

## Joseph Louis Lagrange's Algebraic Vision of the Calculus

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Prior to the development of real analysis in the 19th century, J. L. Lagrange had provided an algebraic basis for the calculus. The most detailed statement of this program is the second edition of his *Leçons sur le calcul des fonctions* (1806). The paper discusses Lagrange's conception of algebraic analysis and critically examines his demonstration of Taylor's theorem, the foundation of his algebraic program. Lagrange's striking algebraic style is further explored in two specific subjects of the *Leçons*: the theory of singular solutions to ordinary differential equations and the calculus of variations. A central theme of the paper concerns Lagrange's treatment of exceptional values in his demonstration of analytical theorems. The paper concludes that Lagrange's algebraic program was a natural one, but that the conception of a functional relation given by a single analytical expression was too restrictive to provide an adequate basis for analysis. © 1987 Academic Press, Inc.

Noch vor der Entwicklung der reellen Analysis im 19. Jahrhundert hatte J. L. Lagrange schon eine algebraische Grundlage für die Infinitesimalrechnung geliefert. Als ausführlichste Darstellung dieses Programms gilt die zweite Ausgabe seiner *Leçons sur le calcul des fonctions* (1806). Der Aufsatz behandelt Lagranges Auffassung von der algebraischen Analysis und prüft kritisch seinen Beweis für Taylors Satz, die Grundlage seines algebraischen Programms. Weiterhin wird Lagranges eindrucksvoller algebraischer Stil am Beispiel zweier bestimmter Themen seiner *Leçons* untersucht, der Theorie der singulären Lösungen für gewöhnliche Differentialgleichungen und der Variationsrechnung. Ein zentrales Thema des Aufsatzes betrifft Lagranges Behandlung der Ausnahmewerte in seinem Beweis für analytische Sätze. Schliesslich wird im Aufsatz darauf hingewiesen, dass Lagranges algebraisches Programm natürlich war, daß aber seine Auffassung von einer funktionalen Beziehung, die durch einen einzigen analytischen Ausdruck gegeben ist, zu begrenzt war, um eine angemessene Grundlage für die Analysis zu liefern. © 1987 Academic Press, Inc.

Avant le développement au XIXe siècle de l'analyse réelle, J.-L. Lagrange avait proposé de fonder le calcul différentiel sur une base algébrique. Ce programme fut exposé avec le plus de détails dans la deuxième édition de son *Leçons sur le calcul des fonctions* (1806). Notre étude porte sur la conception qu'avait Lagrange de l'analyse algébrique, et présente une analyse critique de sa démonstration du théorème de Taylor, fondement de son programme algébrique. Nous explorons par la suite le style algébrique de Lagrange par le biais de deux sujets spécifiques du *Leçons*: la théorie des solutions singulières des équations différentielles ordinaires et le calcul des variations. La façon dont Lagrange traite les valeurs exceptionnelles dans sa démonstration des théorèmes d'analyse constitue un thème important de notre étude. Nous concluons que le programme algébrique de Lagrange, tout en étant naturel, reposait sur une conception trop restrictive de la relation fonctionnelle, réduite à être représentée par une seule expression analytique, pour fournir une base adéquate de l'analyse. © 1987 Academic Press, Inc.

In the early 19th century Augustin-Louis Cauchy and Bernhard Bolzano published researches that formed the basis for the classical arithmetical foundation of

the calculus. Three decades earlier Joseph Louis Lagrange had proposed a quite different, algebraic, basis for the calculus. Among mathematicians of the late 18th century Lagrange was the most successful in developing a consistent algebraic style in analysis. The purpose of the present article is to clarify and to sharpen our understanding of Lagrange's theory by providing a sympathetic discussion of the basic elements of his program.

Lagrange's foundational investigations are presented in two works: *Théorie des fonctions analytiques* and *Leçons sur le calcul des fonctions* [1]. The *Théorie* was published in two editions in 1797 and 1813. In addition to pure analysis this treatise includes extensive applications to geometry and mechanics. The *Leçons* appeared in two editions, in 1801 and 1806, and contains a more detailed treatment of the analytical subjects of the *Théorie*. The second edition of the *Leçons* (1806) is the most advanced statement of Lagrange's program of algebraic analysis. It is therefore the primary text for the study which follows.

### 1. ALGEBRAIC ANALYSIS

Lagrange's plan is to make the calculus part of algebraic analysis by making it a theory of analytical functions. He hopes in this way to rid the calculus of intuitive geometric notions and to avoid procedures involving the logically questionable use of infinitesimal entities. His conception of algebra is more general than the view one occasionally finds in the 18th century of algebra as "universal arithmetic." As he explains in the introduction to a treatise on numerical equations:

The essential character [of algebra] consists in the fact that the results of its operations do not give the individual values of the quantities that are sought, as those of arithmetical operations or geometrical constructions, but represent only the operations, either arithmetic or geometric, that must be performed on the given first quantities in order to obtain the values sought; . . . .

