J. L. Lagrange's Early Contributions to the Principles and Methods of Mechanics

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Introduction

The publication of J. L. LAGRANGE'S *Mécanique Analytique* in 1788 has long been recognized as an important event in the history of science. Although not an exhaustive account of mechanical science of the period, the treatise remains impressive in its attempt to present this subject from a unified viewpoint; it represents the culmination of a distinctive line of development whose origins extend back to LAGRANGE'S very early researches. Thus the basis of the *Mécanique Analytique*, a generalization of the principle of virtual work, first appeared in a memoir LAGRANGE composed in 1763 for a prize competition of the Paris Academy of Sciences. This memoir, devoted to the astronomical problem of lunar libration, in turn heralded a shift in his approach to principles and methods. He had previously endeavored to found dynamics on another mechanical law, the celebrated principle of least action. This research was closely associated with his work in the calculus of variations and was presented to the Turin Academy of Sciences in 1760. LAGRANGE'S interest in foundational questions was therefore an early one, having passed through two stages by 1763, when he was twenty-seven years old.

The *Mécanique Analytique* and the basic facts of LAGRANGE's prior shift from the principle of least action to the principle of virtual velocities are reasonably well known. What is perhaps less known is the extent to which the later treatise is anticipated in his early researches. The following study will focus on the latter and in so doing attempt to contribute to an understanding of the development of LAGRANGE's thought. My goals are three: i) to examine in the writings of EULER and D'ALEMBERT the immediate antecedents to each of the two stages in LAGRANGE's early approach to the foundations of mechanics; ii) to describe these stages in sufficient detail to elucidate the basic theory involved; iii) to investigate some of the reasons for his shift in approach. Since my interest lies primarily in the area of principles and methods I confine our discussion to a consideration of LAGRANGE's treatment of the mechanics of mass points and rigid bodies. This restriction, though unwelcome in a broader study, will nevertheless I hope permit the accomplishment of the above goals.

Part One: The Principle of Least Action (1760)

Section 1: Background

The Memoirs of the Turin Academy for the years 1760–1761 contain two papers by the young Italian geometer J. L. LAGRANGE: "Essai d'une nouvelle méthode pour determiner les maxima et les minima des formules indéfinies" and "Application de la méthode exposée dans le mémoire précédent à la solution de différentes problèmes de dynamique."[1] The theory presented in these memoirs actually originated several years earlier in LAGRANGE's first important work in mathematical science. In a letter of 1755 to LEONHARD EULER the nineteen-year-old LAGRANGE explained that he had derived a general method for treating those problems which had appeared in EULER's classic treatise of 1744 *Methodus Inveniendi lineas curvas maximi minimive proprietate gaudentes* (Method of finding curved lines which show some maximum or minimum property)[2]. The branch of mathematics that studies such problems later became known as the calculus of variations and it was LAGRANGE's achievement to give the subject its characteristic methods and symbolism. In particular, he introduced a new calculus based on the symbol δ which provided a powerful tool for attacking the following general problem: given an expression involving several variables and their derivatives, to find the functional relation between these variables which makes the definite integral of the given expression an extremum.

[1] *Miscellanea Taurinensia* **2** 1760–1761 (1762) = *Oeuvres de Lagrange* **1** (1867) pp. 335–362, 365–468. Throughout this study I adopt the following convention in attaching a date to a work: if the work was presented to a scientific academy for inclusion in its publications, I give the year of presentation; if submitted for a prize competition, the year set for the competition; if published independently, the year of publication.

[2] LAGRANGE's letter to EULER appears in the former's *Oeuvres* **14** (1892) p. 138–144. The *Methodus Inveniendi* appears in EULER's *Opera Omnia* I **24**, C. CARATHÉODORY, ed. (Bern, 1952).

EULER immediately recognized the importance of LAGRANGE'S work and the two men became regular correspondents. LAGRANGE soon applied his energies to the analysis of a problem in dynamics that EULER had investigated in an appendix to the *Methodus Inveniendi.* The problem consisted of deducing the motion of a particle from a variational law, a law which by the 1750's was known as the principle of least action. We learn from the Registers of the Berlin Academy that LAGRANGE had in May of 1756 submitted a memoir to the Academy on the subject of the principle.[3] Although this memoir is lost, it is possible to trace LAGRANGE'S subsequent progress in his letters to EULER.[4] Thus it becomes clear that the two memoirs of 1760 evolved from what was originally intended as a single treatise on dynamics introduced by an exposition of the mathematical method of variations. We also learn that LAGRANGE was very proud of what he had accomplished, which in his view was to have generalized and completed EULER'S earlier research by making the principle of least action the basis of dynamics.

It is clear that during the 1750's EULER was the preeminent influence on LAGRANGE'S investigation of the use of variational techniques in mechanics. I shall therefore devote the remainder of this section to a brief survey of EULER'S research in this subject. The emphasis will be on those aspects that would have been suitable to have influenced LAGRANGE'S approach to the principles and methods of mechanics.

