag{32}$$

[12] Things are no better in modern textbooks (of which (Bolza 1904, Chapter 2, 49) and (Gelfand, Fomin 1963, Chapter 5, 108) are typical). One arrives at an equation of the form (19). It is then asserted with no explanation or process of derivation that we should let $\frac{Q-2\nu}{R} = -\frac{u'}{u}$, where u is of the form (30). Solving for ν and substituting into (18), we arrive at (23), which u satisfies. The last step establishes the validity of the expression obtained for ν.

Setting $\frac{\partial \Omega_2}{\partial z'} = Qz + Rz'$ rearranging terms leads to the identity

$$\left(zR\left(z' + \frac{Q-2\nu}{R} z \right) \right)_{x_0}^{x_1} = \int_{x_0}^{x_1} R\left(z' + \frac{Q-2\nu}{R} \right)^2 \mathrm{d}x. \qquad (33)$$

It is apparent that $z' + \frac{Q-2\nu}{R} z = 0$ is a solution to this equation. Hence we obtain an expression for *ν*. Eliminating it from (18) leads to the linear differential equation (23), which we know to be valid. The present derivation shows more directly than Husserl's how we are led to use variations of the form (30) in the transformation of the second variation[13].

4.2. Mayer's Result

4.2.1 The case $n = 1$

The main question that remains is to show that it is possible to find a suitable transformation function $u = u(x)$ of the form (30) that is non-zero on the interval $[x_0, x_1]$. Husserl provided a discussion of this point but did not include a derivation or proof. In fact, the matter had been effectively handled by Mayer in his *Habiliationsschrift*, as follows. It is necessary to show that if there is no conjugate point on the interval then it is possible to find the desired function $u(x)$. First, we should note that the converse of this statement is true. Assume $u = u(x)$ of the form (30) is non-zero on the interval. We may use this function $u(x)$ in the transformation of the second variation. From equation (21) we have $-\frac{u'}{u} = \frac{Q-2\nu}{R}$. Consider now any other variation $z = z(x)$ or the form (30). Suppose $z(x_0) = 0$ and $z(x) = 0$ for some point x on the interval. If such a variation exists then there is a conjugate point at x. For the given z and the transformation function u we have from (21):

[13] In (Spitzer 1854, 1028-1030) one equates (19) and (25):

$$\int_{x_0}^{x_1} R\left(z' + \frac{Q-2\nu}{R} z \right)^2 \mathrm{d}x + (2\nu z^2)_{x_0}^{x_1} = \int_{x_0}^{x_1} z\left(\frac{\partial \Omega_2}{\partial z} - \frac{d}{dx}\frac{\partial \Omega_2}{\partial z'} \right) \mathrm{d}x + \left(z\frac{\partial \Omega_2}{\partial z'} \right)_{x_0}^{x_1}.$$

In this equation, if z is given by (30) then the integrand in the integral on the right side is equal to zero. Spitzer deduced that the integrand in the integral on the left side of the equation is also zero. In doing so he was following a general principle that he laid out on p. 1027. Evidently this form of inference is not valid.

$$\left(z' - \frac{u'}{u} z\right)^2 = 0. \tag{34}$$

It follows that $z' - \frac{u'}{u} z = 0$ or $\frac{z'}{z} = \frac{u'}{u}$. An integration of this equation gives $z = Au$, where A is a constant. Since $z(x_0) = 0$ and $u(x_0) \neq 0$ it follows that $A = 0$ and z is the trivial zero function. Hence there is no point conjugate to x_0 on the interval $[x_0, x_1]$.

Assume now that Jacobi's condition holds and there is no point conjugate to x_0 on the interval $[x_0, x_1]$. We must find a function $u = \varepsilon_1 c_1 \psi_1 + \varepsilon_2 c_2 \psi_2$ of the form (30) which is non-zero on $[x_0, x_1]$. If x' is conjugate to x_0 then it follows that $x' > x_1$. Let x^* be a number such that $x_1 < x^* < x'$. Let x^0 be any point on the interval $(x_0, x^*]$. Define the function $\Delta(x, x_0)$ as $\Delta(x, x_0) = \psi_2(x_0)\psi_1(x) - \psi_1(x_0)\psi_2(x)$. We have $\Delta(x, x_0) \neq 0$ on $[x^0, x^*]$. From the result in the preceding paragraph it follows that if we set $u(x) = \Delta(x, x^*)$ then $\Delta(x, x^*) \neq 0$ on $\left[x^0, x^*\right)$. But $\Delta(x_0, x^*) = \Delta(x^*, x_0)$ so also $\Delta(x_0, x^*) \neq 0$. Hence $\Delta(x, x^*) \neq 0$ for all x on $[x_0, x^*)$. In particular $\Delta(x, x^*) \neq 0$ on $[x_0, x_1]$. If $u(x) = \Delta(x, x^*)$ then u is the desired function[14].