[Algebra's] object is not to find particular values of the quantities that are sought, but the system of operations to be performed on the given quantities in order to derive from them the values of the quantities that are sought. The tableau of these operations represented by algebraic characters is what in algebra is called a *formula*; and when one quantity depends on other quantities, in such a way that it can be expressed by a formula which contains these quantities, we say then that it is a *function* of these same quantities. [Lagrange 1798, 14–15]

Algebra uses functions to investigate relations that arise in geometry and arithmetic. In its conventional use in the 18th century, the term algebra referred to the theory of polynomial equations, the determination of formulas that express the roots of such equations in terms of the undetermined coefficients appearing in the polynomials. The designation "algebraic analysis" was used by Lagrange to describe the more general branch of mathematics that results when a wider class of functions is permitted into algebra. An "analytical function," he writes at the beginning of the *Leçons sur le calcul des fonctions* [Lagrange 1806, 11], is any "expression de calcul" into which variables and constants enter. Although he does not explain this definition, his conception is fairly clearly indicated in the opening *Leçons*. Besides polynomials and rational functions, developed using the four elementary operations, analytical functions include the exponential func-

tion  $a^x$ , which satisfies the algebraic relation  $a^{x+y} = a^x a^y$ , and the trigonometric functions  $\cos x$  and  $\sin x$ , which satisfy the algebraic relations

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

From the functions  $x^n$ ,  $a^x$ ,  $\cos x$ , and  $\sin x$  we obtain the inverse functions  $\sqrt[n]{x}$ ,  $\log_a x$ ,  $\arccos x$ , and  $\arcsin x$ . Any expression formed from other functions using the operation of functional composition is itself a function. Lagrange employs the now familiar notation  $f(x)$ ,  $f(x, y)$ , etc., to denote functions of  $x$ , of  $x$  and  $y$ , and so on.

Analytical expressions may be generated from the elementary algebraic and transcendental functions using composition and inversion. There is, in addition, another fundamental way in which a function may enter algebraic analysis: as the primitive or antiderivative of a given function, and as the derived function or derivative of the same function. Whether a formula obtained in this way can be expressed independently using elementary operations and known functions is a question Lagrange never discusses. He certainly knew that the function obtained by applying inversion to the primitive of  $1/\sqrt{1-x^2}$  was the sine function. The possibility of inverting elliptic integrals, however, is never mentioned, and the construction of generalized transcendental functions is not pursued. (In an early memoir on elliptic integrals Lagrange [1766, 33] had noted that the investigation of the conditions of integrability of such expressions as  $1/\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4 + \dots}$  opens "a vast field to the researches of the analysts." Thus he clearly appreciated the possibilities of transcendental analysis, though he did not develop these possibilities in the *Leçons*.)

The operation of obtaining the derivative is the fundamental procedure of algebraic analysis. The operation, described in the next section, requires expanding the function  $f(x + i)$  in an infinite power series in  $i$ . The question arises as to the place of infinite series in Lagrange's analysis. Grabiner [1981, 53] has suggested that by "analytical function" Lagrange intended to include infinite series. Insofar as the *Leçons* is concerned, I do not think that this suggestion is correct. Lagrange certainly believed that every function  $f(x + i)$  is representable as a power series. Nevertheless, nowhere does he actually *define* a given function as some particular infinite expansion. Expansions are introduced either as a tool for obtaining the derivative, or as a way of representing a function that is already given. Thus in the third *Leçon* Lagrange [1806, 40–45] defines the sine and cosine functions using the angle-sum formulas. He then shows that these functions possess power series representations in  $x$ . In later lessons, where more complicated infinite processes are employed, the functions in question are always assumed to possess an existence that is independent of the infinite analysis.

It must also be emphasized that in the *Leçons* a function  $y = f(x)$  is given by a *single* analytical expression. In the 18th century such functions were called "continuous." At that time the calculus was concerned preeminently with "continuous" functions. In the 1750s and 1760s, during the celebrated debate over the vibrating string, Leonhard Euler had proposed a revolutionary generalization of

the function concept. (Details and references to this subject are given in [Truesdell 1960].) Euler introduced into mathematics "discontinuous" functions, functions given by more than one analytical expression. The algebraic form of a "discontinuous" function  $f(x)$  would vary, depending on the interval of real numbers to which  $x$  belongs. Euler wanted to allow such functions to be initial solutions to the wave equation.

At an early stage in his career Lagrange attempted in some well-known researches to provide mathematical justification for Euler's position in the vibrating string controversy [Truesdell 1960, 263–273]. Lagrange, however, never pursued these researches beyond the 1760s. Furthermore, Euler's introduction of "discontinuous" functions was itself quite anomalous in the context of his wider contributions to analysis. (This point is documented by Lützen [1983].) It cannot be emphasized too strongly that in the 18th century the calculus was a calculus of functions given by single analytical expressions. It is this conception that is at the base of the *Leçons*.

In Lagrange's world of algebraic analysis a function  $y = f(x)$  is given by a single analytical expression. This expression is constructed from variables and constants using the operations of analysis. The relation between  $y$  and  $x$  is indicated by the series of operations schematized in  $f(x)$ . Each function  $f(x)$  possesses a well-defined, unchanging algebraic form. It is the algebraic form of  $f(x)$  that distinguishes it from other functions and determines its properties. The emphasis on relations embodied in the concept of functional algebraic form is perhaps what most significantly distinguishes Lagrange's approach from modern calculus.

## 2. THE TAYLOR SERIES PROCESS

The idea behind Lagrange's theory of derived functions is to take any function  $f(x)$  and expand it in a power series

$$f(x + i) = f(x) + pi + qi^2 + ri^3 + si^4 + \dots \quad (*)$$

The "derived function" or derivative  $f'(x)$  of  $f(x)$  is defined to be the coefficient  $p(x)$  of the linear term  $i$  in this expansion.  $f'(x)$  is a new function of  $x$  with a well-defined algebraic form, different from but related to the form of the original function  $f(x)$ . The derived function of  $f'(x)$  is in turn denoted  $f''(x)$  and referred to as the second derived function of  $f(x)$ . This procedure may be repeated to obtain the higher-order derived functions  $f'''(x)$ ,  $f^{(iv)}(x)$ , and so on.