EULER'S most extensive contributions to variational mechanics appear in memoirs concerned with problems in statics. In appendix one of the *Methodus Inveniendi* EULER presents what C. TRUESDELL has called "the first treatise on any aspect of the mathematical theory of elasticity."[5] Using ideas and suggestions of JAMES and DANIEL BERNOULLI, EULER obtains important results on the equilibrium properties of thin elastic bands. These results are derived from the condition that a certain quantity, what we would today call the potential energy function of the system, be a minimum. EULER, however, did not in this treatise fully understand the significance of this function, which, moreover, was introduced in rather arbitrary ways into the solution of problems. He would later return to this subject in two memoirs presented to the Berlin Academy of Sciences in 1748.[6] In these

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[3] EDUARD & MARIA WINTER *Die Registres der Berliner Akademie der Wissenschaften* 1746–1766 (Berlin, 1957) p. 223.

[4] See the following letters in *Oeuvres de Lagrange* **14** (1892): 19 May 1756 (pp. 154–156); 4 August 1758 (pp. 157–159); 28 July 1759 (pp. 159–161).

[5] EULER'S appendix is titled "Additamentum I De curvis elasticis" and appears in his *Opera Omnia* I **24** pp. 231–297. It is analyzed in detail by C. TRUESDELL in "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788", EULER's *Opera Omnia* II **11**^{2} (Orell Füssli, 1960) pp. 199–219.

[6] "Recherches sur les plus grands et plus petits qui se trouvent dans les actions des forces" *Mémoires de l'académie des sciences de Berlin* **4** 1748 (1750) pp. 149–188 = *Opera Omnia* II **5** (1957) pp. 1–37.

"Reflexions sur quelques loix générales de la nature qui s'observent dans les effets des forces quelconques" *Mémoires ... de Berlin* **4** 1748 (1750) pp. 189–218 = *Opera Omnia* II **5** (1957) pp. 38–63.

memoirs he develops more explicitly the basis of his method—'Maupertuis's law of rest'—and applies it to further problems in elasticity and hydrostatics.

Of more direct interest for the background to LAGRANGE's dynamical memoir of 1760 is EULER's research on the principle of least action. The second appendix of the *Methodus Inveniendi* contains a remarkable application of techniques from the calculus of variations to the analysis of the motion of a particle moving in a plane and acted upon by central forces.[7] In particular, EULER shows that the path obtained, if one assumes the integral of the velocity of the particle multiplied by the differential element of arc length is a minimum, is the same as that yielded by a direct calculation using known methods. The extremal property invoked here by EULER was also the basis of a memoir published by MAUPERTUIS in the same year (1744).[8] Although MAUPERTIUS's analysis was less exact than EULER's it was he who coined the term 'action' to refer to the product of mass, velocity and length; subsequently it became customary to attribute the principle to him.

EULER resumed his investigation of the principle of least action in November 1751 in a memoir presented to the Berlin Academy: "Harmonie entre les principes généraux de repos et de mouvement de M. de Maupertuis."[9] This treatise contains a nice summary of EULER's ideas on variational methods in mechanics; in addition, I shall argue, it was a probable source of inspiration for LAGRANGE. Before proceeding to an examination of the memoir it is first necessary to comment on the conditions under which it was written. In 1751 the celebrated KÖNIG affair had disrupted the life of the Berlin Academy. The controversy centered on KÖNIG's criticism of MAUPERTUIS's dynamical principle and his additional claim that the principle had in any case first appeared in a letter of LEIBNIZ to HERMANN in 1707. Details of this troublesome affair have been well documented in the literature and require no further comment here, except to mention that EULER vigorously defended MAUPERTUIS, the president of the Academy, throughout the affair.[10] Thus the memoir contains repeated affirmations of the truth and importance of the principle of least action as well as of its priority in the work

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[7] This appendix is titled "Addititamentum II De motu projectorum in medio non resistente. per methodum maximorum ac minimorum determinando" and appears in the *Opera Omnia* I **24** pp. 298–308. A detailed recent account of it is contained in H. GOLDSTINE's *A History of the Calculus of Variations from the 17th through the 19th Century* (Springer-Verlag, 1980) pp. 101–109.

[8] MAUPERTIUS "Accord de différentes loix de la nature qui avoient jusqu'ici paru incompatibles" *Mémoires de l'académie des sciences de Paris* **1744** pp. 417–426. This memoir was followed by another one in 1746: "Les loix du mouvement et du repos déduites d'un principe metaphysique" *Mémoires de l'académie des sciences de Berlin 1746* pp. 267–294. Both memoirs are reprinted in EULER's *Opera Omnia* II **5** (1957) pp. 274–302.

