

# Isoperimetric Problems in the Variational Calculus of Euler and Lagrange

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Historians have documented the main development of the calculus of variations in the 18th century. Although we have a clear overall picture of this subject there is in the literature no connected historical account of the more specialized research carried out during the period on problems of extremization under constraint. Concentrating on the work of Leonhard Euler and Joseph Louis Lagrange between 1738 and 1806, the present study attempts to identify and draw together the different threads that make up this story. © 1992 Academic Press, Inc.

Les historiens ont déjà décrit les développements centraux du calcul des variations au dix-huitième siècle. Quoique nous ayons une vue d'ensemble, il nous manque encore une histoire du développement durant cette période des recherches spécialisées sur les problèmes d'extrémalisation sous contrainte. En nous focalisant sur les travaux de Leonhard Euler et Joseph-Louis Lagrange entre 1738 et 1806, la présente étude cherche à identifier et à ramener ensemble les différents éléments de cette histoire. © 1992 Academic Press, Inc.

Historiker haben die Hauptentwicklung der Variationsrechnung im 18. Jahrhundert dokumentiert. Zwar haben wir eine klare Vorstellung über dieses Thema, doch gibt es in der Literatur keine zusammenhängende historische Beschreibung der eingehenderen Forschung, die in dieser Zeit zu Problemen der Extremalisierung unter Einbeziehung von Nebenbedingungen betrieben wurde. Die vorliegende Untersuchung konzentriert sich auf die Arbeit von Leonhard Euler und Joseph Louis Lagrange, die zwischen 1738 und 1806 durchgeführt wurde, und stellt einen Versuch dar, die unterschiedlichen Gedankengänge dieses Problembereichs herauszuarbeiten und zusammenzuführen. © 1992 Academic Press, Inc.

AMS 1991 subject classification: 01A50, 01A55.

KEY WORDS: history of analysis, calculus of variations, isoperimetric problem, Euler's rule, Lagrange multiplier, problem of Lagrange.

## 1. INTRODUCTION

Historical writers beginning with Lagrange [1806] have documented the development of the calculus of variations in the 18th century. Although we have a clear outline of the major stages in this development, there is in the literature no connected historical account of the more specialized research carried out during the period on problems of extremization under constraint.<sup>1</sup> Concentrating on the work of Leonhard Euler and Joseph Louis Lagrange between 1738 and 1806, the

present study attempts to identify and draw together the different threads that make up this story.

In addition to their historical importance for the early variational calculus the researches discussed here illuminate more generally the question of theory change in mathematics. They provide an example of how a mathematical theory comes to be formed, how its character changes in the course of its development, and how it incorporates and adapts to new ideas. They indicate the degree of theoretical sophistication achieved within analysis by 1800 and point to the increasing internalization that would characterize this subject in the nineteenth century.

## 2. MATHEMATICAL BACKGROUND

The most basic problem of the calculus of variations requires finding the function  $y = y(x)$  from among a class of functions that renders a given definite integral of the form

$$\int_a^b f(x, y, y')dx \quad \left(y' = \frac{dy}{dx}\right) \tag{1}$$

a maximum or minimum. Perhaps the simplest example of this problem is to find the shortest curve joining two points in the plane (a straight line). A necessary condition that must be satisfied by the extremizing function is the so-called Euler or Euler–Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) = 0 \tag{2}$$

first derived by Euler [1741] in a memoir published in the Commentaries of the St. Petersburg Academy of Sciences.

The basic problem may be modified by demanding that the class of potential extremizing functions also satisfy a side condition of the form

$$\int_a^b g(x, y, y')dx = \text{constant.} \tag{3}$$

One thereby obtains the so-called isoperimetric problems that figured prominently in the early history of the subject.<sup>2</sup> The classic example is to find the curve of given perimeter which bounds the greatest area (a circle). The solution of these problems involves an application of “Euler’s rule”, first presented by Euler [1738] in a St. Petersburg memoir. This rule stipulates that the extremization of (1) relative to (3) leads to the same equation as the problem of extremizing the integral

$$\int_a^b (f + \kappa g)dx, \tag{4}$$

where  $\kappa$  is a constant (sometimes called an “undetermined coefficient”) and where there is now no side condition. The extremizing function  $y = y(x)$ , and with

it the precise value of  $\kappa$ , will be determined jointly from (3) and the condition that (4) be an extremum.

The basic problem may be modified in another way by considering a variational integral of the form

$$\int_a^b f(x, y, y', z) dx, \quad (5)$$

where the variable  $z$  in the integrand is itself expressed in terms of an integral

$$z = \int_a^x g(x, y, y') dx. \quad (6)$$

One may wish more generally to suppose that  $g$  contains  $z$ :

$$z = \int_a^x g(x, y, y', z) dx. \quad (7)$$

An example is the problem of the brachistochrone in a resisting medium, in which the time of descent is proportional to  $\int_a^b (1/v)\sqrt{1+y'^2} dx$ . The speed  $v$  in this case satisfies an auxiliary relation of the form  $v(dv/dx) = g - R(v)\sqrt{1+y'^2}$ , where  $g$  is the acceleration due to gravity and  $R$  is some function of  $v$  that measures the resistance. (The variable  $v$  here takes the place of  $z$  in (7).) In the modern subject this type of example is an instance of the theory associated with the very general "problem of Lagrange." Euler first considered the problem in 1741, and provided a fuller analysis in his *Methodus inveniendi curvas lineas* [1744], where the problem achieved a certain prominence. He showed for example that the extremization of (5) with  $z$  given by (6) leads to the equation

$$\frac{\partial g}{\partial y} \int_x^b \frac{\partial f}{\partial z} dx - \frac{d}{dx} \left( \frac{\partial g}{\partial y'} \int_x^b \frac{\partial f}{\partial z} dx \right) + \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0. \quad (8)$$

This last example may also be viewed as one of extremizing the integral  $\int_a^b f(x, y, y', z) dx$  subject to the differential side condition  $g(x, y, y') - z' = 0$ . It may then be exhibited as an instance of a more general multiplier rule. Lagrange [1806] in his *Leçons sur le calcul des fonctions* was the first to envisage the theory in this way. He began with a formulation of the original unconditional problem that results when more than one dependent variable is introduced into the integrand  $f$ . Thus the problem of extremizing

$$\int_a^b f(x, y, y', z, z') dx \quad (9)$$

leads to the two Euler equations

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0, \quad \frac{\partial f}{\partial z} - \frac{d}{dx} \left( \frac{\partial f}{\partial z'} \right) = 0. \quad (10)$$

Assume now that there is a differential side condition of the form

$$h(x, y, y', z, z') = 0. \quad (11)$$

The extremization of (9) subject to (11) leads to the same equations as the problem of extremizing

$$\int_a^b (f + \lambda h) dx, \quad (12)$$

where  $\lambda = \lambda(x)$  is a function of  $x$  (a “Lagrange multiplier”) and where there is now no side condition.<sup>3</sup> In the example from Euler above  $h(x, y, y', z, z') = g(x, y, y') - z'$ . Equations (11) become

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'} \right) = 0, \quad \frac{\partial f}{\partial z} + \frac{d}{dx} (\lambda(x)) = 0, \quad (13)$$

which for  $\lambda(b) = 0$  reduce to equation (8).

