[62] Robinson 1973, p. 14.

[63] For details, see Dauben 1979/1990, pp. 128-132 and pp. 233-239.

[64] See Dauben 1979.

[65] Robinson 1973, p. 16.

[66] Robinson completed his dissertation, "The Metamathematics of Algebraic Systems," at Birkbeck College, London University, in 1949. It was published two years later; see A. Robinson, *On the Metamathematics of Algebra*, Amsterdam: North-Holland Publishing Company, 1951.

[67] S. Kochen, "Abraham Robinson: The Pure Mathematician. On Abraham Robinson's Work in Mathematical Logic," *Bulletin of the London Mathematical Society*, 8 (1976), pp. 312-315, esp. p. 313

History of Mathematics and Education: Ideas and Experience, Ed. H. N Jahnke, N. Knoche and M. Otte. Volume 11 of the series "Studien zur Wissenschafts-, Sozial und Bildungsgeschichte der Mathematik". 1996. Vandenhoeck. Göttingen. Pp. 149-172

Jacobi´s Result (1837) in the Calculus of Variations and its Reformulation by Otto Hesse (1857). A Study in the Changing Interpretation of Mathematical Theorems

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I. Introduction

I.1

Jacobi's interest in the calculus of variations was connected with his work in analytical mechanics, a subject that he had been investigating since at least 1832. His major paper of 1837, *"Zur Theorie der Variations-Rechnung und der Differential-Gleichungen"*, was divided in two parts, the first devoted to the mathematical problem of finding conditions for an extremum, the second to a discussion of the dynamical equations of motion of a system of particles. Although the mathematical and mechanical parts were logically independent there were significant links connecting his work in the two domains. Some of the examples that he presented to illustrate his mathematical ideas were drawn from analytical mechanics, while the latter subject was itself developed in terms of integral variational principles.

From the moment of its publication Jacobi's paper was recognized as an important contribution to mathematics. Joseph Liouville arranged a French translation which was published in 1838 in his *Journal de Mathématiques Pures et Appliquées*. In the years which followed V. A. Lebesgue and C. Delaunay wrote detailed memoirs for this journal in which they provided demonstrations for results Jacobi had stated without proof. In his 1850 *An Elementary Treatise on the Calculus of Variations* John Jellett used Delaunay's memoir in order to give a very complete account of Jacobi's theorem. He supplemented this account with the presentation of many illustrative examples. In 1861 Isaac

Todhunter published an English translation of the mathematical part of Jacobi's paper and gave an historical survey of the various results then available.

Jacobi's results were the subject of detailed investigation by researchers in Italy and Germany. Notable among these was the Königsberg mathematician Otto Hesse. Hesse, a student of Jacobi's in the early 1830s, published in 1857 an important and influential article in Crelle's journal. His results were extended by the young Alfred Clebsch in the following year to problems with side conditions in the form of differential equations; these researches also appeared in Crelle's journal. Adolph Mayer, a leading researcher in the calculus of variations at the end of the century, built on the work of both Hesse and Clebsch.

Hesse, Clebsch and Mayer developed their results in research articles published in leading mathematical journals. Elements of Hesse's approach were also picked up by the authors of two textbooks of the 1860s. Moigno and Lindelöf in volume 4 of their *Leçons de Calcul Différentiel et de Calcul Intégral* (1861) and J. Dienger in his *Grundriss der Variationsrechnung* (1867) both based their exposition on Jacobi's original paper, but incorporated important elements of Hesse's contribution. In an 1881 textbook the American author Lewis Carll adopted Jellett's exposition (the latter itself being based on the work of Delaunay) but also described several of Hesse's innovations.

In the following decades Hesse's version gradually achieved dominance as the preferred account of the classical theory of the second variation. In volume three of his 1896 treatise *Cours d'Analyse* Camille Jordan based his exposition solely on Hesse's and Clebsch's theory. In Oscar Bolza's authoritative textbooks of 1904 and 1909 Hesse's account was adopted as the preferred way of analyzing the second variation.

Hesse's theory became the established approach in advanced continental research into the second variation. It also became closely associated with historical accounts of Jacobi's original paper. At the end of the century, as part of the general historical interest that had developed among German mathematicians, Paul Stäckel published a collection of classic papers in the calculus of variations for the *Ostwald Klassiker* series. Included was Jacobi's article of 1837. In the explanatory notes at the end Stäckel used Hesse's formulation in order to clarify Jacobi's contribution. Bolza, an author concerned with historical sources, indicated that he would draw on Hesse in his account of Jacobi's theorem, indicating that Hesse had provided the "most complete commentary" on Jacobi's result (Bolza 1909, p.60). Goldstine preceded his

historical account of Jacobi's paper by an "excursus" containing a summary of Hesse's theory (Goldstine 1980, p. 151-56). In his discussion of the paper itself he referred to this theory in order to document the successive elements of Jacobi's achievement.