4.2.2. The General Problem

Having considered the simplest variational problem. Husserl proceeded to an examination of the general case. He followed Clebsch's formulation of the theory, which was presented with a fair degree of formal complexity for the most general problem of the calculus of variations. A detailed consideration of this subject is beyond the scope of the present paper, and we will provide only a summary of Husserl's result. For the sake of exposition, we restrict the discussion to the free problem in which there are no constraints present in the form of side differential equations[15].

[14] Mayer's proof is elegant and may be generalized, and makes use of ideas from the calculus of variations. By contrast, Bolza's proof (1904, 57-60) (given for the case $n = 1$) uses Sturm's theorem (a so-called oscillation theorem) concerning the zeros of two independent integrals of a linear second-order ordinary differential equation. It is less simple than Mayer's and is not immediately generalizable.

[15] In fact the presence of the constraint equations leads to no increase in mathematical difficulty in terms of the main result, and omitting them will simplify the discussion.

The problem at hand involves n dependent variables $y_1, \ldots, y_n$ and the variational integral. $\int_{x_0}^{x_1} f(x', y_1, \ldots, y_n, y_1', \ldots, y')$. The goal is to transform the second variation to the form

$$\delta^2 I = \int_{x_0}^{x_1} \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 \Omega}{\partial y_k \partial y_i} U_k U_i \frac{dx}{U^2}. \tag{35}$$

Here the quantity U is a determinant defined in terms of a system of particular variations u_k, while the U_i are determinants defined in terms of the primary variations z_i and the u_k. The u_k are given in terms of partial derivatives of the Euler solutions, as in (9). Any two such systems u_k and $\bar{u}_k$ satisfy a differential equation of the form

$$\sum_{k=1}^{n} \left\{ \bar{u}_k \frac{\partial \Omega_2}{\partial \left(\frac{du_k}{dx} \right)} - u_k \frac{\partial \bar{\Omega}_2}{\partial \left(\frac{d\bar{u}_k}{dx} \right)} \right\} = \text{constant}. \tag{36}$$

There are $n(n-1)/2$ such equations and they impose conditions on the coefficient values that appear in the u_k. The presence of these conditions constitute an essential complication not present in the case $n = 1$.

In this setting it is possible to formulate Jacobi's condition in terms of a determinate function $\Delta(x, x_0)$. By the definition of $\Delta(x, x_0)$ we have $\Delta(x_0, x_0) = 0$. If x is conjugate to x_0 then $\Delta(x, x_0) = 0$. If Jacobi's condition holds there are no conjugate points on the interval and $\Delta(x, x_0) \neq 0$ on $(x_0, x_1]$. Assume that Jacobi's condition holds. Mayer's primary achievement – and it was not an easy one – was to find a system of functions u_k satisfying the requisite conditions such that equation $\Delta(x, x_\omega) = CU(x, x_\omega)$ holds, where C is a constant and x_ω is any value of x at which Δ and U equal zero. To do so he considered the variational equations in canonical form and found a set of coefficients for the u_k that were given in a certain way in terms of the canonical constants. Having derived the equation he was able to use it in a manner that is similar to the process described in § 4.2.1 above to show that one can obtain the function $U(x)$ that is required in the transformation.

A remarkable contribution of Husserl's dissertation was to show that the existence of a given system of constants required in the derivation of U is a consequence of quite general considerations that make no reference to any particular process of selection. As Husserl (1983, 38) put it:

> Although it [Mayer's method] leads to the results in a rigorous way, such a method possesses certain disadvantages. To make a completely special determination of the constants involves necessarily an element of chance, of something arbitrary, and does not make clear the basis of things. Although there is then nothing to add to these results, it would not perhaps be without interest to find a general and natural procedure, free of all secondary calculations, in order to arrive directly at the criteria, starting out from the transformation of Clebsch and Jacobi.