The present study is concerned with the early history of the isoperimetric problem and the “problem of Lagrange”. We examine how these problems were first formulated in the writings of Euler, how relative interest in them on the part of researchers shifted as the subject developed, and how they became unified in Lagrange’s theory of 1806. We document the achievement apparent in Lagrange’s method of multipliers and consider its precise character as an advance over earlier methods and results.

### 3. ISOPERIMETRIC PROBLEM 1738–1766

The basis of Euler’s theory was established in his St. Petersburg memoir of 1738, an investigation that began from the earlier researches of Jakob and Johann Bernoulli.<sup>4</sup> His approach was based on the idea of disturbing the curve at a single ordinate, evaluating the resulting change in the variational integral, and setting the expression obtained equal to zero. Consider a comparison arc obtained from the proposed extremizing curve by increasing the ordinate  $y$  by the small quantity  $b\beta$ . Euler showed that the difference between the integral (1) along the given and comparison curves is an expression of the form  $P(b\beta)dx$ . Since the given curve is assumed to be the one that extremizes (1) we have  $P(b\beta) = 0$  or simple  $P = 0$  as its equation.

Euler [1738] proceeded in the memoir to consider isoperimetric problems. Assume that there is an integral side condition present of the form (3). Let  $R(b\beta)dx$  be the variation of the integral in (3) when the ordinate  $y$  is increased by  $b\beta$ . Because the comparison arc must now satisfy (3) it is no longer possible to obtain it from the given curve by varying a single ordinate. To analyze the problem of extremizing (1) subject to (3) Euler [1738, Sect. 18] therefore supposed that two “consecutive” ordinates are varied by the amounts  $b\beta$  and  $c\gamma$ . The variational problem leads in this case to the relations

$$\begin{aligned} P(b\beta) + (P + dP)(c\gamma) &= 0 \\ R(b\beta) + (R + dR)(c\gamma) &= 0. \end{aligned} \quad (14)$$

By eliminating  $b\beta$  and  $c\gamma$  from (14) he obtained the differential equation

$$RdP = PdR, \quad (15)$$

whose integral is  $P + aR = 0$ ,  $a$  being a constant. This is precisely "Euler's rule."

Later in the 1738 memoir (Sects. 32–33) he considered the case in which there are two integral side conditions:

$$\begin{aligned} \int_a^b g(x, y, y')dx &= (\text{constant})_1 \\ \int_a^b h(x, y, y')dx &= (\text{constant})_2. \end{aligned} \quad (3')$$

Let  $P(b\beta)dx$ ,  $p(b\beta)dx$ , and  $\pi(b\beta)dx$  denote the variations in  $\int_a^b f dx$ ,  $\int_a^b g dx$  and  $\int_a^b h dx$  respectively when  $y$  is increased by  $b\beta$ . In order to obtain a comparison arc that satisfies (3)' Euler varied three "consecutive" ordinates by the small amounts  $b\beta$ ,  $-c\gamma$ , and  $d\delta$ . His variational procedure leads in this case to

$$\begin{aligned} P \cdot b\beta - (P + dP)c\gamma + (P + 2dP + ddP)d\delta &= 0 \\ p \cdot b\beta - (p + dp)c\gamma + (p + 2dp + ddp)d\delta &= 0 \\ \pi \cdot b\beta - (\pi + d\pi)c\gamma + (\pi + 2d\pi + dd\pi)d\delta &= 0. \end{aligned} \quad (16)$$

(Note: The symbol 'd' in 'd\delta' is purely designatory and is unrelated to the differential characteristic  $d$  as it appears in 'dP', 'dd\pi' and so on.) By eliminating the quantities  $b\beta$ ,  $c\gamma$ , and  $d\delta$  he obtained the differential equation

$$pd\pi ddP - \pi dp ddP + \pi dP ddP - Pd\pi ddP + Pd p dd\pi - pdP dd\pi = 0. \quad (17)$$

Euler concluded "Ex qua integrata reperitur  $P + mp + n\pi = 0$ , in qua  $m$  et  $n$  quantitates quascunque constantes designant."

While it is certainly true that  $P + mp + n\pi = 0$  satisfies (17), Euler had not in fact shown that the integration of (17) leads to such a relation. In this respect his analysis is essentially incomplete, a situation that stands in contrast to the earlier case of a single side condition, where  $P + aR = 0$  followed immediately from  $RdP - PdR = 0$  by the quotient rule. In his subsequent study of isoperimetric problems in 1741 and 1744 he never improved upon his analysis here; thus in his major treatise of 1744 he simply noted that  $P + mp + n\pi = 0$  satisfies (17).<sup>5</sup> Although the requisite demonstration would appear to have been available by contemporary principles of the calculus, he was either unable or, more likely, disinclined to supply the necessary details.<sup>6</sup>

Euler's later variational writings did contain extremely important analytical advances. In a given problem it would be necessary to calculate the expressions  $P$ ,  $p$ , and  $\pi$  above in order to derive the variational differential equations. For this purpose Euler in the 1738 memoir prepared tables giving these expressions for various integrands. His efforts constituted an important step in a synthesis of the theory, which he achieved more fully in his paper of 1741.<sup>7</sup> Given an expression  $Z$

composed of  $x$ ,  $y$ , and  $p = dy/dx$  he considered the relation  $dZ = Mdx + Ndy + Pdp$ . This relation provided an expression for the variation in terms of the differential coefficients (what would later be called partial derivatives)  $N (= \partial Z/\partial y)$  and  $P (= \partial Z/\partial y')$ .<sup>8</sup> He showed that the problem of extremizing the integral  $\int_a^b Z(x, y, p)dx$  leads to the general equational form  $N - dP/dx = 0$ , i.e., the Euler equation  $\partial f/\partial y - d(\partial f/\partial y')/dx = 0$  (2) above. He extended this result to the case where there are higher-order derivatives in the integrand.

Euler's *Methodus inveniendi* [1744] provided an extensive and systematic development of the results of the earlier memoirs. The topic of isoperimetric problems was itself separated from the main investigation and presented in the final two chapters. Although several examples involving integral side conditions are worked in detail the theory advanced little beyond the state it had reached in 1738.