I.2

The present essay is devoted to a comparative examination of Jacobi's paper of 1837 and Hesse's of 1857. Our approach is guided by the general historiographical principle that much can be learned in the history of mathematics through a comparative study of the successive formulations that a given theorem or solution receives in the course of its development. Such a critical method, which allows a detailed point-by-point comparison of the corresponding steps in different derivations of the same theorem, results in a clearer perception of the history and leads to a better understanding of the mathematical and conceptual character of the theory in question. This method, which has been applied by the author in several concrete examples[1], provides a useful tool of historical investigation, one whose possibilities have perhaps yet to be fully recognized in contemporary historiography of mathematics.

In the present essay we will for reasons of exposition limit the level of technicality, although it should be noted that additional details would need to be supplied in a more complete account. We have tried to make the relevant points accessible to a mathematically educated reader with the usual access to standard literature who has no special training in the calculus of variations. Our goal it first to identify the relative mathematical character of Jacobi's and Hesse's work and second to consider the implications of our study for an understanding of the place of Jacobi's theorem in the history of the calculus of variations. Our discussion, somewhat critical of traditional historiography, concludes with some reflections on changes that have occurred since 1800 in the basic outlook of the subject.

II. Jacobi's Paper of 1837

II.1

We are in what follows concerned only with the mathematical part of Jacobi's paper. The background to his investigation was provided by work of Legendre and Lagrange on the second variation.[2] In 1786 Legendre had devised a transformation of this variation which led to the establishment of a new necessary condition for the existence of an extremum. Legendre's results were presented by Lagrange in his *Théorie des fonctions analytiques* (1797), who discussed further some technical points concerning the existence of the functions that were required in order to obtain the given condition.

Ostensibly Jacobi formulated his paper as a further refinement and extension of Lagrange's results. While this was the form he gave to his investigation it is clear that he was presenting a whole new approach to the subject, one based on important and original ideas. We will explain his basic conception for the simplest case where the variational integrand f is a function of x, y and $y'=dy/dx$. The variational integral under consideration is

$$(1) \qquad I=\int_{x_0}^{x_1} f(x,y,y')dx$$

where we have inserted the limits of integration x_0 and x_1 - absent in the original- into the formula for this integral. Let us assume that a given function $y=y(x)$ renders the integral I an extremum. We replace y by $y+\delta y$ and consider the value of I along the resulting comparison curve. Here $\delta y=\varepsilon w(x)$, where ε is a small multiplicative constant and w is a function of x with $w(x_0)=w(x_1)=0$. The difference ΔI in the value of I along the actual and comparison arcs is

(2)

$$\Delta I=\varepsilon\cdot\int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}w+\frac{\partial f}{\partial y'}w'\right)dx+\frac{\varepsilon^2}{2}\cdot\int_{x_0}^{x_1}\left(\frac{\partial^2 f}{\partial y^2}w^2+2\frac{\partial^2 f}{\partial y\partial y'}ww'+\frac{\partial^2 f}{\partial y'^2}w'^2\right)dx+\ldots$$

It is clear that the first term in the expansion (2) will dominate. Because I is an extremum we obtain the equation

$$(3) \qquad \int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}w+\frac{\partial f}{\partial y'}w'\right)dx=0$$

By means of a standard procedure that originated with Lagrange (3) may be transformed. Using an integration by parts and the fact that $w(x_0)=w(x_1)=0$ we express (3) in the form

$$(4) \qquad \int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)wdx=0$$

Because $w(x)$ is arbitrary it is clear that the solution $y=y(x)$ to the variational problem will need to satisfy the Euler differential equation

$$(5) \qquad \frac{\partial f}{dy}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

It is evident that in the case of a genuine extremum ΔI must preserve the same sign for all admissible increment functions $w(x)$. Consider now the second term in the expansion for ΔI given in (2)

$$(6) \qquad \frac{\varepsilon^2}{2}\int_{x_0}^{x_1}\left(\frac{\partial^2 f}{\partial y^2}w^2+2\frac{\partial^2 f}{\partial y\partial y'}ww'+\frac{\partial^2 f}{\partial y'^2}w'^2\right)dx$$

Because the first-order term in (2) is zero it is clear that (6) will dominate in the expansion. The third-order term involving ε^3 can in general be made either positive or negative. Thus in order for a genuine extremum to obtain it must be the case that there is no $w(x)$ for which (6) is equal to zero.