[9] *Mémoires de l'académie des sciences de Berlin* **7** 1751 (1753) pp. 160–198 = *Opera Omnia* II **5** pp. 152–176. The main argument of this memoir had been briefly sketched by EULER in the concluding article of his earlier (1748) treatise "Reflexions sur quelques ..." (*op. cit.* n. 6 p. 63).

[10] For an account of the controversy see P. BRUNET's *Étude Historique sur le Principe de la Moindre Action* (Paris, 1938). Further details and references may be found in J. O. FLECKENSTEIN's introduction to EULER's *Opera Omnia* II **5** (1957) pp. vii–l.

of MAUPERTUIS. In point of fact, EULER'S praise seems rather extravagant given that his own intellectual debt in the way of variational methods would appear to lie elsewhere, in the work of JAMES and DANIEL BERNOULLI.[11]

The memoir is written in a more discursive and general fashion than EULER'S earlier treatises. Its purpose is to show that the principle of least action may be "harmonized", *i.e.*, deduced, from the law of rest; since the latter is well established and is "subject to no opposition", the dynamical principle will have been vindicated. The discussion therefore begins with a description of the law of rest followed by an explanation of how it entails the principle of least action. The memoir concludes with some interesting reflections on conditions of equilibrium in constrained systems.

EULER introduces the law of rest for the case of a single mass M attracted to fixed centers by forces that are functions of the distances to these centers. Let $z, z', z'', \dots$ denote these distances and $V = V(z)$, $V' = V'(z')$, $V'' = V''(z'')$, $\dots$ the magnitudes of the corresponding forces. Consider the following expression:

$$(1) \qquad \int V\,dz + \int V'\,dz' + \int V''\,dz'' + \text{etc.}.$$

The law of rest states that in equilibrium this quantity must be an extremum, *i.e.* a maximum or a minimum. Expressed analytically the condition becomes

$$V\,dz + V'\,dz' + V''\,dz'' + \dots = 0.$$

If M belongs to a larger system, we form an expression similar to (1) for each mass. The law of rest then states that the sum of all these expressions must be an extremum.

Note that (1) is simply minus the work function of the system, *i.e.* the potential energy (the minus sign results from the fact that V, V', V'' are magnitudes and the forces are attractive). MAUPERTUIS had first presented the law of rest in 1740 for the case in which the force functions have certain special forms ($V(z) = z^n$, $V'(z') = z'^n, \dots$, n an integer).[12] He had demonstrated its validity in several examples by showing that it followed from accepted principles of equilibrium. EULER himself considered the law well established by his own earlier researches, where he had demonstrated its "truth by an infinity of entirely different cases."

EULER now proposes that that the word "effort" be used to describe (1). He observes that the effort will be a maximum when a slight disturbance serves to destroy the equilibrium; otherwise—and this he says is the more usual case—it will be a minimum.[13] EULER proceeds to argue that the law of rest may be

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[11] A discussion of the relation of EULER'S research to that of the BERNOULLIS is contained in C. TRUESDELL'S "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788" (*op. cit.* n. 5 pp. 199–203).

[12] "Loix du repos du corps" *Mémoires de l'académie des sciences de Paris 1740* pp. 170–176. This memoir is reprinted in EULER'S *Opera Omnia* II **5** (1957) pp. 268–273.

[13] EULER uses the example of a cone resting on a surface to illustrate this distinction: if the cone rests on its base the "effort" will be a minimum; if it rests on its point the "effort" will be a maximum. See *Opera Omnia* II **5** p. 173.

generalized to motion. Since this argument is important, and since its plausibility depends on quite general considerations, I quote in whole the relevant passage:

> Having established the principle for rest, or equilibrium, what is more natural than to maintain that this same principle is also valid in the motion of bodies, solicited by similar forces? For if the intention of Nature is to economize as much as is possible in the sum of the efforts, she must also extend herself to motion, provided that we take the efforts, not only as they subsist in an instant, but in all the instants together which the motion lasts. Thus the effort, or the sum of the efforts, being for any instant of motion $= \Phi$, and letting the element of time $= dt$, it is necessary that the integral formula $\int \Phi \, dt$ be a *minimum*. So that if in the cases of equilibrium the quantity Φ must be a *minimum*, the same laws of Nature seem to require that in motion this formula $\int \Phi \, dt$ be the smallest.[14]

Although EULER in this passage refers only to a minimum, it is clear from his other comments that what in fact is involved is an extremum. Indeed, he later states that if a quantity Z is a maximum, $-Z$ will be a minimum, so presumably something is always "economized" in nature. We shall, incidentally, follow EULER in denoting the potential or "effort" by the symbol Φ.