Using the principles of algebraic analysis, Lagrange attempts in the second *Leçon* to show that every function  $f(x)$  has an expansion of the form (\*). He hopes in this way to establish the existence of the derived function, without any appeal to infinitesimals or to such geometric notions as slope or tangent. He recognizes that for a given function  $f(x)$  the expansion (\*) may fail at isolated values of the argument  $x$ . His goal is to show that the representation (\*) is nevertheless valid "in general," i.e., algebraically, and he wishes to construct the demonstration so as to provide a plausible account within his algebraic framework for the possibility of exceptional values. In considering his derivation one should remember that he

is dealing with a very concrete notion of function, a single algebraic object built up from variables and constants using analytical operations. The following expansions, worked in detail at the completion of his demonstration of Taylor’s theorem in the *Théorie des fonctions analytiques* [Lagrange 1797, 25–26], are typical of the examples that motivate his theory:

$$\sqrt{x + i} = \sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{i^2}{8x\sqrt{x}} + \frac{i^3}{16x^2\sqrt{x}} - \dots, \quad x \neq 0$$

$$\sqrt{x + i} = i^{1/2}, \quad x = 0;$$

$$\frac{1}{x + i} = \frac{1}{x} - \frac{i}{x^2} + \frac{i^2}{x^3} - \frac{i^3}{x^4} + \frac{i^4}{x^5} - \dots, \quad x \neq 0$$

$$\frac{1}{x + i} = i^{-1}, \quad x = 0.$$

Lagrange first argues that any expansion of  $f(x + i)$  in powers of  $i$  can contain no fractional or negative powers of  $i$ . His argument is inspired by examples like  $f(x) = \sqrt{x}$  and  $f(x) = 1/x$ , and I shall use these examples to illustrate his reasoning. The series expansion of  $\sqrt{x + i}$  could not be of the form

$$\sqrt{x + i} = \sqrt{x} + pi + qi^2 + \dots + i^{m/n},$$

for, if it were, it would establish a relation of dependency between the two-valued quantity on the left side and the  $2n$ -valued quantity on the right side, a consequence that is clearly absurd. (The square root on the left side has two values; the square root and the  $n$ th root on the right side have two and  $n$  values for a total of  $2n$  values.) The series expansion of  $1/(x + i)$  could not contain negative powers of  $i$ ; by setting  $i = 0$  the presence of such powers would imply that  $1/x$  is infinite everywhere, an evident contradiction.

Lagrange [1806, 14] notes that the “generality” and “rigor” of these arguments require that  $x$  and  $i$  be indeterminate. In particular cases we will have fractional or negative powers of  $i$  ( $\sqrt{x + i} = i^{1/2}$  and  $1/(x + i) = i^{-1}$  at  $x = 0$ ). Such exceptional instances arise because of the disappearance at isolated values of  $x$  of certain formal features of the functions in question. (Thus the radical  $\sqrt{x}$  disappears at  $x = 0$ , and  $1/x$  assumes the indeterminate form  $1/0$  at  $x = 0$ .) In these exceptional cases the reasoning that led to the original conclusion concerning integral exponents no longer applies.

The remainder of Lagrange’s demonstration rests on a result that I shall refer to as the “factor lemma”:

*If  $g(x, i)$  is a function of  $x$  and  $i$  and  $g(x, i) = 0$  when  $i = 0$  then  $g(x, i) = i^\alpha h(x, i)$ , where  $\alpha$  is a positive number and  $h$  is a function of  $x$  and  $i$  that assumes a finite nonzero value at  $i = 0$ .*

This result is nowhere explicitly formulated by Lagrange; it appears in his analysis as a self-evident truth about analytical functions. He uses it as follows [2]. Since  $f(x + i) - f(x) = 0$  when  $i = 0$  we must have

$$f(x + i) = f(x) + i^\alpha P(x, i),$$

where  $\alpha$  is positive and  $P$  is a function of  $x$  and  $i$  that assumes a finite nonzero value at  $i = 0$ . By the argument described above we may suppose  $\alpha$  is an integer; in the most general case we may suppose  $\alpha = 1$ .  $P(x, i)$  at  $i = 0$  will itself be a function  $p(x)$  of  $x$ . We may therefore repeat the process and obtain "by a reasoning similar to the preceding"

$$P(x, i) = p(x) + iQ(x, i),$$

where  $Q(x, i)$  is a function of  $x$  and  $i$  that assumes a finite value at  $i = 0$ . The continuation of this process leads to the desired expansion (\*).