Jacobi's basic insight was to formulate the preceding analysis in terms of the variational operation δ. Using the δ-process the increment ΔI may be expressed in the form

(7) $\Delta I = \delta I + \frac{1}{2}\delta^2 I + \frac{1}{3!}\delta^3 I....$

Here δI, $\delta^2 I$, $\delta^3 I$,... denote the first, second, third... variation of I. We have

(8) $$\delta I = \delta \int_{x_0}^{x_1} f\,dx = \int_{x_0}^{x_1} \delta f\,dx = \int_{x_0}^{x_1} (\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y')dx$$

In the same way in which equation (4) was obtained (8) may be expressed in the form

(9) $$\delta I = \int_{x_0}^{x_1} V\delta y dx$$

where

(10) $$V = \frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'})$$

Using the operational character of the δ-process we find that

(11) $$\delta^2 I = \delta(\delta I) = \delta(\int_{x_0}^{x_1} V\delta y)dx = \int_{x_0}^{x_1} \delta V \delta y dx,$$

or simply

(12) $$\delta^2 I = \int_{x_0}^{x_1} \delta V \delta y dx.$$

We have seen that the solution $y=y(x)$ to the variational problem must satisfy the Euler differential equation (5). The general solution to this second-order equation will contain two arbitrary constants a and b. If the given solution is to

be a genuine extremum then the second variation $\delta^2 I$ must be different from zero for all admissible variations δy. We are therefore led naturally to consider conditions under which $\delta^2 I=0$. From (12) it is apparent that $\delta^2 I=0$ if

(13) $\delta V = 0$.

Let $y=y(x,a)$ be a solution of the Euler equation (5), where the notation indicates the dependency of the solution on the arbitrary constant a. We have

(14) $$V(a) = \frac{\partial f(x,y(x,a),y'(x,a))}{\partial y} - \frac{d}{dx}(\frac{\partial f(x,y(x,a),y'(x,a))}{\partial y'}) = 0$$

Let us increase a by the increment $a+\delta a$. We have again

(15) $V(a+\delta a) = \varnothing$

$$= \frac{\partial f(x,y(x,a+\delta a),y'(x,a+\delta a))}{\partial y} - \frac{d}{dx}(\frac{\partial f(x,y(x,a+\delta a),y'(x,a+\delta a))}{\partial y'}) = 0$$

We now consider a variational process in which $\delta y=(\partial y/\partial a)\delta a$. By subtracting (14) from (15) we see that $\delta V=0$. Similarly if $\delta y=(\partial y/\partial b)\delta b$ we have $\delta V=0$. Since $\delta V=0$ is a second-order linear differential equation in $\delta y=\varepsilon w(x)$ the general solution will be of the form

(16) $$w(x) = \alpha \cdot \frac{\partial f}{\partial a} + \beta \cdot \frac{\partial f}{\partial b}$$

where α and β are constants.

Jacobi was able to derive an important result concerning Legendre's transformation of the second variation (he referred in this connection to treatises of Lagrange and Dirksen). The second variation is

(17) $$\delta^2 I = \varepsilon^2 \int_{x_0}^{x_1} (\frac{\partial^2 f}{\partial y^2}w^2 + 2\frac{\partial^2 f}{\partial y \partial y'}ww' + \frac{\partial^2 f}{\partial y'^2}w'^2)dx$$

Let $v=v(x)$ be a function of x and consider the expression

$$(18)\qquad \frac{d}{dx}(w^2v).$$

Because $w(x_0)=w(x_1)=0$ the integral of (18) is zero:

$$(19)\qquad \int_{x_0}^{x_1}\frac{d}{dx}(w^2v)dx=0.$$

Thus if we add (19) (multiplied by ε^2) to (17) there results no change in the value of the second variation:

$$(20)\qquad \delta^2 I=\varepsilon^2\int_{x_0}^{x_1}((\frac{\partial^2 f}{\partial y^2}+\frac{dv}{dx})w^2+2(\frac{\partial^2 f}{\partial y\partial y'}+v)ww'+\frac{\partial^2 f}{\partial y'^2}w'^2)dx$$

The integrand is a quadratic expression in w and w'. Legendre observed that it will become a perfect square if

$$(21)\qquad \frac{\partial^2 f}{\partial y'^2}(\frac{\partial^2 f}{\partial y^2}+\frac{dv}{dx})=(\frac{\partial^2 f}{\partial y\partial y'}+v)^2$$

For $v(x)$ satisfying this differential equation the second variation becomes

$$(22)\qquad \varepsilon^2\int_{x_0}^{x_1}\frac{\partial^2 f}{\partial y'^2}(w'+\frac{\frac{\partial^2 f}{\partial y\partial y'}+v}{\frac{\partial^2 f}{\partial y'^2}}w)^2dx$$

Thus in order for the solution to be a minimum it is clear that we must have

$$(23)\qquad \frac{\partial^2 f}{\partial y'^2}\geq 0,$$

which is simply Legendre's necessary condition.