EULER's next step is to derive the law of conservation of vis viva. He does so for the case of a single particle M. Assume T is the tangential component of the sum of the forces acting on M. From the "principles of mechanics" we have the relation

$$M \, du = T \, dt, \tag{2}$$

where u is the speed. This of course is NEWTON's second law, which, however, had only received its first general formulation by EULER in the previous year.[15] Multiplying (2) by $u = ds/dt$ and integrating yields the conservation of vis viva:

$$\frac{1}{2} Mu^2 = \text{constant} + \int T \, ds$$

or

$$\frac{1}{2} Mu^2 = \text{constant} - \Phi. \tag{3}$$

Let us now multiply (3) by dt:

$$\frac{1}{2} Mu \, ds = (\text{constant}) \, dt - \Phi \, dt.$$

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[14] *Opera Omnia* II **5** p. 156.

[15] "Découverte d'un nouveau principe de mécanique" *Mémoires de l'académie des sciences de Berlin* **6** *1750* (1752) pp. 185–217 = *Opera Omnia* II **5** (1957) pp. 81–108. In both this memoir and the one presently under consideration EULER for some unexplained reason employs special units (they do not appear in his earlier treatises). In describing his analysis we do not follow him in this choice of units. The interested reader will find an explanation of them in C. TRUESDELL's essay "Rational Fluid Mechanics 1687–1765" published in EULER's *Opera Omnia* II **12** (1954) pp. xli–xliv.

We integrate this equation to obtain the final relation

$$\frac{1}{2} Mu\, ds = (\text{constant})\, t - \Phi\, dt. \tag{4}$$

With (4) we have reached a critical point in EULER's analysis. He wishes to examine the extremal properties of the quantities appearing in this equation. He states that terms like (constant) t do not enter into "the consideration of maxima or minima." EULER is therefore envisaging a variational process in which time itself is not varied.[16] Hence (4) combined with the fact that $\int \Phi\, dt$ is a maximum or minimum implies that $\int Mu\, ds$ (ignoring the factor of $\frac{1}{2}$) is respectively a minimum or maximum. EULER calls the quantity $\int Mu\, ds$ the "action"; he has shown that this quantity is an extremum. This is simply MAUPERTUIS's dynamical principle. It is, however, important to note that EULER's treatment is new and that he is the first to designate the action by the expression $\int Mu\, ds$.

The previous analysis applied to a single mass M. To extend the argument to arbitrary systems it would be necessary to generalize (3), the law of conservation of vis viva. EULER states that to do this it is sufficient to examine the motion of two bodies M and N attached to the ends of a massless rigid rod and attracted to a single fixed center. His approach here is in keeping with the spirit of the memoir, which is to suggest general lines of development through the consideration of special cases. In the example at hand I shall merely outline EULER's solution. At a given instant consider the tangents to the paths of the two bodies. In each case we take the projection onto the tangent of two forces: i) the external force to the fixed center, and ii) the force of tension in the rod maintaining rigidity. The values obtained are equated to $M\, du/dt$, $N\, dv/dt$ (u, v being the speeds of M and N). We proceed to multiply each side of the equations thus formed by u and v respectively, multiply both by dt, add the resultant relations and integrate to obtain

$$\frac{1}{2} Mu^2 + \frac{1}{2} Nv^2 = C - \Phi,$$

where Φ is the sum of the efforts and C is a constant.

With the presentation of this result EULER ends his discussion of the principle of least action. The remainder of the memoir consists of an analysis of equilibrium. EULER shows that the static rule of composition of forces follows from the law of rest and he proceeds to derive conditions of equilibrium in simple machines. The latter investigation, though not relevant to LAGRANGE's work on least action, is of interest for his later approach to foundations and we shall return to it in Part Two.

To conclude I repeat that the principle of least action appears in two places in EULER's research: in 1744, in an appendix to a treatise on the calculus of variations, and in 1751, in a memoir devoted to justifying the principle as a general law of dynamics. In his treatise of 1760 on least action LAGRANGE refers only to the earlier appendix. There is, however, considerable evidence to suggest he was also familiar with EULER's later research. First, it is reasonable to assume he had

[16] EULER also drew explicit attention to this fact in the memoir of 1748 "Reflexions sur quelques loix ..." (*op. cit.* n. 6 p. 63).

access to the recent published memoirs of the Berlin Academy of Sciences.[17] Second, his choice of notation in presenting the principle is identical with that employed by EULER in the memoir of 1751, an unlikely coincidence in the absence of any contact. Third, he does refer to the memoir in his piece of 1764 on libration. Finally, there is a deeper sense in which his treatment of least action is closer to EULER's later account. The last point will be developed in more detail in the next section.