Husserl realized that Mayer's result did not in fact require a particular selection process in which the coefficients in the expressions for the variations are expressed as functions of the constants of integration appearing in solutions of the Euler equations. Essentially his underlying insight involved properties of linear algebraic equations that belong to what is known today as linear algebra. It is not actually necessary to derive an equation of the form $\Delta(x, x_\omega) = CU(x, x_\omega)$. All that we need to know is that every root of $U(x, x_\omega) = 0$ is also a root of $\Delta(x, x_\omega) = 0$. This fact may be deduced from fairly basic facts about systems of linear equations, showing (in modern terms) that the space of solutions of $U = 0$ is contained in the space of solutions of $\Delta = 0$. Husserl (1983, 43) himself perceptively observed that his analysis was not only more general than Mayer's, yielding the latter as a special case, but was also simpler. (At the conclusion of his discussion of Mayer's result he indicated without proof a simple method involving linear equations of selecting the constants which gives rise to a relation of the form $U(x, x_\omega) = C\Delta(x, x_\omega)$.

In retrospect one could say that the particular approach adopted by Mayer was essentially contingent and arose from his tacit understanding that the coefficients appearing in the u_b must be given in terms of the constants of integration, and more particularly in terms of those constants involved in the integration of the variational equations expressed in canonical form. Clebsch's second paper (1858b) would have reinforced this point of view, providing as it did a general characterization of the transformation functions in terms of the canonical constants of integration.

4.3. Reception of the Mayer Theory

Husserl's dissertation was not published and had no apparent influence.

The later accounts of the Clebsch-Mayer theory, by Jordan (1896), von Escherich (1899) and Bolza (1909), developed the subject along the lines laid down by its original authors, Clebsch and Mayer. These works show high level of generality: the exposition is characterized by formal complexity and elaborate notation. All three researchers made canonical integration methods fundamental to their derivation of Mayer's theorem. Jordan, although he gave no references to sources, followed the second method employed by Mayer (1866) in his *Habilitationsschrift*.

The theory of Mayer and Clebsch and indeed all work on the problem of sufficiency up to that point may be characterized generally in terms of what are called expansionist methods. The differential or increment of the variational integral is expanded as a Taylor series and the question of sufficiency is investigated by examining the second variation. In the late 1870s and early 1880s Weierstrass gave lectures at the University of Berlin on the calculus of variations, copies of which circulated in the last years of the century and which were eventually published in 1927. Husserl's own set of notes on these lectures along with other sets were used in compiling this edition[16]. The final result Husserl presented in his dissertation concerned a necessary condition formulated by Weierstrass. Working at a high theoretical and critical level, Weierstrass set out a new approach to the problem of sufficiency. This approach became the subject of a substantial research program in the period 1890 to 1920. Stimulated by some contributions of David Hilbert it led to the creation of what is known as field theory in the calculus of variations. (For a detailed history of this development see (Thiele 2007); (Fraser 2009) provides an English-language essay review of Thiele's book.) The success of the Weierstrassian program tended to overshadow the earlier contributions of Clebsch and Mayer and may have contributed to the relative neglect of their work. There were prominent researchers such as (von Escherich 1899) who continued to uphold the older expansionist methods and found the Weierstrassian field theoretic approach to be somewhat artificial. Nevertheless, proponents of field theory dominated the study of sufficiency in the first half of the twentieth century, and the early work of Mayer receded into history.

[16] On Husserl and Weierstrass see (Biermann 1969).

Conclusion

As a contribution to the theory of the second variation, Husserl's dissertation was a substantial work, notable for its attention to foundational matters and mathematically incisive in its exposition of Mayer's major result. As a doctoral dissertation it was not typical of this genre of mathematical writing, which tends to focus on novel solutions of well-defined problems or on elaborating the details of known methods. By contrast, Husserl was reflecting on the fundamental logical and theoretical character of the subject at hand, with trenchant observations about leading contemporary work in the subject. One does not normally find this sort of critical sense in research memoirs or textbook expositions, either in the nineteenth century or today. As he drifted from the calculus of variations to philosophy Husserl may have felt that the qualities he brought to the mathematical subject were not an ideal fit for a career as a research mathematician.

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