Let $u(x)$ be given by (16), i.e., $u(x)=\alpha(\partial y/\partial a)+\beta(\partial y/\partial b)$, where $y(x,a,b)$ is the general solution of the Euler equation (5). Jacobi observed that the desired solution v to (21) will be given as

$$(24)\qquad v=-(\frac{\partial^2 f}{\partial y\partial y'}+\frac{1}{u}\frac{\partial^2 f}{\partial y'^2}\frac{du}{dx}).$$

To understand this note that $w(x)=u(x)$ makes the second variation equal to zero. Introducing this value for w into (22) and equating the integral to zero we obtain

$$(25)\qquad \frac{du}{dx}+(\frac{\frac{\partial^2 f}{\partial y\partial y'}+v}{\frac{\partial^2 f}{\partial y'^2}})u=0.$$

Solving this equation for v we obtain the expression given in (24).

A criticism Lagrange had made of Legendre's procedure concerned the question of whether finite solutions v to (21) exist on the interval in question. By expressing v by means of (24) Jacobi had reduced this question to a study of the known function $u(x)=\alpha(\partial y/\partial a)+\beta(\partial y/\partial b)$.

The preceding account follows the reasoning of Jacobi's original paper closely. We have rendered in more detail the reasoning which allows one to conclude that $\delta y=(\partial y/\partial a)\delta a$ is a solution of $\delta V=0$. In addition, Jacobi at the beginning of the paper asserted, with no explanation at all, that equation (24) gives the function v needed in Legendre's transformation of the second variation. It is only rather later in the paper that the reasoning behind this somewhat mysterious announcement becomes clear.

That our account is faithful to the underlying sense of the paper is clear from Jacobi's own very lucid summary:

The metaphysic of the results obtained (if I may use a French expression) depends nearly upon the following considerations. The first variation is known to take the form $\int V\delta y dx$, where $V=0$ is the equation to be integrated. The second variation then takes the form $\int \delta V \delta y dx$. If then the second variation is to be incapable of changing its sign, it must be incapable of vanishing; so that the equation $\delta V=0$, which is linear in δy, must have no integral which satisfies the conditions to which by the nature of the problem δy is subjected. Thus we see that the equation $\delta V=0$ plays an important part in these investigations, and we soon perceive its connexion with the differential equations which must be integrated in order to obtain the criteria for maxima and minima. For we easily see that a partial differential coefficient of y with respect to any constant which occurs in y as the solution of $V=0$, will be a suitable value of δy satisfying the differential equation $\delta V=0$. Thus the general expression for δy as the integral of the equation $\delta V=0$ will be a linear function of all the partial differential coefficients of y with respect to the constants which it involves.[3]

II.2

In the paper Jacobi also outlined a general method different from Legendre's for changing the second variation to a form in which the integrand consists of a term multiplied by a square. His method was based on a certain result in the theory of linear differential equations. Although we will not here describe the analysis it is worth noting that his derivation of what later became known as "Jacobi's differential equation" made use of the operational character of the d and δ symbols.

Having presented this transformation Jacobi turned to the question of how in a particular problem one decides whether a given solution is a genuine maximum or minimum. He formulated the discussion in terms of neighbouring solutions to the Euler equation. Although he did not relate the main idea to his earlier analysis his reasoning seems to have been more or less the following. Possible extremalizing arcs will be solutions to the Euler equation $V=0$. Let its general solution $y(x,a)$ contain the arbitrary constant a. If we let $\delta y=(\partial y/\partial a)\delta a$ then $\delta V=0$ and the second variation is zero. Assume further that both $y(x,a)$ and $y(x,a+\delta a)$ satisfy the given end conditions. Then δy is an admissible variation, i.e. one for which $\delta y(x_0)=\delta y(x_1)=0$. It is clear in this situation that there can be no maximum or minimum, because the sign of the third variation can in general be made either positive or negative. Hence to establish that we have a genuine extremum we must ensure that it is not possible to pass two neighbouring arcs $y(x,a)$ and $y(x,a+\delta a)$ through the endpoints. As illustration of these ideas Jacobi considered the elliptical motion of a planet governed by the principle of least action. He also mentioned the case of a particle constrained to move on the surface of a sphere. For the purposes of the present paper it is not necessary to work through these examples. As in the rest of his discussion he provided only an outline, suppressing much of the reasoning and rather considerable calculational work that would have been involved in a more complete account.

III. Hesse's Paper of 1857

III.1

As noted earlier much subsequent research in the calculus of variations centred on providing a fuller treatment of the transformation of the second variation for the general case where the integrand contains derivatives of arbitrary order of y with respect to x. Hesse's distinctive achievement in his paper of 1857 consisted of the particular analytical formulation he gave to this question. An influential line of later writers would base their presentation of Jacobi's theorem on Hesse's account. These authors were typically concerned with exposition, with establishing the basic results and principles of the calculus of variations, and so restricted their initial discussion to the simplest case where the variational integrand is of the form $f(x,y,y')$. It thus occurred somewhat paradoxically that Hesse's formulation, whose rather specific and technical purpose was to facilitate the general analytical transformation of the second variation, became the accepted approach to the basic ideas underlying Jacobi's theorem.