Section 2: The Basic Theory

i) Introductory Remarks

The appearance of LAGRANGE's treatise "Application de la méthode exposée dans le mémoire précédent à la solution de diffèrentes problèmes de dynamique" in the Turin Miscellanea marks the first publication of his work in variational mechanics. As its title announces it will apply results from the "preceding memoir" on the calculus of variations to dynamics. It would, however, be wrong to suppose that the treatise is intended only as an illustration of how these mathematical techniques may be employed in mechanics. Its very length—103 pages compared to 27 for the previous memoir—belies such a conclusion. The young LAGRANGE's own conception is better revealed in his letters to EULER, where he emphasizes that he has constructed a general foundation for all of dynamics. LAGRANGE's stress on the mathematics may simply reflect the great importance he attached throughout his career to his earliest work in the calculus of variations.[18]

LAGRANGE opens the memoir with a reference to the second appendix of EULER's *Methodus Inveniendi* and a statement of his own generalization of the principle of least action:

> General Principle. Let there be as many bodies as one would wish M, M', M'', ..., which mutually interact in any manner, and which are moreover, if one wishes, animated by central forces proportional to any functions of these distances; let s, s', s'', ... denote the spaces travelled by these bodies in time t and let u, u', u'', ... be their speeds at the end of this time; the formula
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> $$M \int u\, ds + M' \int u'\, ds + M'' \int u''\, ds'' + \ldots$$
>
> will always be a maximum or a minimum.[19]

[17] For example, in Problem VIII of the memoir of 1760 LAGRANGE takes up the study of the motion of a rigid body. Although he makes no reference to EULER, his treatment is simply a reworking by his own variational method of theory that had appeared originally in EULER's "Découverte d'un nouveau principe de mécanique", published in the Memoirs of the Berlin Academy for the year 1751 (*op. cit.* n. 15).

[18] In a letter to D'ALEMBERT dated 20 November 1769 LAGRANGE states that he considers his work in the calculus of variations to be his finest contribution to mathematics. See LAGRANGE's *Oeuvres* **13** (1882) p. 154.

[19] *Oeuvres* **1** (1867) p. 365.

I should note at this point that LAGRANGE throughout the memoir proceeds very formally, with little in the way of explanation, and it is often difficult to ascertain the precise basis of his understanding. However, the intended meaning of the principle of least action seems clear. Given two configurations of the system, the bodies move in such a way that the (first) variation of the quantity $\int \Sigma Mu\, ds$ is zero. Hence the principle allows us to choose the actual motion from among all possible ones. In LAGRANGE'S treatment this means obtaining the differential equations that describe the motion; once these equations are derived he regards the problem as solved. When there are constraints in the system LAGRANGE proceeds in one of two ways: i) the constraints are replaced by the forces to which they give rise and no restriction is placed on the variation; or ii) we ignore these forces but require that the variations be compatible with the constraints. In actual practice LAGRANGE tends to follow the second approach.

Our subsequent discussion of the theoretical foundations of the memoir is divided into two parts. We begin with LAGRANGE'S analysis of the motion of a single particle (Problem I). A detailed study of this problem will permit a deeper understanding of his method and some of the issues associated with it. We then turn to LAGRANGE'S presentation of the principle of least action for arbitrary dynamical systems (Problem II) and examine his attempt to relate it to other general laws of mechanics.

ii) Problem I: The Motion of a Single Particle

Before embarking on a study of Problem I it is first necessary to review briefly the mathematics of LAGRANGE'S δ process.[20] Although this process is ostensibly introduced as a means for effecting the comparison of curves in space, it is nonetheless presented in a very formal manner. The symbol δ has properties analogous to the usual d of the differential calculus. Thus $\delta(xy) = x\,\delta y + y\,\delta x$ etc. In addition, d and δ are interchangeable (they commute) as are δ and the integral operator $\int$. The interchangeability of the d and the δ is important because it is used in LAGRANGE'S most basic analytic device: integration by parts. As an example we would have

$$\int (y\,\delta dx) = \int (y\, d\delta x) = y\,\delta x - \int (\delta x\, dy).$$

Note finally that in considering arcs in space LAGRANGE proceeds parametrically: if the path followed by the particle is represented by three spatial co-ordinates x, y, z, these variables must be understood as functions of an (unspecified) independent parameter.

Problem I consists of finding the motion of a particle M attracted to fixed centers by forces $P, Q, R, \ldots$ that are functions of the distances $p, q, r, \ldots$ to these centers. For simplicitly we shall assume only the forces P and Q are present.

[20] For a detailed account of the researches of both EULER and LAGRANGE in the calculus of variations see H. GOLDSTINE'S *A History of the Calculus of Variations* ... (*op. cit.* n. 7).