It should be noted that much of this final part of Euler's treatise is open to serious criticism. He seemed to have been suggesting possible approaches and trying out tentative lines of development, rather than presenting mature and considered theory. The theorem that opens Chapter Six is a case in point. Euler's stated intention is to provide an approach to isoperimetric problems that provides an alternative to his established practice of working with differentials. We are given the expression  $\alpha A + \beta B$ , where  $\alpha$  and  $\beta$  are constants and  $A$  and  $B$  denote formula specifying properties of a curve. (In the usual variational problem  $A$  and  $B$  would be definite integrals of the form  $\int_a^b f(x, y, y')dx$ .) The theorem asserts that if for a given curve the expression  $\alpha A + \beta B$  is a maximum or minimum then the same curve will extremize  $B$  with respect to the class of curves that possess property  $A$ . Euler demonstrates this result for the case where the curve  $Q$  in question renders  $\alpha A + \beta B$  a maximum. Let  $R$  be any other curve along which  $A$  has the same value as it has along  $Q$ . Since the value of  $\alpha A + \beta B$  along  $Q$  is greater than along  $R$  it follows that the value of  $B$  along  $Q$  is greater than along  $R$  ( $\beta$  being implicitly assumed to be positive).

An evident limitation on this result is that in the usual isoperimetric problem of extremizing (1) subject to (3) the constant in (3) is given in advance. There is nothing in Euler's theorem concerning the range of values assumed by  $A$  as the constants  $\alpha$  and  $\beta$  change in value. The interest of his theorem is therefore limited. The converse, given as Corollary One, although more directly applicable to the isoperimetric variational problem, is even more problematic. It asserts that if  $B$  is a maximum or minimum along a given curve among all those curves for which  $A$  has a specified value then  $\alpha A + \beta B$  is a maximum or minimum along the said curve, no restriction now being placed on the comparison class of curves. The questionable character of this reasoning is evident.

Lagrange's early variational calculus based on his famous  $\delta$ -algorithm was presented in a letter to Euler of 1755 and in two memoirs published in the *Miscellanea* of the Turin Society in 1762 and 1773. One of the most striking features of these writings was the absence of any mention of isoperimetric problems. Lagrange's silence indicates that he was primarily concerned at this stage with presenting his  $\delta$ -algorithm as a significant mathematical method, rather than in systematically developing the subject on the proposed new basis. Problems with integral side-

conditions, although they posed no major challenge to this theory, were not particularly compelling examples in support of it either.

Lagrange’s neglect of isoperimetric problems may also have been connected to the mathematical philosophy implicit in his writings. A very strong algorithmic and algebraic sense guided his understanding of the calculus of variations. To reproduce Euler’s derivation of the isoperimetric rule would have required explicitly regarding the integral  $\int Zdx$  as a sum of the form . . . +Z,  $dx + Zdz + Z'dx + . . .$ . That such a conception was mathematically consistent with the adoption of the  $\delta$ -method is evident in the subsequent work of Euler and other researchers of the period.<sup>9</sup> Although these authors accepted Lagrange’s innovation and indeed emphasized the analytical character of the new calculus they continued to regard the integral as a sum. For them the procedures of the subject were not—as they were for Lagrange—understood exclusively in term of algorithmic relationships.<sup>10</sup>

Euler in his memoir *Elementa calculi variationum* of 1766, his first writing employing the  $\delta$ -algorithm, provided at the end a brief discussion of isoperimetric problems. He considered [1766, 91–93] the problem of finding among all relations between  $x$  and  $y$  (note that he now writes of relations rather than curves) the one that renders the integral  $\int_0^a Udx$  a maximum or minimum, subject to the side condition  $\int_0^a Vdx = \text{constant}$ . By means of the  $\delta$ -process he established the two equations

$$(V)\delta y + (V)'\delta y' + (V)''\delta y'' + (V)'''\delta y''' + . . . = 0 \tag{18}$$

$$(A)\delta y + (A)'\delta y' + (A)''\delta y'' + (A)'''\delta y''' + . . . = 0, \tag{19}$$

where (letting  $p = dy/dx$ )  $(V) = \partial V/\partial y - d(\partial V/\partial p)/dx$ ,  $(A) = \partial U/\partial y - d(\partial U/\partial p)/dx$ , and  $(V)', (V)'', (V)''', . . . , (A)', (A)'', (A)''', . . .$  are the values of  $(V)$  and  $(A)$  respectively at the ordinates  $y', y'', y''', . . .$ . He proceeded to reason as follows. (Our discussion is somewhat more explicit than Euler’s original account.) It is clear that the class of permissible variations must satisfy equation (18). If  $y = y(x)$  is such that the differential equation  $(A) = n(V)$  ( $n$  a constant) is valid then it is apparent that (19) will also be satisfied, and it follows that the given  $y = y(x)$  is an extremizing function. But the condition on  $y = y(x)$  that  $(A) = n(V)$  be valid is precisely the one which obtains in the unconditional variational problem of extremizing  $\int_0^a (U - nV)dx$ .

19th-century writers cited similar arguments in order to make Euler’s rule plausible.<sup>11</sup> Let the notation  $[f]$  denote  $\partial f/\partial y - d(\partial f/\partial y')/dx$  ( $y' = dy/dx$ ). Consider the usual isoperimetric problem of extremizing

$$\int_a^b f(x, y, y')dx$$

subject to the integral side-condition

$$\int_a^b g(x, y, y')dx = \text{constant}.$$

By means of the rule  $\delta(dy/dx) = d(\delta y)/dx$ , an integration by parts, and the assumption of unvaried endpoints we obtain

$$\int_a^b [f] \delta y dx = 0, \quad \int_a^b [g] \delta y dx = 0.$$

If  $y = y(x)$  satisfies  $[f] = \kappa[g]$  ( $\kappa$  a constant) then the second of these equations implies the first, and  $y(x)$  is consequently an extremizing function. But the condition that  $[f] = \kappa[g]$  (or, equivalently,  $[f - \kappa g] = 0$ ) is precisely the one which obtains in the unconditional problem of extremizing  $\int_a^b (f - \kappa g) dx$ . (A similar argument applies when there is a second integral side condition, say of the form  $\int_a^b h(x, y, y') dx = \text{constant}$ . If the differential equation  $[f] = \kappa[g] + \mu[h]$  is valid for  $y = y(x)$  then the conditions  $\int_a^b [g] \delta y = 0$  and  $\int_a^b [h] \delta y = 0$  imply  $\int_a^b [f] \delta y = 0$ , and the given  $y(x)$  is consequently an extremizing function. But the validity of  $[f] = \kappa[g] + \mu[h]$  will obtain in the unconditional problem of extremizing  $\int_a^b (f - \kappa g - \mu h) dx$ .)

In this argument the isoperimetric rule fails to emerge as a necessary condition of the variational problem; it is logically possible that the solution  $y = y(x)$  is one for which the equation  $[f] = \kappa[g]$  does not hold. The reasoning in question is therefore less than entirely satisfying. It is perhaps not surprising that it appears nowhere in Lagrange's writings, where formalistic considerations and the presentation of results take precedence over discursive discussions of plausibility.

It is worth emphasizing the very marginal attention that isoperimetric problems receive in those researches of Euler that employ the  $\delta$ -algorithm. In the memoir discussed above the subject is relegated to the final few sections. It is only briefly mentioned in his subsequent variational writings and does not appear at all in his longer treatise *De calculo variationum* of 1770.