A full appreciation of Hesse's contribution can only be achieved by considering his treatment of the general variational problem, in which the integrand function f is of the form $f(x,y,y',y'',...y^{(n)})$. Since our interest here is only in the implications of his approach for the ideas of Jacobi's theorem we shall - with some reservation - follow the precedent of modern textbook writers and consider his formulation for the simplest case, in which f is of the form $f(x,y,y')$.[4]

III.2

Hesse considered functions y of x on the interval $a \le x \le b$. The variational integral is

$$(26) \quad \int_a^b f(x,y,y')dx.$$

The variation of y is denoted by z, where z is assumed to consist of an increment function multiplied by a small constant. The Euler equation for the problem is

$$(27) \quad f'(y) - \frac{df'(y')}{dx} = 0.$$

Solutions to (27) will contain two arbitrary constants α_1, α_2. Hesse presents the second variation in the standard form

$$(28) \quad \Delta_2 = \int_a^b 2\Psi dx$$

where

$$(29) \quad 2\Psi = a_{00}zz + 2a_{01}zz' + a_{11}z'z'.$$

Here

$$(30) \quad a_{00} = \frac{\partial^2 f}{\partial y^2}, \; a_{01} = \frac{\partial^2 f}{\partial y \partial y'}, \; a_{11} = \frac{\partial^2 f}{\partial y'^2}$$

In §6 of his paper he showed how (28) can be transformed by means of an integration by parts. We write (29) in the form

$$(31) \quad 2\Psi = a_0 z + a_1 z',$$

where

$$(32) \quad a_0 = \Psi'(z) = \frac{\partial \Psi}{\partial z}, \; a_1 = \Psi'(z') = \frac{\partial \Psi}{\partial z'}.$$

(31) is now re-expressed in the form

$$(33) \quad a_0 z + a_1 z' = (a_0 - a_1')z + \frac{da_1 z}{dx}.$$

Hesse set

$$(34) \quad \Psi(z) = a_0 - a_1' = \Psi'(z) - \frac{d\Psi'(z')}{dx}.$$

It is assumed that the variation z vanishes at the endpoints a and b. Integrating (33) we obtain the desired expression for the second variation

$$(35) \quad \Delta_2 = \int_a^b z\Psi(z)dx.$$

Hesse re-expressed Ψ in the form

$$(36) \quad \Psi(z) = (a_{00} - a_{01}')z - \frac{da_{11}z'}{dx}.$$

From (36) we immediately have

$$(37) \quad u\Psi(z) - z\Psi(u) = -\frac{d\{a_{11}(u'z - zu')\}}{dx}.$$

Integration of (37) yields

$$(38) \quad \int \{u\Psi(z) - z\Psi(u)\}dx = -a_{11}(u'z - zu').$$

Recall that the general solution to the Euler equation (27) contains the constants α_1 and α_2. Hesse observed (§8) that if one differentiates (27) with respect to α_λ (λ=1,2) one obtains

(39) $$\Psi(\frac{\partial y}{\partial \alpha_\lambda}) = 0.$$

He set

(40) $$r_1 = \frac{\partial y}{\partial \alpha_1},\ r_2 = \frac{\partial y}{\partial \alpha_2}.$$

Since $\Psi(z)=0$ is a second-order linear differential equation in z its general solution is $u=a_1r_1+a_2r_2$ (a_1 and a_2 are new constants). Substituting this value for u into (38) we have

(41) $$\int u\Psi(z)dx = -a_{11}(uz' - zu').$$

Using (41) we integrate (35) by parts and obtain

(42) $$\Delta_2 = \int_a^b a_{11}\frac{d(\frac{z}{u})}{dx}(uz' - zu')dx,$$

or

(43) $$\Delta_2 = \int_a^b a_{11}\frac{(uz' - zu')^2}{u^2}dx.$$

(43) is the desired transformation of the second variation, yielding the Legendre condition $a_{11}\geq 0$ for a minimum. Hesse concluded with a discussion of the relationships among the constants appearing in the solution to the problem.

III.3

There are two defining elements of Hesse's approach that would be widely adopted by later writers in the calculus of variations. The first is to express the quantity 2ψ in the integrand of the second variation Δ_2 in the form

$$2\Psi = \frac{\partial \Psi}{\partial z}z + \frac{\partial \psi}{\partial z'}z',$$

and to integrate

$$\Delta_2 = \int_a^b 2\Psi dx$$

by parts to obtain

$$\Delta_2 = \int_a^b z\Psi(z)dx,$$

where $\Psi(z) = \partial\psi/\partial z - d(\partial\psi/\partial z')/dx$. The second is to take a solution $y=y(x,\alpha)$ of Euler's equation and to verify by differentiating this equation with respect to α that $z=\partial y/\partial\alpha$ satisfies the equation $\Psi(z)=0$. (The equation $\Psi(z)=0$ is often called "Jacobi's differential equation" in modern textbooks.)