LAGRANGE begins his analysis with the principle of least action

(1) $$\delta\left(\int u\, ds\right) = 0,$$

where u denotes speed, $ds = u\, dt$ and the mass M cancels. LAGRANGE then writes

$$\delta(u\, ds) = u\, \delta ds + \delta u\, ds,$$

and interchanges the $\int$ and δ to obtain

(2) $$\int (u\, \delta ds + \delta u\, ds) = 0.$$

He now lays down the conservation of vis viva in the form

(3) $$\frac{1}{2} u^2 = \text{const} - \int (P\, dp + Q\, dq),$$

a result which "all geometers know." Let us take the variation of each side of (3):

(4) $$u\, \delta u = -\delta \int (P\, dp + Q\, dq).$$

After some manipulations from the calculus of variations (to which we return later) LAGRANGE transforms the right side of (4) to obtain

(5) $$u\, \delta u = -P\, \delta p - Q\, \delta q.$$

Recognizing that $u\, \delta u\, dt = \delta u\, ds$, we combine (2) and (5) to arrive at the following important relation, designated (A) in the original text:

(6) $$\text{(A)} \quad \int (u\, \delta ds - P\, dt\, \delta p - Q\, dt\, \delta q) = 0.$$

LAGRANGE proceeds to evaluate the integrals $\int u\, \delta ds$ and $\int (P\, dt\, \delta p + Q\, dt\, \delta q)$ separately. In each case he expresses the analysis in terms of the Cartesian position co-ordinates x, y, z of M. Since $ds = \sqrt{dx^2 + dy^2 + dz^2}$, we obtain after some reductions the result

(7) $$\int u\, \delta ds = \int ((u\, dx/ds)\, d\delta x + (u\, dy/ds)\, d\delta y + (u\, dz/ds)\, d\delta z),$$

where we have used the fact that $d\delta = \delta d$. Integrating the right side of (7) by parts yields

(8) $$\int u\, \delta ds = -\int (d(u\, dx/ds)\, \delta x + d(u\, dy/ds)\, \delta y + d(u\, dz/ds)\, \delta z) + (u\, dx/ds)\, \delta x + (u\, dy/ds)\, \delta y + (u\, dz/ds)\, \delta z.$$

LAGRANGE turns now to the integral $\int (P\, dt\, \delta p + Q\, dt\, \delta q)$. In Cartesian coordinates we have

$$\delta p = (\partial p/\partial x)\, \delta x + (\partial p/\partial y)\, \delta y + (\partial p/\partial z)\, \delta z,$$

with a similar expression for δq.[21] Hence

(9) $$P\, \delta p + Q\, \delta q = \Pi\, \delta x + \omega\, \delta y + \Psi\, \delta z,$$

[21] For reasons of notational convenience I have departed from the original text and used partial differentials to denote the coefficients of δx, δy and δz.

where

$$\Pi = P(\partial p/\partial x) + Q(\partial q/\partial x),$$

$$\omega = P(\partial p/\partial y) + Q(\partial q/\partial y),$$

$$\Psi = P(\partial p/\partial z) + Q(\partial q/\partial z).$$

Integrating (9), we may write

$$\text{(10)} \qquad \int (P\,dt\,\delta p + Q\,dt\,\delta q) = \int (\Pi\,\delta x + \omega\,\delta y + \Phi\,\delta z)\,dt.$$

Using the results contained in (8) and (10) and substituting into (6), LAGRANGE arrives at his second important relation, designated as (B) in the original:

$$\begin{aligned}\text{(11)} \qquad \text{(B)} \quad \int \{&(d(u\,dx/ds) + \Pi\,dt)\,\delta x + (d(u\,dy/ds) + \omega\,dt)\,\delta y \\ &+ (d(u\,dz/ds) + \Psi\,dt)\,\delta z\} + (u\,dx/ds)\,\delta x + (u\,dy/ds)\,\delta y \\ &+ (u\,dz/ds)\,\delta z = 0.\end{aligned}$$

Since (11) is valid "whatever values one supposes for the differences δx, δy, δz" LAGRANGE obtains as the final result the following three equations:

$$\text{(12)} \qquad \begin{aligned} d(u\,dx/ds) + \Pi\,dt &= 0, \\ d(u\,dy/ds) + \omega\,dt &= 0, \\ d(u\,dz/ds) + \Psi\,dt &= 0.\end{aligned}$$

He subsequently makes the important remark that he is "supposing the extremities be given in position", *i.e.* he is assuming the variations of the co-ordinates at the endpoints are zero. Since the terms in (11) outside the integral sign are to be evaluated at these points they will be zero and thus do not affect the inference from (11) to (12).

I have described LAGRANGE's derivation in such detail because it is the model for all subsequent problems presented in the memoir. Before turning to a critical analysis of his method let us briefly summarize what he has done. Equations (12) are simply the Newtonian equations of motion for a single particle, expressed however in terms of a potential function. In vector notation they would be written

$$\ddot{\vec{r}} + \vec{\nabla} V = 0,$$

where V is the potential (per unit mass), *i.e.* minus the work function for the system. To pass from (11) to (12) and obtain these equations LAGRANGE has invoked the so-called 'fundamental lemma of the calculus of variations',[22]

[22] This term is a modern one. H. GOLDSTINE has remarked that the fundamental lemma "is certainly not as self evident as Lagrange seemed to feel it to be." (*A History of the Calculus of Variations*, p. 288). The first rigorous proof of the lemma, due to DU-BOIS-REYMOND in 1879, is described by GOLDSTINE on pages 289–293 of his book.

allowing him to conclude that the coefficients of δx, δy, δz in the integrand of (11) are zero.