Having established the existence of (\*) Lagrange [1806, 14–15] relates the coefficients  $q, r, s, \dots$  to the higher-order derived functions  $f''(x), f'''(x), f^{(iv)}(x), \dots$ . He does so by means of a formal argument. If we replace  $i$  by  $i + o$  in (\*) we obtain

$$\begin{aligned} f(x + i + o) &= f(x) + (i + o)p + (i + o)^2q + (i + o)^3r + (i + o)^4s + \dots \\ &= f(x) + ip + i^2q + i^3r + i^4s + \dots + op + 2ioq + 3i^2or \\ &\qquad\qquad\qquad + 4i^3os + \dots, \end{aligned}$$

where we have displayed only the first two terms of each power of  $(i + o)$ . Suppose now that we replace  $x$  by  $x + o$ .  $f(x), p, q, r, \dots$  then become

$$f(x) + op + \dots, p + op' + \dots, q + oq' + \dots, r + or' + \dots, \dots$$

If we next increase  $x + o$  by  $i$  we obtain (using  $x + i + o = (x + o) + i$ )

$$\begin{aligned} f(x + i + o) &= f(x) + op + \dots + i(p + op' + \dots) + i^2(q + oq' + \dots) \\ &\qquad\qquad\qquad + i^3(r + or' + \dots) + \dots \\ &= f(x) + ip + i^2q + \dots + op + iop' + i^2oq' + i^3or' + \dots \end{aligned}$$

Equating these two expressions for  $f(x + i + o)$  we see that

$$q = \frac{1}{2}p', r = \frac{1}{3}q', s = \frac{1}{4}r', \dots$$

The derived functions  $f'(x), f''(x), f'''(x), \dots$  are the coefficients of  $i$  in the expansions of  $f(x + i), f'(x + i), f''(x + i), \dots$ . Hence

$$q = \frac{1}{2}f''(x), r = \frac{1}{2 \cdot 3}f'''(x), s = \frac{1}{2 \cdot 3 \cdot 4}f^{(iv)}(x), \dots$$

Thus the expansion (\*) becomes

$$f(x + i) = f(x) + if'(x) + \frac{i^2}{2}f''(x) + \frac{i^3}{2 \cdot 3}f'''(x) + \frac{i^4}{2 \cdot 3 \cdot 4}f^{(iv)} + \dots,$$

which is the Taylor series for  $f(x + i)$ .

The Taylor series process is the foundational basis for Lagrange's theory of derived functions. Once introduced, derived functions may be viewed indepen-

dently of this process as objects whose formation follows certain general rules. The key point in understanding the passage from a function  $f(x)$  to its derived function  $f'(x)$  is that the relation in question is one of *algebraic form*. Lagrange's theory should be contrasted here with the modern calculus, where the derivative of  $f(x)$  is defined at each numerical value of  $x$  by a limit process. In the modern calculus the relationship of the derivative to its parent function is essentially *arithmetic*.

Lagrange's understanding of derived functions is illuminated by his discussion in the eighteenth *Leçon* of the method of finite increments. This method was of historical interest in the background to his program. Brook Taylor's original derivation in 1715 of Taylor's theorem, described by Feigenbaum [1985], was based on a passage to the limit of an interpolation formula involving finite increments. Lagrange wishes in the eighteenth *Leçon* to distinguish clearly between a foundation of the calculus that uses finite increments and his own quite different and quite superior algebraic theory of derived functions. In taking finite increments, he notes, one considers the difference  $f(x_{n+1}) - f(x_n)$  of the same function  $f(x)$  at two successive values of the independent argument. In the differential calculus the object Lagrange refers to as the derived function was traditionally obtained by letting  $dx = x_{n+1} - x_n$  be infinitesimal, setting  $dy = f(x + dx) - f(x)$ , dividing  $dy$  by  $dx$ , and neglecting infinitesimal quantities in the resulting reduced expression for  $dy/dx$ . Although this process leads to the same result as Lagrange's theory, the connection it presumes between the method of increments and the calculus obscures a more fundamental difference between these subjects: in taking  $\Delta y = f(x_{n+1}) - f(x_n)$  we are dealing with one and the same function  $f(x)$ ; in taking the derived function we are passing to a new function  $f'(x)$  with a new algebraic form. Lagrange explains this point thus:

. . . the passage from the finite to the infinite requires always a sort of leap, more or less forced, which breaks the law of continuity and changes the form of functions. [Lagrange 1806, 270]

. . . in the supposed passage from the finite to the infinitely small, functions actually change in nature, and . . .  $dy/dx$ , which is used in the differential Calculus, is essentially a different function from the function  $y$ , whereas as long as the difference  $dx$  has any given value, as small as we may wish, this quantity is only the difference of two functions of the same form; from this we see that, if the passage from the finite to the infinitely small may be admitted as a mechanical means of calculation, it is unable to make known the nature of differential equations, which consists in the relations they furnish between primitive functions and their derivatives. [Lagrange 1806, 279]

The centrality of the notion of algebraic form in Lagrange's analysis is further reflected in the methods of justification he employs in the *Leçons*. He prefers demonstrations that make no assumptions concerning the individual values of the variables in question. An analytical relation may fail at isolated values; a "rigorous" demonstration is one that establishes its general, algebraic correctness. The study of exceptional values, Lagrange [1806, 84] observes in one place, "has no influence on the theory of functions, in so far as one considers there only

the form and the derivation of functions." Lagrange's formal algebraic conception of analytical justification lends a distinctive style to the inferential patterns of the *Leçons*. Some of the issues associated with this aspect of his program are discussed again in the next section.

### 3. SELECTED TOPICS

Two subjects investigated by Lagrange in the *Leçons* are of particular interest as illustration of his algebraic program: the theory of singular solutions to ordinary differential equations and the calculus of variations. Issues central to the foundation of the calculus—the nature of mathematical existence, the use of formal analogy, the relationship between local and global analysis—are illuminated by a study of his presentation of these topics.

#### (i) *Singular Solutions to Differential Equations*

Much of the *Leçons* is devoted to the study of ordinary and partial differential equations, a prominent topic in late-18th-century analysis. The theory as it was developed during this period was strongly algebraic, with no concern for the considerations of existence that arise in the modern subject. When the formalism did receive an interpretation it was not in real analysis but in differential geometry: the investigation of curves using their description by means of differential equations.