III.4

Unlike Jacobi Hesse provided no examples from dynamics or geometry to illustrate the analysis. The perspective of his investigation was limited. The paper was concerned with a study of a certain class of linear differential equations and with the formal transformation of the integrand in the second variation. The conceptual development of the subject received little attention.[5]

IV. Comparison

IV.1

The really new idea of Jacobi's, what imparted to the subject a fundamental new direction, was to analyze the second variation in terms of the partial derivatives of the general solution to the Euler equation with respect to the arbitrary constants appearing in this solution. If one traces the source of this idea it will be found in Jacobi's exploitation of the operational character of the variational process. By initially expressing the first variation in the form $\int V\delta y dx$, where $V=0$ is the Euler equation, he was able to express the second variation as $\int \delta V \delta y dx$. The condition that the second variation is zero could then be analyzed in terms of variations that arose from varying the constants appearing in the general solution to $V=0$.

In Hesse's formulation this fundamental idea, what Jacobi called the "metaphysic" of his result, is absent. We are instructed to treat the integrand in the second variation as a quadratic form in the increment function and its derivative and to integrate this form by parts; the resulting expression equated to zero is shown, a posteriori, to be satisfied by the partial derivatives of the general solution to the Euler equation. Rather than being regarded as a more "complete" account of Jacobi's result, Hesse's formulation should be seen as a distinct variant, one lacking the cogency of Jacobi's original exposition but possessing certain attractive analytical features in the formal transformation of the general variational integrand.

IV.2

Jacobi's research was rooted in an approach to the calculus of variations in which the conception of the δ-process was fundamental. The operational character of this process had been at the base of Lagrange's original innovation and was emphasized by subsequent writers of the period. It should be noted that at a more general level the calculus of variations was embedded in a well-established eighteenth-century tradition of algebraic analysis. The ordinary calculus had from its very beginning in Leibniz relied heavily on the operational nature of the differential process, evident in the algorithmic properties of the *d* operation. With the increasing de-geometrization of the calculus that occurred

in the eighteenth century, most prominently in the textbooks of Euler, a conception of algebraic analysis developed, stressing the formal, relational, and purely algebraic character of the subject. In the writings of Euler and Lagrange this programme of algebraic analysis received detailed and highly authoritative development.[6]

Interest in algebraic analysis was apparent in the attempts at the end of the eighteenth and the first part of the nineteenth centuries to develop various sorts of operational calculi.[7] Within the calculus of variations authors such as Robert Woodhouse (1810) and E. H. Dirksen (1823) explicitly construed the subject in terms of the application of a general δ-process to problems of maxima and minima. Jacobi's paper of 1837 can be seen as a brilliant instance of algebraic analysis in which recognition of the operational character of the δ-process led to a major advance, one that reoriented the subject and resulted in a seminal new line of research.

The more general trend of nineteenth-century mathematics was however to reject algebraic analysis as the logical basis for the calculus. Beginning with Cauchy's textbooks of the 1820s the older conception was replaced by a newer understanding which emphasized the numerical continuum and the properties of functions defined on this continuum. The algorithmic, formal features of the subject were no longer regarded as part of its essential defining character.

The researches of Hesse in the calculus of variations were consistent with this larger movement in nineteenth-century analysis. In his approach the operational character of the subject was suppressed in favour of an a posteriori analytical transformation of the second variation. This feature of Hesse's investigation was present in the subsequent researches of Clebsch and Mayer. The period 1875 to 1900 witnessed the development of field methods in the writings of Scheeffer, Du Bois-Reymond, Weierstrass, Zermelo, Mayer, Schwarz, Kneser, and Hilbert. It should be noted that here as well the operational character of the δ-process was suppressed. There was no consideration of the second variation as such; instead of expanding the difference of the variational integral along two comparison arcs in a power series one analyzed instead this difference directly in terms of a field of extremals consisting of solutions to the Euler equation.

In the modern subject the δ operation has come to be seen as something which has a certain practical value but which is in fact extraneous to the essential logical and conceptual character of the subject. The situation is clear in comments scattered throughout the literature. Osgood writes "One of the weak points in the use of the calculus of variations in physics lies in the tacit assumption that the variations require no particular definition, for everyone knows what δy, δJ, etc. mean. As a matter of fact, their definition...is an extremely delicate matter..." (Osgood 1925, p. 432). In his textbook Bolza prefaces his account of the δ formalism with the note, "Wir empfehlen den Leser, diesen Paragraphen vorläufig zu überschlagen und erst bei Bedarf darauf zurückzugreifen." (Bolza 1909, p. 45 n.1). In referring to research of the early 1800s Bliss observes that the "The analogies between the variations of Lagrange and the differentials of the ordinary calculus absorbed the interest of students of the subject, who elaborated them, with doubtful rigor and without great profit to the theory." (Bliss 1925, p. 176-177). In a work published two decades later the same author notes "At the present time many of the results of the theory can be obtained more readily without extensive use of the δ notations." (Bliss 1946, p. 6). Pars observes "The operator δ played a great part in the early researches. Nowadays it is possible to develop the subject without using δ at all.." (Pars 1962, p. 17). Ewing notes "The name calculus of variations comes from a procedure of Lagrange involving an operator δ called a variation, but this restricted meaning has long been overgrown." (Ewing 1969, xi). In his *Lectures on the calculus of variations and optimal control theory*. Young writes, "Already the title of our subject is a purely historical one; it refers to a particular method, due to Euler and based on so-called variations, that was at one time important in the subject, but which is of very secondary interest today." (Young 1969, p. 3).[8]