Given the prominent place of isoperimetric problems in the writings of Jakob and Johann Bernoulli, their increasingly subordinate role in the midcentury theory is remarkable. The mathematical substance of the Bernoullis' investigation was centered in the detailed analysis of individual problems. Although specific examples continued to occupy an important place in Euler's researches we also see in his work the emergence of a theoretical structure for the subject. A shift had begun to take place away from problems as such to a study of the theory that they generate. The reorientation of the subject that occurred with the establishment of Lagrange's  $\delta$ -calculus reinforced the prevailing emphasis on analytical generality. At this level of development the question of integral side conditions was not one of major interest or mathematical complexity.

#### 4. THE "PROBLEM OF LAGRANGE" 1741-1773

In his memoir of 1741 Euler first grappled with the problem of extremizing integrals of the form  $\int_a^b f(x, y, y', z) dx$  (5) where  $z = \int_a^x g(x, y, y') dx$  (6). The essay is indeed something of a work-in-progress with respect to this type of problem. In the opening sections he failed to realize that if the integrand of the variational integral contains terms of the form  $z = \int_a^x g(x, y, y') dx$  then it is necessary to take



account of the variation at all values of  $x$  that exceed the value corresponding to the given altered ordinate. (Thus the equation at the end of his Section 6 must be supplemented by an additional term of the form  $-(\int_x^b (\partial f/\partial s) dx) d(dy/ds)/dx$ .) His later analysis of motion in a resisting medium (Sects. 16–18) is in consequence in error. Similar difficulties recur in subsequent sections (Sects. 22, 23, 25, and 27). Finally, at the end he returned to the initial subject of the memoir and provided a correct analysis, including a derivation of the correct form of the equation that had appeared (Sect. 6) earlier. The final sections of the memoir would become the starting-point for the investigations presented in *Methodus inveniendi*, the entire Chapter Three of which is devoted to the “problem of Lagrange.”<sup>12</sup>

Euler’s most notable achievement in Chapter Three of *Methodus inveniendi* was to obtain a complete solution in terms of differential equations for two problems involving the motion of a body in a resisting medium. In the first, it is required to find the curve joining two points in a vertical plane along which a heavy body should be constrained to move in order to achieve maximal terminal speed. In the second, the problem of the brachistochrone, it is required to find the curve along which the body will descend in the least time. With Euler’s treatise the “problem of Lagrange” became a central part of variational mathematics, occupying indeed a considerably more prominent place than the historically venerable isoperimetric problems.

Euler’s theory was based on the idea of disturbing the curve  $y = y(x)$  at a single ordinate, evaluating the resulting change in the variational integral, and setting the expression obtained equal to zero. When the integrand was of the form  $f(x, y, y')$  the entire change could be calculated locally in the neighborhood of the given altered ordinate. When the integrand was of the form  $f(x, y, y', z)$  with  $z = \int_a^x g(x, y, y') dx$  it became necessary to consider the change in the variational integral over the entire range of values from  $x$  to  $b$ .

Euler’s procedure was notationally and computationally very complicated, especially when applied to examples with higher-order derivatives  $y''$ ,  $y^{(3)}$ ,  $y^{(4)}$ , . . . in the integrand. Lagrange’s  $\delta$ -algorithm effected an immediate and dramatic simplification of the theory. Based on a technique in which all of the ordinates are varied simultaneously, it used integration by parts to arrive at a global variational process that was particularly suited to handle the examples of Chapter Three of the *Methodus inveniendi*.<sup>13</sup> The interest and evident superiority of his algorithm was based to a very considerable degree on its effectiveness in dealing with this type of problem.

It is important to note that neither Euler nor Lagrange in his early researches treated the “problem of Lagrange” mathematically as one of extremization under constraint. The new variable  $z$  in the integrand was regarded as a function of  $x$ ,  $y$ , and  $y'$  and the variational process was extended to calculate the additional variation introduced by it into the integral. In the more general examples that these authors considered the differential equations were always obtained by means of direct computation of the requisite variations.

Euler used the adjective “absolute” for problems in which there was no isoperi-

metric condition and the adjective “relative” for ones in which such a condition was present. Variational integrals where expressions of the form (6) or (7) appear in the integrand were always regarded by him as instances of an absolute problem.

In none of his writings did Euler derive the isoperimetric rule of his 1738 memoir from the theory associated with the “problem of Lagrange.”<sup>14</sup> In the modern subject the expression  $\int_x^b (\partial f/\partial z) dx$  that appears in an equation such as (8) is recognizable as a multiplier function (evident from Eqs. (13) above). On this basis several historians have suggested that the method of multipliers should be credited to Euler [1744].<sup>15</sup> Such an attribution is however mistaken. Possession of the method would require at the very least some development of the variational theory for the case of two dependent variables and their derivatives in the integrand of the variational integral. Nowhere in the *Methodus inveniendi* does Euler introduce multiple dependent variables into his general formulation of the variational problem. (This circumstance may be explained in part by certain characteristics of his approach in that treatise. Although he recognized that the analytical core of the subject was independent of any particular geometrical interpretation, its contents were nevertheless motivated throughout by geometrical examples and applications, and none of these suggested introducing multiple dependent variables.)

The situation is less clear in Euler’s post-1755 writings, in which he consciously emphasized the development of analytical aspects of the theory. In his treatise *De calculo variationum*, published in 1770 as an appendix to the third volume of his *Institutiones calculi integralis*, he introduced [1770, 549–564] variational integrals of the form (9) and derived Eqs. (10). He never, however, considered examples in which the second dependent variable  $z$  is given by a relation of the form (6), (7), or (11). Instead he investigated such integrals as  $\int_a^b f(x, y, y', z, z', v) dx$  in which the new variable  $v$  is given by  $v = \int_a^x g(x, y, y', z, z') dx$ . The resulting variational equations were obtained in an exactly analogous manner to his earlier derivation of (8).

Although Euler was the creator of the calculus of variations his conception of this subject was ultimately a limited one. His synthetic sense and feeling for generality were largely confined to a study of the general forms that appear in the derivation of the differential equations of individual problems. The sort of considerations that would have motivated a unified treatment of the different problems of the *Methodus inveniendi* required at once new ideas as well as a more developed theoretical sense than he possessed.

## 5. SYNTHESIS IN LAGRANGE’S LEÇONS SUR LE CALCUL DES FONCTIONS (1806)

Lagrange’s early variational writings consisted of research papers intended to reveal the power of his  $\delta$ -algorithm and the possibilities of his abstract formal conception of variational calculus. In his two late treatises *Théorie des fonctions analytiques* (1797) and *Leçons sur le calcul des fonctions* (2nd ed., 1806) he embarked on a discursive, systematic formulation of the differential, integral, and

variational calculus. These writings were characterized by the distinctive notation he adopted as well as by the way in which formal patterns were identified and employed in the deductive development of the subject. The guiding principle throughout was to avoid infinitesimals by defining the processes of the subject in terms of algebraic procedures and algorithms.