Basic to Lagrange's theory is the notion of a primitive [Lagrange 1806, 166–167]. Consider the first-order differential equation

$$f(x, y, y') = 0. \quad (1)$$

A solution to (1) will consist of an analytical relation between  $x$ ,  $y$ , and an arbitrary constant  $a$ ,

$$F(x, y, a) = 0. \quad (2)$$

Equation (2) is known as the primitive equation for (1). By differentiating (2) with respect to  $x$  we obtain

$$\partial F/\partial x + (\partial F/\partial y)y' = 0,$$

which Lagrange writes

$$F'(x) + F'(y)y' = 0. \quad (3)$$

We can derive (1) from (2) and (3) by eliminating the constant  $a$  between these equations. Thus the differential equation  $y - xy' + 1 = 0$  has the primitive  $y + ax + 1 = 0$ . Differentiation of the primitive gives rise to the relation  $y' + a = 0$ , which combined with the primitive itself yields the original equation  $y - xy' + 1 = 0$ . A more complicated example, one Lagrange returns to repeatedly to illustrate his analysis, is the differential equation

$$y'\sqrt{x^2 + y^2 - b} - yy' - x = 0, \quad (4)$$



where  $b$  is a parameter. The primitive equation for (4) is

$$x^2 - 2ay - a^2 - b = 0, \quad (5)$$

with arbitrary constant  $a$ . If we differentiate (5) with respect to  $x$  we obtain

$$x - ay' = 0. \quad (6)$$

Substitution of  $a = x/y'$  into (5) yields

$$x^2 - \frac{2xy}{y'} - \frac{x^2}{y'^2} - b = 0,$$

which reduces to the original equation (4).

Lagrange devotes the fourteenth to eighteenth *Leçons* to a detailed study of singular solutions to ordinary differential equations. A singular solution is one not obtainable from the primitive by specification of the arbitrary constant. Thus  $x^2 + y^2 - b = 0$  is a singular solution to the differential equation  $y'\sqrt{x^2 + y^2 - b} - yy' - x = 0$  because it is a solution that is not furnished by the primitive  $x^2 - 2ay - a^2 - b = 0$  for any value of  $a$ . Singular solutions were recognized early in the history of the calculus and were regarded as "paradoxical" because they showed that the apparent generality of the primitive, indicated by the presence of the arbitrary constant, was incomplete. Lagrange intends in the *Leçons* to dispel any mystery surrounding such integrals by presenting them as a natural consequence of the analytical theory.

To obtain singular solutions Lagrange [1806, 167–169] employs a "variation of arbitrary constants" procedure. The differential equation  $f(x, y, y') = 0$  (1) was obtained from the primitive  $F(x, y, a) = 0$  (2) by eliminating the arbitrary constant between (2) and the equation  $F'(x) + F'(y)y' = 0$  (3). Let us suppose now that  $a$  is "variable," i.e., that it is some as yet unspecified function of  $x$  and  $y$ . Differentiation of (2) now yields

$$F'(x) + F'(y)y' + F'(a)a' = 0, \quad (7)$$

where  $F'(a) = \partial F/\partial a$  and  $a' = da/dx$ . If we suppose that the equation  $F'(x) + F'(y)y' = 0$  (3) remains valid then the elimination procedure will proceed as before, and  $F(x, y, a) = 0$  will be a solution to (1). Given (7), this condition will be satisfied if

$$F'(a)a' = 0.$$

If  $a' = 0$  then  $a$  is a constant and we obtain the original primitive equation  $F(x, y, a) = 0$ . If

$$F'(a) = 0, \quad (8)$$

then we use (8) to solve  $a$  as a function of  $x$  and  $y$ . Substitution of this function into  $F(x, y, a) = 0$  yields the desired singular solution. In the example  $y'\sqrt{x^2 + y^2 - b} - yy' - x = 0$ ,  $F(x, y, a) = x^2 - 2ay - a^2 - b$ . The condition  $F'(a) = 0$  becomes

$a = -y$ . Substitution of this value of  $a$  into the primitive  $F(x, y, a) = 0$  yields the singular solution  $x^2 + y^2 - b = 0$  to the differential equation.

Lagrange's theory of singular solutions had originated in work first published in the 1770s. In these researches he had shown that the theory receives a natural geometrical interpretation as a description of envelopes of families of curves in the plane. (An account of Lagrange's early work is presented by Engelsman [1980].) In the *Leçons* he summarizes the geometrical applications and extends and systematizes the analysis itself. He devotes considerable attention to the investigation of conditions that would enable one to determine by inspection of a differential equation whether a given solution is a possible singular solution. One of these conditions is of interest from a foundational viewpoint because of the connection it establishes between exceptional values of analytical functions and singular solutions to differential equations. Assume we express the differential equation (1) in the form

$$y' + f(x, y) = 0. \tag{9}$$

Lagrange [1806, 212] shows that a necessary condition for  $y = X(x)$  to be a singular solution is that  $f'(x)$  ( $=\partial f/\partial x$ ) and  $f'(y)$  ( $=\partial f/\partial y$ ) be infinite. Thus in the example  $y'\sqrt{x^2 + y^2} - b - yy' - x = 0$  we have

$$y' - \frac{x}{\sqrt{x^2 + y^2} - b - y} = 0,$$

and it is easily verifiable in the case of the singular solution  $x^2 + y^2 - b = 0$  that  $f'(x) = \infty$ ,  $f'(y) = \infty$ . The proposition may be established generally by taking a primitive  $F(x, y, a) = 0$  of the given differential equation. We solve the equation  $F'(x) + F'(y)y' = 0$  for  $a$  as a function of  $x, y$  and  $y'$ :  $a = \phi(x, y, y')$ . We substitute  $a = \phi(x, y, y')$  back into the primitive, set  $y' = -f(x, y)$ , and take partial differentials with respect to  $x$  and  $y$ :

$$\begin{aligned} F'(x) + F'(a)(a'(x) - a'(y')f'(x)) &= 0 \\ F'(y) + F'(a)(a'(y) - a'(y')f'(y)) &= 0 \end{aligned} \tag{10}$$