It is interesting to compare LAGRANGE'S treatment of the principle of least action here with EULER'S earlier analysis in the memoir of 1751 "Harmonie entre les principes généraux de repos et de mouvement". EULER started from the assumption that the time integral of the potential is an extremum and used this fact and the law of conservation of vis viva to establish the principle. LAGRANGE on the other hand begins with the principle and uses the conservation of vis viva to bring the variation of the potential into the action integral. In addition, while EULER'S account is general and suggestive, LAGRANGE'S method is specifically directed to the derivation of equations of motion.

Despite these differences there is an important similarity in the two approaches. Both EULER and LAGRANGE assume that time is not varied in the variational process. EULER drew explicit attention to this fact and it is implicit in all of LAGRANGE'S procedures. Both men also assume the validity of the law of conservation of vis viva; LAGRANGE further requires that the variation of the total energy be zero (used to get equation (4) above). Because LAGRANGE develops his analysis in greater detail it is possible to arrive at a more precise estimation of his method. Recall that he had assumed that the variations of the co-ordinates vanish at the endpoints. This imposes an additional restriction on the class of variations. Now it must be said that the principle of least action developed in this way has some important drawbacks. The restrictions described above when taken together sharply limit the range of possible variations. This fact, though not fatal in itself, does render problematic the last step in the derivation, the inference from (11) to (12). Furthermore this difficulty cannot be dismissed simply as a quibble over rigor for elementary examples reveal that the class of admissible δx, δy, δz is not large enough to support such an inference.[23]

The difficulty I have described in LAGRANGE'S account of the principle of least action was noticed neither by him nor his contemporaries. Indeed, his formulation became standard in French mechanics and appears in the treatises of LAZARE CARNOT, LAPLACE and POISSON.[24] In the 19^{th} century LAGRANGE'S treatment was critically evaluated by various researchers at various times; by the end

[23] Thus the point at issue here is more serious than the question of rigor raised in the preceding note. The conditions of LAGRANGE'S anlysis require that the endpoint, total energy and time all remain unvaried. Furthermore, because the variation of the potential (δV) is given by $\vec{\nabla} V \cdot \delta\vec{r}$, the varied motion must be accomplished by the introduction of frictionless constraints. In the case of a particle moving freely under the action of no forces, or a particle falling from rest under the action of gravity, these conditions imply that all comparison arcs must coincide with the path the particle actually follows. In other situations one would have to examine the particular force law under consideration and investigate the mathematical question of how extensive is the class of comparison arcs.

[24] LAPLACE'S formulation appears in the *Mécanique Céleste* V. **1** (Paris, 1799) = *Oeuvres de Laplace* **1** (Paris, 1878) pp. 23–26, 72–73. For LAZARE CARNOT see *Principes fondamentaux de l'équilibre et du mouvement* (Paris, 1803) pp. 218–221. POISSON'S discussion is contained in his *Traité de Mécanique* (Paris, 1811) V. **1** pp. 460–463, V. **2** pp. 304–306.

of the century a rather involved body of literature had accumulated.[25] Among these later developments one is of particular interest here: JACOBI's formulation in 1842 of the principle that bears his name.[26] I mentioned earlier that EULER had presented the principle of least action in two places, the first in the *Methodus Inveniendi* of 1744 and the second in the "Harmonie entre ..." of 1751. JACOBI's principle, in which the parameter time is eliminated from the analysis, is a direct generalization of the first of these treatments. LAGRANGE's principle of least action, where this parameter is retained but not varied in the δ-process, is a generalization of the second. Though there is no evidence that JACOBI was influenced by EULER we do know he departed from LAGRANGE for reasons related to the difficulty described in the previous paragraph.[27]

If one sought an explanation for the reason LAGRANGE proceeds in the way he does, it would probably be found in the very strong formalistic tendency which characteristizes all of his work in the exact sciences. In applying his δ-process to the principle of least action LAGRANGE appears to be guided more by his success in algebraic manipulation than by any physical or geometric insight.[28] Given its goal, the derivation of equations of motion, LAGRANGE'S method is indeed quite successful. In fact, from a formal point of view it is very similar to the method associated with the later variational law known as 'HAMILTON'S

[25] A partial list of those during the 19th century who concerned themselves with LAGRANGE's formulation of the principle of least action would include O. RODRIGUES. W. R. HAMILTON, C. G. J. JACOBI, M. OSTROGRADSKY, I. TODHUNTER, E. J. ROUTH, A. MAYER and P. JOURDAIN. A large number of lesser known figures also contributed to the discussion over the principle. JACOBI and OSTROGRADSKY were especially critical of LAGRANGE; TODHUNTER and JOURDAIN defended him. JOURDAIN in particular attempted at some length (mistakenly in my view) to justify LAGRANGE's treatment of the principle. Additional details with references will be found in the following two sources:

Abhandlungen über die Prinzipien der Mechanik von Lagrange, Rodrigues, Jacobi und Gauss; PHILIP E. B. JOURDAIN, Ostwald's Klassiker Nr. 167 (Leipzig, 1908).