The *Théorie* [1797, 200–220] contained a brief indication of results in variational calculus, which were developed much more fully in the twenty-first and twenty-second *Leçons* [1806, 401–501], offered by Lagrange as a “traité complet du calcul des variations.” The twenty-first lesson included a discussion of the integrability of functions, the derivation of the basic variational equations, and a survey of the history of the subject. In the twenty-second lesson Lagrange presented his method of variations “déduite de la considération des Fonctions.” His definition of the variation and the notation he employs are described in [Fraser 1985]. In terms of the notational conventions of the present essay his approach to problems with side conditions proceeds as follows.<sup>16</sup>

Lagrange first considered the case in which there is more than one dependent variable  $y$  in the integrand of the variational integral, as in (9) above. He derived by means of his variational process the relation

$$\delta f = [f]_y \delta y + [f]_z \delta z + \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \delta y \right) + \frac{d}{dx} \left( \frac{\partial f}{\partial z'} \delta z \right), \quad (20)$$

where the notations  $[f]_y$  and  $[f]_z$  here denote  $\partial f / \partial y - d(\partial f / \partial y') / dx$  and  $\partial f / \partial z - d(\partial f / \partial z') / dx$ . In the variational problem the “primitive” of  $\delta f$  between  $a$  and  $b$  is by assumption zero, i.e.,  $\int_a^b \delta f dx = 0$ . He inferred from this fact and (20) the equations

$$[f]_y \delta y + [f]_z \delta z = 0 \quad (21)$$

$$\frac{\partial f}{\partial y'} \delta y \Big|_b^a + \frac{\partial f}{\partial z'} \delta z \Big|_b^a = 0. \quad (22)$$

If no relation is assumed between  $y$  and  $z$  (21) reduces to

$$[f]_y = 0, \quad [f]_z = 0. \quad (10)$$

Equations (10) allow one to determine the extremizing functions  $y = y(x)$  and  $z = z(x)$ , while (22) provides the conditions that must be satisfied at the endpoints. (The reasoning by which Lagrange passed from  $\int_a^b \delta f dx = 0$  and (20) to (21) is rather curious. He writes [1806, 460] in reference to (20): “Les termes  $[f]_y \delta y + [f]_z \delta z$ , qui ne sauraient être des fonctions dérivées exactes, tant que  $\delta y$  et  $\delta z$  ont des valeurs arbitraires, doivent être détruits, ce qui donnera d’abord l’équation générale  $[f]_y \delta y + [f]_z \delta z = 0$ . . . .” Thus he assumes that the expression for  $\delta f$  given by (20) must be an exact differential, and hence that the part  $[f]_y \delta y + [f]_z \delta z$  itself must be an exact differential. Since  $\delta y$  and  $\delta z$  are arbitrary this can only happen (he suggests) if (20) holds. This style of reasoning, which he had employed extensively earlier in the treatise, is analyzed in [Fraser 1985, 181–185].)

Consider now the case where the variables  $y$  and  $z$  are connected by a relation of the form  $F(x, y, z) = 0$ . Lagrange set  $\delta F = (\partial F/\partial y)\delta y + (\partial F/\partial z)\delta z = 0$  and used this equation to eliminate  $\delta y$  and  $\delta z$  from (21):

$$[f]_y \left( \frac{\partial F}{\partial z} \right) = [f]_z \left( \frac{\partial F}{\partial y} \right). \quad (23)$$

Equation (23) and  $F = 0$  are the equations of the variational problem.

Assume further that the side relation  $F = 0$  contains the derivatives of  $y$  and  $z$  with respect to  $x$ :  $F(x, y, z, y', z') = 0$ . It would in principle be possible to follow the same procedure here as in the derivation of (23). Lagrange suggested that it would be simpler to use the method of multipliers, first introduced by him in his *Mécanique analytique* [1788, 44–58], to investigate problems of static equilibrium. His procedure may be illustrated by the case of the equilibrium of a single particle (with spatial coordinates  $x, y,$  and  $z$ ) acted upon by an external force (with components  $X, Y,$  and  $Z$ ). Lagrange took as a condition for equilibrium the relation  $X\delta x + Y\delta y + Z\delta z = 0$ , where  $\delta x, \delta y,$  and  $\delta z$  are virtual displacements of the particle consistent with the constraints that are present. If the particle is unconstrained then  $\delta x, \delta y,$  and  $\delta z$  are independent and we obtain the equations of equilibrium  $X = Y = Z = 0$ . Assume now the particle is constrained to lie on the surface  $F(x, y, z) = 0$ . We take the equation  $\delta F = (\partial F/\partial x)\delta x + (\partial F/\partial y)\delta y + (\partial F/\partial z)\delta z = 0$ , multiply it by the constant  $\lambda$ , and add the result to  $X\delta x + Y\delta y + Z\delta z = 0$ :

$$X\delta x + \lambda \frac{\partial F}{\partial x} \delta x + Y\delta y + \lambda \frac{\partial F}{\partial y} \delta y + Z\delta z + \lambda \frac{\partial F}{\partial z} \delta z = 0. \quad (24)$$

The introduction of the multiplier allows us to assume that  $\delta x, \delta y,$  and  $\delta z$  are independent. The equations of equilibrium are therefore  $F = 0$  and

$$\begin{aligned} X + \lambda \frac{\partial F}{\partial x} &= 0 \\ Y + \lambda \frac{\partial F}{\partial y} &= 0 \\ Z + \lambda \frac{\partial F}{\partial z} &= 0. \end{aligned} \quad (25)$$

The method of multipliers would prove to be an important tool of variational analysis. In understanding how Lagrange arrived at the method it is significant to note that the underlying idea originated in mechanics. In the static problem the terms  $\lambda(\partial F/\partial x)$ ,  $\lambda(\partial F/\partial y)$ , and  $\lambda(\partial F/\partial z)$  in (25) have a natural physical interpretation as forces of constraint. Equations (25) assert that in equilibrium the total constraint force acts normally to the surface and exactly balances the applied force. Given that the constraint is given mathematically by an equation of the form  $F = 0$  it would in fact be reasonable to consider an analytical expression for this

force in terms of  $F$ . In this way one would be led to a solving procedure different from the usual direct one involving the elimination of unknown variables.<sup>17</sup>

In the *Théorie* [1797, 197–198] Lagrange indicated how the method of multipliers could be used in ordinary calculus to handle problems of extremization under constraint. His procedure is the one that is common today in multivariable calculus. He followed an analogous approach in the *Leçons* [1806, 462–469] to variational problems with side conditions of the form  $F(x, y, y', z, z') = 0$ . He took the variation  $\delta F(x, y, y', z, z')$ , multiplied it by the multiplier  $\lambda(x)$  (now a function of  $x$ ), and expressed the result in the form

$$\lambda \delta F = [\lambda F]_y \delta y + [\lambda F]_z \delta z + \frac{d}{dx} \left( \lambda \frac{\partial F}{\partial y'} \delta y \right) + \frac{d}{dx} \left( \lambda \frac{\partial F}{\partial z'} \delta z \right). \quad (26)$$