In the case of the singular solution  $y = X$  we have  $F'(a) = 0$ . Hence it is clear in this case that  $f'(x) = \infty$ ,  $f'(y) = \infty$ .

Lagrange [1806, 215–218] proceeds to connect the condition  $f'(x) = \infty$ ,  $f'(y) = \infty$  to the behavior of the series expansion of  $f(x, y + z)$  ( $z$  a function of  $x$ ) in powers of  $z$ . He shows that in the case of the singular solution  $y = X$  the usual Taylor series fails; the development of  $f(x, X + z)$  in powers of  $z$  here leads to fractional or negative exponents. The situation is analogous to the one involving analytical functions and their derivatives. The Taylor series of  $f(x + i)$  in powers of  $i$  may fail at isolated values of  $x$ . In such exceptional instances the derived function  $f'(x)$  becomes infinite. Thus if  $f(x) = \sqrt{x}$  then  $f(x + i) = i^{1/2}$  at  $x = 0$  and  $f'(x) = 1/2\sqrt{x} = \infty$  at  $x = 0$ . If  $f(x) = 1/x$  then  $f(x + i) = i^{-1}$  at  $x = 0$  and  $f'(x) = -1/x^2 = -\infty$  at  $x = 0$ . A connection exists, at least at the level of formal analogy, between

exceptional values of analytical functions and singular solutions to differential equations. Both are anomalies occasioned by the breakdown of the Taylor series process, the foundational basis of Lagrange's algebraic system of analysis.

To establish this result Lagrange uses the fact that a singular solution  $y = X(x)$  is not furnished by the primitive for any value of the arbitrary constant [3]. Thus if we try solutions to the differential equation (9) of the form  $X + z$  we should not obtain expressions for  $z$  that yield  $z = 0$  as a special case. Suppose now that  $X + z$  is a solution to (9) and that  $f(x, X + z)$  possesses a Taylor expansion,

$$f(x, X + z) = f(x, X) + f'(X)z + \frac{1}{2}f''(X)z^2 + \dots, \tag{11}$$

where  $f'(X), f''(X), \dots$  denote  $\partial f/\partial y(x, X), \partial^2 f/\partial y^2(x, X), \dots$ . Since  $X$  is a solution to (9) we have  $X' + f(x, X) = 0$ . Thus if we substitute the series (11) into (9) we obtain

$$z' + z f'(X) + \frac{z^2}{2} f''(X) + \dots = 0. \tag{12}$$

Lagrange [1806, 216] solves (12) by successive approximation. If we assume  $z$  is small then a good first approximation will be given by the equation

$$z' + z f'(X) = 0. \tag{13}$$

The general solution to (13) is  $z = a e^{-\bar{X}}$ , where  $a$  is an arbitrary constant and  $\bar{X}$  is a primitive function or integral of  $f'(X)$  regarded as a function of  $x$ . We next try a second, third, etc., approximation to  $z$ , obtaining in each case a function involving an arbitrary constant and equaling zero when  $a = 0$ . Lagrange concludes from this process that (12) itself possesses a primitive containing a constant  $a$  and equaling zero when  $a = 0$ . The existence of such a primitive contradicts our hypothesis concerning  $X$ . Hence our assumption that  $f(x, X + z)$  has the Taylor series (11) is false.

Lagrange [1806, 217] shows that there are further restrictions on the exponents of  $z$  in any series expansion of  $f(x, X + z)$ . Let us write Eq. (12) in the form

$$z' + Pz^m + Qz^n + \dots = 0, \tag{14}$$

where the exponents  $m, n, \dots$  are ordered by increasing magnitude, and the coefficients  $P, Q, \dots$  are functions of  $x$ . Lagrange supposes that the equation

$$z' + Pz^m = 0 \tag{15}$$

will yield a good approximation to  $z$ ; any conclusion based on this approximation will also hold for the general primitive to (14). Since we have already excluded the case  $m = 1$ , the solution to (15) is

$$\frac{z^{1-m}}{1-m} + V = k, \tag{16}$$

where  $V$  is a function of  $x$  and  $k$  is an arbitrary constant. If  $1 - m < 0$  then the solution  $z = 0$  will be given by the primitive (16) in the limiting case  $k = \infty$ . The

case  $1 - m < 0$  is therefore excluded by our hypothesis concerning  $X$ . Suppose now that  $1 - m > 0$ . The solution  $z = 0$  would in this case imply that  $V = k$ , an impossibility because  $V$  is a function of  $x$  and  $k$  is a constant. Hence  $1 - m > 0$  and the exponent  $m$  is either negative or a fraction less than one.

