Preface to J. L. Lagrange, *Analytical Mechanics*. Lagrange's *Mécanique analytique, nouvelle édition* of 1811 translated into English and edited by Auguste Boissonnade and Victor N. Vagliente. Volume 191 of Boston Studies in the Philosophy of Science. Dordrecht and Boston, Mass.: Kluwer Academic Publishers; 1997. Preface is on pp. vii-x.

PREFACE

to the English translation of Lagrange's Mécanique Analytique

Lagrange's *Mécanique Analytique* appeared early in 1788 almost exactly one century after the publication of Newton's *Principia Mathematica*. It marked the culmination of a line of research devoted to recasting Newton's synthetic, geometric methods in the analytic style of the Leibnizian calculus. Its sources extended well beyond the physics of central forces set forth in the *Principia*. Continental authors such as Jakob Bernoulli, Daniel Bernoulli, Leonhard Euler, Alexis Clairaut and Jean d'Alembert had developed new concepts and methods to investigate problems in constrained interaction, fluid flow, elasticity, strength of materials and the operation of machines. The *Mécanique Analytique* was a remarkable work of compilation that became a fundamental reference for subsequent research in exact science.

During the eighteenth century there was a considerable emphasis on extending the domain of analysis and algorithmic calculation, on reducing the dependence of advanced mathematics on geometrical intuition and diagrammatic aids. The analytical style that characterizes the *Mécanique Analytique* was evident in Lagrange's original derivation in 1755 of the δ-algorithm in the calculus of variations. It was expressed in his consistent attempts during the 1770s to prove theorems of mathematics and mechanics that had previously been obtained synthetically. The scope and distinctiveness of his 1788 treatise are evident if one compares it with an earlier work of similar outlook, Euler's *Mechanica sive Motus Scientia Analytice Exposita* of 1736.[1] Euler was largely concerned with deriving the differential equations in polar coordinates for an isolated particle moving freely and in a resisting medium. Both the goal of his investigation and the methods employed were defined by the established programme of research in Continental analytical dynamics. The key to Lagrange's approach by contrast was contained in a new and rapidly developing branch of mathematics, the calculus of variations. In applying this subject to mechanics he developed during the period 1755–1780 the concept of a generalized coordinate, the use of single scalar variables (action, work function), and standard equational forms (Lagrangian equations) to describe the static equilibrium and dynamical motion of an arbitrary physical system. The fundamental axiom of his treatise, a generalization of the principle of virtual work, provided a unified point of view for investigating the many and diverse problems that had been considered by his predecessors.

In what was somewhat unusual for a scientific treatise, then or now, Lagrange preceded each part with an historical overview of the development of the subject. His study was motivated not simply by considerations of priority but also by a genuine interest in the genesis of scientific ideas. In a book on the calculus published several years later he commented on his interest in past mathematics.

He suggested that although discussions of forgotten methods may seem of little value, they allow one "to follow step by step the progress of analysis, and to see how simple and general methods are born from complicated and indirect procedures."[2]

Lagrange's central technical achievement in the *Mécanique Analytique* was to derive the invariant-form of the differential equations of motion

$$\frac{\partial T}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial V}{\partial q_i},$$

for a system with n degrees of freedom and generalized coordinates q_i $(i = 1, \ldots, n)$. The quantities T and V are scalar functions denoting what in later physics would be called the kinetic and potential energies of the system. The advantages of these equations are well known: their applicability to a wide range of physical systems; the freedom to choose whatever coordinates are suitable to describe the system; the elimination of forces of constraint; and their simplicity and elegance.

The flexibility to choose coordinates is illustrated in the simplest case by a calculation of the inertial reactions for a single mass m moving freely in the plane under the action of a force. It is convenient here to use polar quantities r and θ to analyze the motion. We have $x = r\cos\theta$ and $y = r\sin\theta$, where x and y are the Cartesian coordinates of m. The function T becomes

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2).$$

Hence

$$\frac{\partial T}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial T}{\partial \dot{r}}) = m(r\dot{\theta}^2 - \ddot{r}),$$

$$\frac{\partial T}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial T}{\partial \dot{\theta}}) = -\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}).$$

By equating these expressions to $\partial V/\partial r$ and $\partial V/\partial \theta$ we obtain the equations of motion in polar coordinates. If the force is central, $V = V(r)$, this procedure leads to the standard form

$$m(r\dot{\theta}^2 - \ddot{r}) = V'(r),$$

$$mr^2\dot{\theta} = \text{constant}.$$

Lagrange derived his general equations from a fundamental relation that originated with the principle of virtual work in statics. The latter was a well-established rule to describe the operation of such simple machines as the lever, the pulley and the inclined plane. The essential idea in dynamics — due to d'Alembert — was to suppose that the actual forces and the inertial reactions form a system in equilibrium or balance; the application of the static principle leads within a variational framework to the desired general axiom. Historian Norton Wise has called attention to the pervasiveness of the image of the balance in Enlightenment scientific thought.[3] Condillac's conception of algebraic analysis emphasized the balancing of terms on each side of an equation. The high-precision balance was a central laboratory instrument in the chemical revolution of Priestley, Black and Lavoisier. A great achievement of eighteenth-century astronomy, Lagrange and Laplace's theory of planetary perturbations, consisted in establishing the stability of the various three-body systems within the solar system. The *Mécanique Analytique* may be viewed as the product of a larger scientific mentality characterized by a neoclassical sense of order and, for all its intellectual vigour, a restricted consciousness of temporality.