Adding together  $\int_a^b \delta f dx = 0$  and  $\int_a^b \lambda \delta F dx = 0$  we obtain  $\int_a^b (\delta f + \lambda \delta F) dx = 0$ . On the basis of this relation and (20) and (26) he arrived at the equations

$$[f + \lambda F]_y \delta y + [f + \lambda F]_z \delta z = 0 \quad (27)$$

$$\left( \frac{\partial f}{\partial y'} + \lambda \frac{\partial F}{\partial y'} \right) \delta y \Big|_b^a + \left( \frac{\partial f}{\partial z'} + \lambda \frac{\partial F}{\partial z'} \right) \delta z \Big|_b^a = 0. \quad (28)$$

He asserted that by virtue of the introduction of the multiplier we may now suppose that  $\delta y$  and  $\delta z$  are independent. Hence the variational equations of the problem are  $F(x, y, y', z, z') = 0$  and

$$\begin{aligned} [f + \lambda F]_y &= 0 \\ [f + \lambda F]_z &= 0. \end{aligned} \quad (29)$$

Equations  $F = 0$  and (29) suffice to determine the multiplier function  $\lambda(x)$  and the extremizing functions  $y = y(x)$  and  $z = z(x)$ . Lagrange noted that the procedure is generalizable to the case where there is more than one constraint equation present by introducing additional multipliers.

The multiplier rule presented here by Lagrange afforded a powerful and versatile tool of variational analysis. We encountered an example of the method at the end of Section 2 above. Lagrange himself illustrated it with a more general version of a result that had figured prominently in Chapter Three of Euler's *Methodus inveniendi* [1744, 120, Corollary 5] as well as in his own early writings [1762, Problem 3]. Given an equation of the form  $F(x, y, y', z, z') = 0$  the problem is to find the relation between  $y$ ,  $z$ , and  $x$  that maximizes or minimizes  $z$  evaluated between specified values of  $x$ . The variational problem then becomes that of extremizing  $\int_a^b z' dx$  subject to the side condition  $F = 0$ . (The classic example is to find the curve joining two points in a vertical plane along which a heavy particle moving through a resisting medium should be constrained to follow in order to achieve maximal terminal speed  $v$ . It is assumed that the resistance is a function of the velocity. In this example  $z = \frac{1}{2}v^2$  and the side relation  $F = 0$  is  $z' = g - R(z)\sqrt{1 + y'^2}$ , where  $g$  is the acceleration due to gravity and  $R(z)$  is the function of

$z$  that measures the resistance.<sup>18</sup>) Following the multiplier rule we consider the variational integral  $\int_a^b (z' + \lambda F) dx$ . Then  $\partial(z' + \lambda F)/\partial y = \partial \lambda F/\partial y$ ,  $\partial(z' + \lambda F)/\partial y' = \partial \lambda F/\partial y'$ ,  $\partial(z' + \lambda F)/\partial z = \partial \lambda F/\partial z$ ,  $\partial(z' + \lambda F)/\partial z' = 1 + \partial \lambda F/\partial z'$ , and equations (29) become

$$\begin{aligned}\lambda \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \lambda \frac{\partial F}{\partial y'} \right) &= 0 \\ \lambda \frac{\partial F}{\partial z} - \frac{d}{dx} \left( \lambda \frac{\partial F}{\partial z'} \right) &= 0.\end{aligned}$$

The preceding derivation is more general and much simpler than the ones that had appeared in Euler's and the early Lagrange's writings.<sup>19</sup> In the latter a particular case of the relation  $F = 0$  was used in integrated form in order to obtain an expression for the variable  $z$  which appeared in the integrand of the variational integral; the variation of this integral was then obtained by direct calculation. In the present investigation, by contrast, the equations are derived by taking an auxiliary differential relation, multiplying it by an unknown multiplier function, and adding the product to the integrand in the variational problem. The use of multipliers represented a new mathematical method involving the introduction of a novel and fertile idea into the calculus of variations.<sup>20</sup>

Lagrange [1806, 469–470] proceeded in the twenty-second *Leçon* to the classic isoperimetric problem of extremizing  $\int_a^b f(x, y, y') dx$  (1) subject to  $\int_a^b g(x, y, y') dx = \text{constant}$  (3). He showed here that his method of multipliers leads to Euler's rule. He let  $z = \int_a^x g(x, y, y')$  (6) and treated this as a differential side condition of the form  $z' - g(x, y, y') = 0$  in the extremization of (1). According to the multiplier rule the variational integral under consideration is

$$\int_a^b (f + \lambda(z' - g(x, y, y')))) dx. \quad (30)$$

The total variation of (30) is

$$\begin{aligned}\int_a^b \left( \left[ \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'} \right) \right] \delta y + \left[ - \frac{d}{dx} (\lambda) \right] \delta z \right) dx \\ + \left( \frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'} \right) \delta y \Big|_b^a + \lambda \delta z \Big|_b^a\end{aligned} \quad (31)$$

Equations (29) in this case are

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'} \right) = 0, \quad - \frac{d}{dx} (\lambda) = 0. \quad (32)$$

The second of these equations implies that  $\lambda$  is a constant. The isoperimetric condition implies that  $\delta z(b) - \delta z(a) = 0$ . Hence (31) is identical with the total variation of  $\int_a^b (f - \lambda g) dx$  (4), where  $\lambda$  is a constant and where no auxiliary condition is now being assumed. The extremization of (1) subject to (3) is there-

fore shown to be equivalent to the unconditional problem of extremizing (4). This is precisely Euler's rule for isoperimetric problems. It is evident that the rule may be extended to problems with more than one side condition by introducing additional multipliers.

By introducing the side condition  $z' - g(x, y, y') = 0$  into the variational problem  $\delta \int_a^b f(x, y, y') dx = 0$  and deducing that the multiplier function  $\lambda(x)$  is constant Lagrange had shown that Euler's rule may be obtained as a special instance of the "problem of Lagrange." His theory therefore afforded a natural unification of two classes of problems—the isoperimetric problem and the "problem of Lagrange"—that had hitherto been unconnected in variational mathematics.

At the end of the *Leçons* Lagrange considered several problems in which the endpoints of the extremalizing curve are allowed to vary. (This subject had since 1773 occupied an important place in his variational calculus.) Although these investigations were not directly related to the question of extremization under constraint, it is noteworthy that he used his method of multipliers here to derive the differential equations for the problem of the brachistochrone in a resisting medium.

## 6. CONCLUSION

The multiplier rule introduced into the calculus of variations a theoretical orientation absent in Lagrange's earlier writings. It afforded a unification of the isoperimetric problem and the "problem of Lagrange" and provided the basis for an integrated theory of considerable deductive power.