Lagrange [1806, 218] illustrates this conclusion with the paradigm example  $y'\sqrt{x^2 + y^2 - b} - yy' - x = 0$ . We have

$$y' - \frac{x}{\sqrt{x^2 + y^2 - b} - y} = 0$$

and  $f(x, y) = -x/(\sqrt{x^2 + y^2 - b} - y)$ . In the case of the singular solution  $y = \sqrt{b - x^2}$ ,  $f(x, y + z)$  equals

$$\frac{-x}{\sqrt{2z}\sqrt{b - x^2 + z^2} - \sqrt{b - x^2} - z}$$

Expanded in powers of  $z$  this expression becomes

$$\frac{x}{\sqrt{b - x^2}} + \frac{\sqrt{2z}}{(b - x^2)^{3/4}} - \dots$$

and the exponent of  $z$  in the second term is  $\frac{1}{2}$ .

(ii) *Calculus of Variations*

In classical analysis functions are mappings “defined” on domains of real numbers. Under appropriately specified conditions a theorem is valid over a given domain. Specification of the domain is basic to the way one proceeds.

In the late 18th century considerations of domain arose in applications rather than in fundamental theory. Mathematicians were interested in the behavior of functions at particular numerical values. Considerations of domain also arose in the investigation of solutions to partial differential equations. Nevertheless, when it concerned the establishment of some general result or theorem of analysis, questions of domain were hardly considered at all [4]. A theorem of analysis was regarded as true because of the formal correctness of the underlying algebra. The fact that the theorem might fail at isolated values was not considered a serious drawback; indeed, these isolated failures were often viewed as the exceptions that, so to speak, proved the rule.

The difference between 18th-century and modern analysis is strikingly illustrated in the classical result [Courant & Hilbert 1953, 185] known as the fundamental lemma of the calculus of variations. Let  $\Phi$  be a continuous function defined on the interval  $[a, b]$ . Consider the class of functions  $\zeta(x)$  defined on  $[a, b]$  with continuous second derivatives and  $\zeta(a) = \zeta(b) = 0$ . Suppose the definite integral of the product  $\Phi(x)\zeta(x)$  evaluated between  $a$  and  $b$  is zero for all functions  $\zeta(x)$  in this class. The fundamental lemma asserts in this situation that  $\Phi$  is identically equal to zero on  $[a, b]$ . The proof begins by supposing that there exists a point at which the function  $\Phi$  is nonzero, say positive. The continuity of  $\Phi$  is then used to

expand this point to a subinterval  $[a_1, b_1]$  of  $[a, b]$ . By choosing a function  $\zeta(x)$  that is positive on this subinterval, zero elsewhere, and satisfies the conditions of differentiability of the lemma, we obtain a contradiction. (One example of such a function is given by the function that is equal to  $(x - a_1)^4(x - b_1)^4$  on  $[a_1, b_1]$  and zero elsewhere.)

The central ideas of classical real analysis are nicely illustrated in the demonstration of the fundamental lemma: the continuity of  $\Phi$  is used to expand the point into a continuum of values  $[a_1, b_1]$ ; the function  $\zeta(x)$  is given by more than one analytical expression. The basic idea behind the proof, the assumption of the existence of a point at which the lemma fails, is counter to the entire late-18th-century understanding of analytical theorems. A clear indication of this understanding is provided by Lagrange's demonstration of Taylor's theorem [5]. Lagrange's goal was to establish for any analytical function the general validity of the Taylor series expansion. The failure of the expansion at isolated values was attributed to the collapse at these values of the algebraic form of the analytical relations in question. The "rigor" of the demonstration resided in the requirement of an essentially algebraic formulation. In the conception of analytical justification presupposed here the existence of an exception is scarcely a meaningful possibility.

The fundamental lemma is used today in the calculus of variations to obtain the Euler–Lagrange equations as a necessary condition for the vanishing of the first variation of the extremalizing integral. As the name suggests, these equations were first derived by Euler and Lagrange in the 18th century. Lagrange's early theory, which became standard, was based on the introduction into the calculus of the variational characteristic  $\delta$ . The use of the  $\delta$ -symbol in conjunction with the method of integration by parts enabled him to devise a general algorithm for deriving the equations of the calculus of variations. In the twenty-first and twenty-second *Leçons* Lagrange [1806, 364–451] reformulates this algorithm in terms of his theory of derived functions. To obtain the variational equations he uses algebraic reasonings to justify the inference sanctioned today by the fundamental lemma. The basis of his argument consists of a formal analogy between the variational equations and equations that express necessary conditions in the theory of integrability. This analogy emerges when both subjects are developed using Taylor series.

An account of Lagrange's derivation of the variational equations in the *Leçons* is contained in [Fraser 1985]. This derivation is a clear example of Lagrange's use of formal analogy in analysis. Analogies appear in his mathematics both as a method of discovery and as a means of justification. They assume a special significance in his work. In the absence of an interpreted base for the calculus in real analysis, analogies suggest the presence of unity and cohesion, of underlying connections between different branches of the subject. They support his conviction that in his formal algebraic presentation of analysis he has discovered the "true metaphysics" of the calculus.

## 4. AN ASSESSMENT: THE FATE OF LAGRANGE'S VISION

An assessment of Lagrange's program must begin with the recognition that he was dealing with a collection of analytical functions given by mathematical experience and commonly familiar to any practitioner of the calculus on the Continent in the late 18th century. He was attempting to show that the calculus could be developed in a way that was both logically sound and consistent with the algebraic spirit of the period. We should remember that although the algebraic aspect of the calculus receives little attention in the modern foundation it is no less a prominent feature of the subject. Lagrange's program possesses a certain reasonableness even today.