A comparison of Lagrange's general equations with the various laws and special relations that had appeared in earlier treatises indicates the degree of formal sophistication mechanics had reached by the end of the century. The *Mécanique Analytique* contained as well many other significant innovations. Notable here were the use of multipliers in statics and dynamics to calculate the forces of constraint; the method of variation of arbitrary constants to analyze perturbations arising in celestial dynamics (added in the second edition of 1811); an analysis of the motion of a rigid body; detailed techniques to study the small vibrations of a connected system; and the Lagrangian description of the flow of fluids.

In addition to presenting powerful new methods of mechanical investigation Lagrange also provided a discussion of the different principles of the subject. The *Mécanique Analytique* would be a major source of inspiration for such nineteenth-century researchers as William Rowan Hamilton and Carl Gustav Jacobi.[4] The seminal character of Lagrange's theory is evident in the way in which they were able to use it to derive new ideas for organizing and extending the subject. Combining results from analytical dynamics, the calculus of variations and the study of ordinary and partial differential equations Hamilton and Jacobi constructed on Lagrange's variational framework a mathematical-physical theory of great depth and generality. Within the calculus of variations itself the Hamilton-Jacobi theory would become a source for Weierstrassian field theory at the end of the century; within physics it took on new importance with the advent of quantum mechanics in the 1920s.

Beyond its historical and scientific interest the *Mécanique Analytique* is a work of considerable significance in the philosophy of science. It embodies a type of empirical investigation which emphasizes the abstract power of mathematics to link and to coordinate observational variables. The concepts of an idealized constraint, a generalized coordinate and a scalar functional allow one to describe the system without detailed hypotheses concerning its internal physical structure and working.[5] In the third part of his *Treatise on Electricity and Magnetism* James Clerk Maxwell (1892) stressed this aspect of Lagrange's theory as he used it to create a "dynamical" theory of electromagnetism.[6] Beginning with Auguste Comte and continuing with such later figures as Ernst Mach and Pierre Duhem, Lagrange's analytical mechanics has attracted the attention of leading positivist philosophers of physics.[7] In 1883 Mach praised Lagrange for having brought the subject to its "highest degree of perfection" through his introduction of "very simple, highly symmetrical and perspicuous schema."[8]

Lagrange's book remains valuable today as an exposition of subjects of ongoing utility to engineering physics and applied mathematics. Its value to the historian of mechanics, its intrinsic interest to the practising scientist and its contribution to the philosophy of physics ensure its place as an enduring classic of exact science.

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[1] Euler's work was published as volumes 1 and 2 of series 2 of his *Opera Omnia* (Leipzig and Berlin: Teubner, 1912)

[2] These remarks appear in the section on calculus of variations in Lagrange's *Leçons du calcul des fonctions* (1806), p. 315 of volume 10 of his *Oeuvres* (1884).

[3] M. Norton Wise and Crosbie Smith, "Work and Waste: Political Economy and Natural Philosophy in Nineteenth Century Britain", *History of Science* 27 (1989), pp. 263-301. Wise contrasts earlier scientific thought with the emerging consciousness of temporality (change, evolution, dissipation) that took place in British natural philosophy in the 1840s.

[4] Hamilton, William Rowan, "On a general method employed in Dynamics, by which the study of the motions of all free Systems of attracting or repelling points is reduced to the search and differentiation of one central solution or characteristic function", *Philosophical Transactions of the Royal Society of London* 124, (1834) 247-308; and "Second essay on a general method in Dynamics", *Philosophical Transactions of the Royal Society of London* 125 (1835), 95-144. Carl Gustav Jacobi, "Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systemes gewohnlicher Differentialgleichungen", *Journal für die reine und angewandte Mathematik* 17 (1838), 97-162.

[5] See Mario Bunge, "Lagrangian Formulation and Mechanical Interpretation", *American Journal of Physics* 25 (1957), pp. 211-218.

[6] J. Clerk Maxwell, *A Treatise on Electricity and Magnetism*, 3rd edition (Oxford, 1892).

[7] Auguste Comte, *Cours de Philosophie Positive*, Volume 1 (1830). Ernst Mach, *The Science of Mechanics A Critical and Historical Account of its Development* (Open Court, 1960). The English translation appeared in 1893. The German first edition was published in 1883 as *Die Mechanik in Ihrer Entwicklung Historisch-Kritisch Dargestellt.* Pierre Duhem, *The Aim and Structure of Physical Theory* (New York: Athenum, 1974). Duhem's book appeared originally in French in 1906 as *La Théorie physique, son object et son structure.* The English translation is of the second 1914 edition.

[8] Ibid Mach, pp. 561-2.