The idea of a multiplier was suggested to Lagrange by his work in mechanics. His subsequent variational researches, carried out when he was seventy years old, illustrated how an external source could stimulate and reorient the development of a mathematical theory. These researches also displayed a sensitivity to questions concerning the internal constitution of the theory itself. The calculus of variations had evolved in his writings to the point where it had acquired its own structure and identity. It had become meaningful to consider the deductive organization of the subject and to explore links connecting its different parts.

## 7. POSTSCRIPT: THE METHOD OF MULTIPLIERS IN LATER VARIATIONAL CALCULUS

Lagrange's research on problems of extremization under constraint was carried out from a larger foundational perspective that was distinctively algebraic in character.<sup>21</sup> The conceptual revolution in analysis initiated by Cauchy in the 1820s called into question his general outlook as well as many of the specific reasonings he had employed. Throughout the 19th century writers continued to understand variational mathematics in terms of the concepts and methods of operator and formal calculus.<sup>22</sup> Cauchy's program, however, was eventually consolidated in the calculus of variations, in the writings during the 1870s and 1880s of Karl Weierstrass, Paul Du Bois-Reymond, Adolph Mayer, and others. (A comparison

of the mean-value theorem of the ordinary calculus and the fundamental lemma of the calculus of variations illustrates clearly the slowness with which arithmetical conceptions entered variational mathematics. Although Cauchy [1823, Leçon 7] presented his proof of the mean-value theorem in 1823 it was not until 1879 that Du Bois-Reymond first formulated and proved the fundamental lemma. (Various earlier versions of this lemma are described in [Huke 1930].) The relatively late date at which researchers became interested in an arithmetical foundation is a distinctive feature of the development of the calculus of variations in the 19th century.)

Mayer [1886] was the first to attempt a general proof of the multiplier rule, although there are difficulties with his demonstration.<sup>23</sup> The derivation of the rule that became generally accepted appears in [Bolza 1909, 551–553]. Consider the problem of extremizing  $\int_a^b f(x, y, y', z, z')dx$  subject to the side relation  $F(x, y, y', z, z') = 0$ . Multiply  $F = 0$  by the function  $\lambda(x)$ , integrate from  $a$  to  $b$  and take the variation of the resulting equation,  $\delta \int_a^b \lambda F dx = 0$ . Add this to  $\delta \int_a^b f dx = 0$  to get  $\delta \int_a^b (f + \lambda F) dx = 0$ . By an integration by parts and the assumption of unvaried endpoints we arrive at  $\int_a^b ([f + \lambda F]_y \delta y + [f + \lambda F]_z \delta z) dx = 0$  (\*). Let  $\lambda(x)$  be a solution of the differential equation  $[f + \lambda F]_z = 0$ . The preceding integral equation becomes  $\int_a^b [f + \lambda F]_y \delta y dx = 0$ . Because the variation  $\delta y$  is arbitrary we may now invoke the so-called “fundamental lemma of the calculus of variations” and conclude that  $[f + \lambda F]_y = 0$ . In this way we arrive at the variational equations (29). (This derivation is especially noteworthy in providing a mathematically clear explanation of the inference whereby we pass from (\*) to Eqs. (29).)

Although the techniques and conceptions of real analysis replaced Lagrange’s algebraic understanding of variational calculus modern authors continue to acknowledge the theoretical insight evident in his method of multipliers. Mayer [1886, 74] judged the method to be the veritable foundation of the subject, and Pars [1962, 238] more recently has stressed its deductive power and versatility.<sup>24</sup>

### ACKNOWLEDGMENTS

I am indebted to Edward Barbeau for informative discussions on the subject of Lagrange multipliers. I thank Jesper Lützen for helpful discussions as well as comments on an earlier draft of the present article. I am grateful to Ivor Grattan-Guinness for suggestions on revising this draft.

### NOTES

1. Specific results are cited by Kneser [1904] and Bolza [1909] and selected work is described in detail by Woodhouse [1810], Carathéodory [1952], Goldstine [1980], and Fraser [1985]. Lagrange [1806] remains a good source for the 18th-century history.

2. In the 18th century the term “isoperimetric problem” was sometimes used in a general manner to refer to the entire subject of what would be later called the calculus of variations. Thus the full title of [Euler 1744] is “Method of discovering curved lines which display some property of maximum or minimum, or the solution of the isoperimetric problem taken in its widest sense”. Lagrange [1806] writes of the “famous problem of isoperimeters . . . taken in all its extension” to refer to the calculus of variations. In the present study the term will be used more narrowly to refer to variational problems in which integral side conditions involving fixed limits of integration are present.



3. It is necessary to clarify here a point of terminology. Although there is a natural temptation to refer to the constant  $\kappa$  that appears in Eq. (4) as a "multiplier", in the classical literature on the calculus of variations (e.g., [Bolza 1909; Kneser 1900]) this term is reserved for the function  $\lambda(x)$  that appears in Eq. (12). The rule whereby the extremization of (12) is postulated as yielding the extremum of (9) subject to the condition (11) is known as the "multiplier rule" or "multiplier method." The constant  $\kappa$  in Euler's rule, by contrast, has no particular terminological status or name. In the present article we adhere to the conventions of the classical subject, employing the term "multiplier" for the function  $\lambda(x)$  which appears in the solution of the "problem of Lagrange."

4. Selected papers of the Bernoullis are published in German translation with notes by Stäckel [1894]. An account of their researches is presented by Woodhouse [1810 Chaps. 1 and 2] and Goldstine [1980 Chap. 1].

5. Carathéodory [1952, xxvi] considers this point to be a substantial defect in Euler's *Methodus inveniendi*. Lagrange [1806, 432] when he considers the question in his historical survey of the calculus of variations in the *Leçons* progresses little beyond Euler. He writes "comme cette équation [ $P + aQ + bR = 0$  in his notation] contient deux constantes arbitraires  $a$  et  $b$ , il s'ensuit qu'elle sera nécessairement l'intégrale complète de l'équation du second ordre dont il s'agit."

6. Eighteenth-century analysts tended to regard an  $n$ th-order ordinary differential equation as solved when a solution was exhibited containing  $n$  arbitrary constants (cf. the remark of Lagrange's in the preceding note). Although the existence of singular solutions was recognized, these were regarded as anomalous or exceptional. By observing that  $P + mp + n\pi = 0$  satisfied (17) Euler may have regarded the mathematical question in point as settled.