Lagrange's analytical approach was also plausible given the usual application of the calculus to the geometry of curves. In studying the curve the calculus is concerned with the connection between local behavior (slope) and global behavior (area, pathlength). If the curve is representable by a function  $y = f(x)$  given by a single analytical expression then the relation between  $y$  and  $x$  is permanently established in the form of  $f$ . Local and global behavior become identified in this relation. It then seems reasonable, from a foundational perspective, to make the analytical relation itself the primary object of study. This is what Lagrange did.

One can level specific criticism at Lagrange's demonstration of Taylor's theorem. The expansion

$$\frac{1}{x+i} = \frac{1}{i} - \frac{x}{i^2} + \frac{x^2}{i^3} - \frac{x^3}{i^4} + \dots$$

contains negative powers of  $i$ , but it is not true that  $1/(x+i)$  at  $i=0$  is everywhere infinite. The function  $f(x) = e^{-1/x^2}$  is zero at  $x=0$ , but it is not true that we can factor out an  $x^\alpha$  ( $\alpha > 0$ ) from  $e^{-1/x^2}$ . The function  $P(x, i) = \sqrt{x} + \sqrt{i}$  gives rise to the relation  $P(x, i) - P(x, 0) = i^{1/2}$  and the exponent of  $i$  is nonintegral. To be sure, Lagrange's proof was inspired by a study of particular examples and he may have considered these criticisms artificial, if they had occurred to him at all. The fact remains that as a general argument applicable to all analytical functions his demonstration requires steps of questionable mathematical validity.

Apart from specific technical weaknesses there were broader problems in Lagrange's algebraic program. From a foundational viewpoint the cogency and naturalness of this program derived from its conception of a function as something given by a *single* analytical expression. The theory of partial differential equations had always posed troublesome questions for such a conception. In his foundational writings Lagrange responded to these questions by ignoring them; there was little else he could do. By the early 19th century it had become clear in the work of such mathematicians as Joseph Fourier that the foundation envisaged by Lagrange was poorly suited to deal with advanced research in analysis. The subsequent radical restructuring of the basis of analysis by Cauchy seems in retrospect to have been almost inevitable. (Fourier's research and its relation to Cauchy are

discussed by Grattan-Guinness [1970]. This book also contains an extensive bibliography.)

In this paper I have emphasized the contrast between Lagrange's program and the modern foundation of the calculus. There were nonetheless significant elements in his mathematics that were used by later researchers. By focusing attention on the coefficients in the Taylor series of a function he showed that the derivative, regarded itself as a function, was the important object to study. From the expansion (\*),  $f(x + i) = f(x) + if'(x) + i^2(f''(x)/2 + \dots)$ , it was clear that the expression  $(f(x + i) - f(x))/i$  is an approximation to the derived function  $f'(x)$  that can be made arbitrarily close to  $f'(x)$  by making  $i$  sufficiently small. Lagrange correctly viewed this fact as the "fundamental principle" that forms the basis for the application of the calculus to geometry and mechanics. In the ninth *Leçon* he uses inequalities derived from (\*) to deduce results about a function from the behavior of its derivative. The resulting theory provided a fertile source of techniques for Cauchy's later synthesis of the calculus in real analysis.

An account of Lagrange's ninth *Leçon*, as well as related sections of his *Théorie des fonctions analytiques* [1797], is provided by Ovaert [1976] and Grabiner [1981]. Grabiner uses the term "algebra of inequalities" to refer to this part of Lagrange's theory. She sees in Cauchy's later employment of inequalities evidence of continuity in the history of mathematics: Cauchy completed Lagrange's algebraic program, she suggests, by reducing the calculus to the "algebra of inequalities" [1981, 54]. The appeal of Grabiner's thesis clearly depends on the extent to which one regards Lagrange's ninth *Leçon* as representative of his algebraic style. To me the theory developed there seems an *application* of his algebraic program; it is not itself "algebraic" in the sense in which that word is normally used. It also seems unorthodox to assert that the classical arithmetical foundation is "algebraic" because it employs inequalities. For a fuller discussion of Grabiner's thesis the reader is referred to her book [6].

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#### NOTES

1. An account of the circumstances that led Lagrange to write these treatises and a discussion of their relation to his earlier work is provided by Ovaert [1976] and Grabiner [1981].

2. This step in Lagrange's demonstration is laid out in the *Théorie des fonctions analytiques* [1797, Chapt. 1, Sect. 3]. In the *Leçons* he omits this step and infers directly the existence of (\*).

3. Lagrange assumes that a singular solution derived from a given primitive cannot be a particular solution of another primitive of the same differential equation. This assumption requires a general result, obtained from something like the theorem on functional dependence, concerning the dependence of primitives to differential equations. Authors in the 18th century seem not to have concerned themselves with this aspect of the theory; such results as were needed were implicitly assumed to be valid.

4. In the ninth *Leçon* Lagrange investigates the behavior of a function and its derivative on a

specified interval of real numbers. The resulting theory, however, is not itself "analytical" in the sense of Lagrange's algebraic program; it is rather an *application* of this program.

5. For another example see Engelsman's [1984, 9–13] discussion of the theorem on the equality of mixed partial differentials.

6. Grabiner [1981, Chap. 3] also documents the importance of Euler's and Lagrange's work in numerical approximation for later foundational research in analysis. She perceptively notes that 18th-century techniques of approximation became theorems of existence in 19th-century rigorous analysis [1981, 69]. This conceptual transposition indicates once again how radically Cauchy and Bolzano recast the older analysis.

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