Researchers of the early nineteenth century placed the δ-algorithm at the foundation of the subject because of their belief in the fundamental role of the operational variational process. This process lay at the heart of Lagrange's invention and was basic to Jacobi's brilliant research. In their attempt to extend and refine the concept of an extremalizing arc subsequent mathematicians embarked on a very careful delineation of the class of functional solutions and provided a clearer characterization of the precise variational processes under consideration. They were motivated not so much by a concern with rigour as a desire to open up for closer study an extended and fascinating realm of functional entities, one that was mathematically self-contained and capable of detailed study by sophisticated techniques of real analysis. The approach of the older variational formalism, with its relatively liberal manipulation of operations and expressions, seemed to detract from this mission. The concept of a variation - understood as an operation - was no longer regarded as a fundamental entity of the theory. A certain incommensurability developed between the modern outlook and the way of thinking which had prevailed in the subject a century earlier.

V. Further discussion

V.1

In sections I.1 and III.3 it was noted that modern authors typically follow Hesse in their presentation of Jacobi's theorem. This is the case for textbooks as well as for writings that are explicitly historical in nature. In the textbook literature Hesse is seldom actually mentioned or cited; his approach has become in a generalized and conventional sense part of the commonality of the subject.[9] On those occasions where he is credited it is done in a way that would suggest that what is being presented is simply a refinement or an elaboration of the original theory, an advance in style and technique but not really a distinct formulation of the theory.[10]

A possible explanation of this practice arises from Jacobi's well known failure in the 1837 paper to provide details and full demonstrations for his propositions and examples. According to this view Hesse simply supplied ideas and results that were already implicit in the earlier paper. Given this and given that Hesse's formulation became dominant in the modern subject it seems natural to interpret Jacobi's results in terms of the later work.

Plausible as this explanation may seem it fails to withstand closer scrutiny. Although it is true that Jacobi wrote in a compressed and elliptical style it is also the case that his basic approach was clear. The preceding study has documented the differences in mathematical content in the papers of the two men. The Hessian theory fails to provide an appropriate framework for either a mathematical or historical appreciation of Jacobi's achievement.

This criticism is not meant to denigrate the very impressive historical contribution made by modern authors in the calculus of variations. Among the various branches of analysis this subject has beginning with Lagrange been characterized by a continuous and vigorous interest in its history. There has nevertheless since the late nineteenth century been a certain definite alignment to historical writings. The present study provides a concrete example of how a given mathematical achievement is reinterpreted in the course of its development. Advanced research at the end of the nineteenth century adopted Hesse's formulation of Jacobi's theorem. The modern historical tradition has been embedded in this tradition, bringing to its historical outlook various presuppositions and judgments of value. An almost unconscious resistance developed to recognizing or appreciating older viewpoints and ways of thinking. A conceptual gulf is evident in Goldstine's (1980, vii) judgement of Todhunter's history that it is "*hopelessly* archaic"[11], a criticism directed primarily at the Englishman's older mathematical heritage.

V.2

From a purely didactic standpoint, detached from any historical concern, it is also possible to criticize the Hessian formulation of Jacobi's theorem. The essential idea of this theorem is that an increment function given as the partial derivative of the solution to the Euler equation with respect to an arbitrary constant annihilates the second variation. The source of this idea is located in recognition of the operational relationship between the first and second variation. Apprehension of the full meaning of the theorem involves a clear appreciation of this situation, which, as we observed in IV.1, is not conveyed in Hesse's account. In order to arrive at real understanding a student is best advised to return to Jacobi's original paper. It would indeed still be necessary to consult later writings for a detailed treatment of the geometrical and dynamical examples that Jacobi described only in outline. Nevertheless, the suggestion "read the masters", sometimes urged but not taken seriously, here possesses genuine and undeniable import.