The fact remains that it would be desirable to show by integration that (17) necessarily leads to  $P + mp + n\pi = 0$ . This integration may be effected as follows. (The proof was developed by the author.) We first rewrite (17) in the form

$$ddP(pd\pi - \pi dp) + ddp(\pi dP - Pd\pi) + dd\pi(Pdp - pdP) = 0.$$

Multiplying this equation by  $p$  and noting that

$$p\pi dP - pPd\pi = P(\pi dp - pd\pi) + \pi(pdP - PdP)$$

we obtain

$$pddP(pd\pi - \pi dp) + Pddp(\pi dp - pd\pi) + \pi ddp(pdP - PdP) + pdd\pi(Pdp - pdP) = 0$$

This equation may be rewritten

$$\frac{pddP - PddP}{pdP - PdP} = \frac{pdd\pi - \pi ddp}{pd\pi - \pi dp}$$

or

$$\frac{d(pdP - PdP)}{pdP - PdP} = \frac{d(pd\pi - \pi dp)}{pd\pi - \pi dp}.$$

Integration then yields the relation  $pdP - PdP = n(pd\pi - \pi dp)$ , where  $n$  is a constant. This relation may be written in the form

$$d\left(\frac{P}{p}\right) = nd\left(\frac{\pi}{p}\right)$$

Integration of this equation yields

$$\frac{P}{p} = n \frac{\pi}{p} + m,$$

where  $m$  is a second constant. The integral of (17) is therefore  $P = n\pi + mp$ , where  $n$  and  $m$  are arbitrary constants.

7. [Euler 1738, 1741] provide a striking study in the formation of a mathematical theory. Goldstine's [1980, 68] comments in reference to the *Methodus Inveniendi* also apply to these earlier writings: ". . . [Euler] change[d] the subject from the discussion of essentially special cases to a discussion of very general classes of problems. . . he took the fairly special methods of James and John Bernoulli and transformed these into a whole new brand of mathematics." A good account of the writings is presented by Woodhouse [1810 Chaps. 3, 4].

8. This manner of representing partial derivatives was common during the period in the work of the Bernoullis and Euler (see Engelsman 1984). Woodhouse [1810, 30] notes that Brook Taylor in his *Methodus incrementorum* of 1715 had introduced the expression  $\dot{V} = M\dot{x} + N\dot{y} + L\dot{s}$  ( $\dot{x}$ ,  $\dot{y}$ ,  $\dot{s}$  here denote Newtonian fluxions) in his study of isoperimetric problems. Woodhouse observes that Euler "skilfully availed himself of [this mode of expression]." Thus it would seem that some of the important ideas in [Euler 1741] originated with Taylor.

9. In addition to [Euler 1766a, 1766b] (discussed below) see also [Borda 1770].

10. In his historical survey of the calculus of variations in the *Leçons*, Lagrange [1806, 437] followed his praise of Euler's *Methodus inveniendi* with the observation: "Mais la décomposition que l'auteur y fait des différentielles et des intégrales dans leurs éléments primitifs détruit la mécanique de ce calcul, et lui fait perdre ses principaux avantages, la simplicité et la généralité de son algorithme."

11. See for example [Carll 1890, 114–115].

12. Fraser [1985, 158–160] provides an account of Euler's original analysis.

13. Lagrange's original development of his  $\delta$ -algorithm is described by Goldstine [1980, 110–129] and Fraser [1985, 160–172].

14. In Proposition V of Chapter V of the *Methodus inveniendi* Euler [1744, 204–205] considers the variational integral  $\int_a^b f(x, y, y', z)dx$  with  $z = \int_x^b g(x, y, y')dx$ , where it is further assumed that the isoperimetric condition  $\int_a^b g(x, y, y')dx = \text{constant}$  holds. However, he simply invokes "Euler's rule" and deduces as a result the equation obtained from (8) by replacing  $\int_x^b (\partial f/\partial z)dx$  by  $\int_x^b (\partial f/\partial z)dx + \alpha$ ,  $\alpha$  being a constant.

15. On the basis of Euler's derivation of (8) Kneser [1900, 580], Bolza [1909, 566], and Goldstine [1980, 74] credit a special case of the multiplier rule for the "problem of Lagrange" to Euler [1744]. Such an attribution is unacceptable in that it imputes to Euler a specific procedure and a theoretical outlook that he did not possess. See [Fraser 1985, 160] and note 20 of this article.

16. For Lagrange's  $\dot{x}$  we write  $\delta x$ , for  $f'(y) - [f'(y)']$  we write  $\partial f/\partial y - d(\partial f/\partial y')/dx$ , and so on.

17. The point here is that in the static problem the analysis possesses a natural physical interpretation which would have led to the method of multipliers. Another possibility is that Lagrange in his mechanical researches recalled Euler's isoperimetric rule of the calculus of variations and that this inspired the idea of a multiplier. Since the mechanical problem was not directly related to the isoperimetric rule, and since in fact Lagrange never mentions isoperimetric problems in his variational researches of the 1760s and 1770s, the suggestion of such a link must remain speculative.

18. Euler [1744, 122–126] subjects this example to detailed analysis. For an account that closely follows the original see [Goldstine, 79–82]. Woodhouse [1810, 138–141] derives the equations for this problem according to the methods of [Lagrange 1762] and [Lagrange 1806].

19. Since Lagrange's treatment of 1762 was itself a radical revision and simplification of Euler's formulation in the *Methodus Inveniendi*, we see in fact that between 1744 and 1806 there were three mathematically distinct solutions to this type of problem.

20. Kneser [1904, 580] wishes to credit Euler with the method of multipliers for problems with side conditions in the form of differential equations. He suggests "Lagrange multipliers" should be renamed "Euler-Lagrange multipliers," a suggestion endorsed by Bolza [1909, 556] and Goldstine [1980, 74]. Their conclusion is based on the following fact: In the equations which Euler derives in Chapter Three of the *Methodus inveniendi* certain expressions appear that we are able to identify as the multiplier functions that would result if the method of multipliers was used to derive the equations. While it may be of interest to note that Euler's variational process yields some of the results usually

associated with the later subject, it does not follow that he in fact possessed the later methods. To attribute the method of multipliers, or even a special case of this method, to him would be to attribute to him something that he did not possess. The method (in the calculus of variations) is first mentioned in Lagrange's *Théorie* [1797] and is developed more fully in his *Leçons* [1806] and the designation "Lagrange multiplier" is therefore accurate.

21. For a discussion of Lagrange's foundation see [Fraser 1987, 1989].

22. See for example [Jellett 1850].

23. Although Mayer [1886, 76] appears to have the demonstration in hand with his Eq. (8), he proceeds with an unusually complicated argument involving three pages of analysis and ten further equations before he reaches the desired conclusion. His derivation is described by Goldstine [1980, 282–285].

24. Mayer [1886, 74] writes

Es ist vielmehr nur kein Beispiel bekannt, in welchem das Lagrange'sche Verfahren zu einem falschen Resultate geführt hätte, und alle diejenigen besonderen Regeln der Variationsrechnung die, wie die isoperimetrische, sich auch noch auf anderem, directen Wege beweisen lassen, gehen als blossе Anwendungen aus demselben hervor. Daher wurde die Lagrange'sche Methode von einem Theile der Mathematiker gewissermassen als Axiom acceptirt, während ein anderer Theil es vorzog, alle diejenigen Aufgaben der Variationsrechnung, zu deren Lösung man keine anderen Methoden kennt, überhaupt einfach zu ignoriren.

Im Anschlusse an Clebsch habe ich selbst mich immer zu dem ersten Theile gehalten und die Lagrange'sche Regel allen meinen Arbeiten über Variationsrechnung zu Grunde gelegt.

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