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Notes

[1] In [Fraser 1985] D'Alembert's original derivation of the dynamical differential equations in two problems contained in his *Traité de Dynamique* (1743) is described in detail along with the corresponding modern or "classical" deduction of the same equations. The resulting comparison reveals the distinctiveness as well as the scientific and historical interest of D'Alembert's theory. In [Fraser, 1989] Euler's demonstration of the theorem on the equality of mixed partial derivatives is presented along with the classical (post-Cauchy) proof of the same result in order to bring out differences between Eulerian and modern conceptions of analysis. In a paper presented at a conference on Newton in Moscow Fraser [1987] provides a detailed account of Newton's derivation of Proposition 11 of Book One of the Principia Mathematica (1687) and Varignon's derivation of the same result in his treatise of 1700 on central forces. The resulting comparison indicates significant differences in their understanding of the nature of analysis.

[2] For references and discussion see [Todhunter, 1861, p. 1-5, p. 229-233] and Goldstine [1981, p. 139-147].

[3] English translation taken from [Todhunter 1861, p. 248].

[4] Hesse explicitly considered this case in §9.

[5] Hesse would go on to do extensive research on the theory of invariants in algebraic geometry, the part of mathematics for which he is best known. Felix Klein's comment on this work (reported in [Haas 1972, p. 357]) is not irrelevant to the paper of 1857 under consideration here: Hesse's methods of presenting material fortified and disseminated the feeling for elegant calculations expressed in symmetrical formulas.

[6] Fraser [1989] discusses Euler's and Lagrange's conception of algebraic analysis. For a discussion of algebraic analysis in the context of German mathematics 1780-1840 see [Jahnke, 1993].

[7] Koppelman [1971-1972] presents a survey of work on operational calculi at the end of the eighteenth and the first part of the nineteenth centuries. She is primarily concerned with examples where the symbol of operation was itself manipulated as an algebraic symbol (for example, in differential equations where expressions representing operations were built up from the D-symbol.) Her study documents the very considerable degree of interest during the period in the calculus of operations.

[8] An interesting example of use of the δ-symbol is provided by the book of Gelfand and Fomin [1963, 99]. The second variation of the functional $J[y]$ is denoted by $\delta^2 J[y]$ and equated to the entire second-order term in the expansion of ΔJ. Thus the second variation as given here is one half of the second variation as it was more traditionally defined.

This is not a particularly good definition. It would be like defining the second derivative of $f(x)$ as $\frac{1}{2}f''(x)$. Nevertheless, the convention expresses the implicit understanding of the authors. The δ symbol has a purely nominal role in denoting certain expressions and lacks any conceptual or operational significance.

[9] See for example [Bliss 1925, p. 162-163], Funk [1962, p. 139-142], [Pars 1962, p. 56-57], [Akhiezer 1962, p. 68-81], [Brechtken-Manderscheid 1991, p. 60-72].

[10] Bolza [1909, p. 60 n.1] credits Hesse but regards his paper as a "completion" of Jacobi's theory rather than an independent development. The turn-of-the-century author Pascal [1899, p. 6-7] writes in the historical introduction at the beginning of his textbook, concerning Hesse, "dem es in seiner Abhandlung gelang, fast eine ganze neue Theorie über eine besondere Darstellbarkeit von linearum Ausdrücken der successiven Derivativen einer und derselben Funktion aufstellen." He is however

here referring only to Hesse's method for transforming the general integrand in the second variation. Pascal's account of Jacobi's result is itself based on the older treatment in Jellett [1850; German edition 1860] and Moigno-Lindelöf [1861].

[11] Emphasis added.

Set and Measure as an Example of Complementarity

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I. History and hermeneutics

There are different ideas on how and why history of mathematics should be included into the teaching of mathematics. One is based on the idea of *hermeneutics* (cf. Jahnke, 1994). History of mathematics is essentially a hermeneutic effort: Theories and their creators are interpreted. Interpretation comprises a circular process of forming hypotheses and checking them against the material given, and interpreters should be aware of the hypothetical, even intuitive character of their interpretations. If history of mathematics is not to deteriorate into a dead dogma, a mere addition to the dogmas of mathematics (as it appears to a lot of students), it should be clear to teachers and their students that historical knowledge is not given from outside, but has to be constructed out of the available sources. Therefore, teachers introducing history should know something about historical sources and the basically hypothetical character of large parts of our historical knowledge.

Hermeneutics is not only the central component of the historical method, but could also gain a *pedagogical dimension*. Historical texts can be used in teaching periods when a new concept or technique has been introduced and the students are working to get acquainted with it. Understanding a historical text requires the *application* of the new concept in a problem context that is different from the usual exercises. In most cases, the ideas of the historical author are different from those of today, and this provokes a rethinking of one's own notions. In a natural way then, historical texts lead to *reflection*. For Galilei, velocity was an intensive magnitude and composed of infinitely small particles. This is opposed to a modern view that might consider instantaneous velocity as a local rate of change of the distance-time function. An interpretation of Galilei's views in the framework of the modern conception is not at all obvious. However, it may provoke interesting reflections about our ideas of what instantaneous velocity is. Guiding questions will be: Which ideas does the historical author have about the respective concept? Which applications is he