Etude Historique sur le Principe de la Moindre Action PIERRE BRUNET (Paris, 1938).

[26] JACOBI's formulation appeared in the sixth in a series of lectures delivered at the University of Königsberg in 1842–1843. These lectures were published posthumously in *Vorlesungen über Dynamik* (Berlin, 1886). The sixth lecture appears on pages 43–51.

[27] JACOBI states that a difficulty exists in the usual textbook presentations of the principle, "even in the best, in those of Poisson, Lagrange and Laplace." (*Ibid.*, p. 44.) His solution is to eliminate at the outset the parameter time from the integral given by the principle.

[28] In particular, LAGRANGE fails to notice that the formal interchangeability of the d and the δ operators presupposes that time is an independent unvaried parameter. Thus if the examples in note 23 had occurred to him at all, it is likely he would have permitted paths traversed in different times, without realizing this creates difficulties for his analysis. In this regard his solution to the brachistochrone problem in the previous mathematical memoir is interesting. In this problem, where it is of course necessary to vary the time, LAGRANGE nevertheless appears to treat this variable as an independent parameter. He is, however, able to obtain the correct equations relating the spatial co-ordinates. This is because the integrand in the problem, $\sqrt{dx^2 + dy^2 + dz^2}/\sqrt{x}$, is such that it is immaterial—from a formal point of view—whether one adopts time as the parameter, or, as a correct procedure would require, another independent variable in terms of which time is expressed.

principle'. In terms of the steps described above there is no difference from equation (6) onward. It is surprising how much of the theory normally associated with this principle may be found in LAGRANGE's memoir. To give one example, he is able to apply his method to obtain what are in fact instances of the celebrated 'Lagrangian' equations of motion, the discovery of which is usually attributed to his work of a later date.[29] I shall document this point at greater length in Section 3 when we examine some of the subsequent problems presented in the memoir.

I take up now another issue which emerges from LAGRANGE's analysis of Problem I, an issue that may have important implications for his future approach to the foundations of mechanics. In describing the derivation of the equations of motion I skipped over the manipulations which allowed LAGRANGE to set $\delta \int (P\, dp + Q\, dq)$ equal to $P\, \delta p + Q\, \delta q$ and therefore to pass from (4) to (5). I now examine in more detail this step in his derivation. LAGRANGE notes that

$$\delta \int (P\, dp + Q\, dq) = \int \delta(P\, dp + Q\, dq) = \int (\delta P\, dp + P\, \delta dp + \delta Q\, dq + Q\, \delta dq).$$

He then interchanges the d and δ and integrates by parts to obtain

$$\delta \int (P\, dp + Q\, dq) = P\, \delta p + Q\, \delta q + \int (\delta P\, dp - dP\, \delta p + \delta Q\, dq - dQ\, \delta q).$$

It was assumed at the start that P and Q are functions of p and q respectively. Hence

$$\delta P = (dP/dp)\, \delta p, \; dP = (dP/dp)\, dp.$$

Thus $\delta P\, dp = dP\, \delta p$, and the corresponding result $\delta Q\, dq = dQ\, \delta q$ follows in the same way. Consequently $\delta \int (P\, dp + Q\, dq) = P\, \delta p + Q\, \delta q$, which is the desired equivalence.

What LAGRANGE has done here in effect is to show that $\int (P\, dp + Q\, dq)$ is a finite function of p and q whose variation equals $P\, \delta p + Q\, \delta q$. He is, however, aware that this is a special case (P, Q must be functions of p, q respectively) and that the final equations (12) are more general. He later comments at some length on this matter. Thus he shows that if we assume the forces are functions of both p and q, $P = P(p, q)$, $Q = Q(p, q)$, then the equivalence $\delta \int (P\, dp + Q\, dq) = P\, \delta p + Q\, \delta q$ reduces to the condition that $P\, dp + Q\, dq$ be an exact differential:

$$\partial P/\partial q = \partial Q/\partial p.^{30}$$

[29] Many authors write as if these equations first appear in the *Méchanique Analitique* of 1788. The procedure required to obtain them was first presented in 1764 and they were explicitly introduced in their modern form in 1780. See the discussion in Part Two Section 2.

[30] As before we have

$$\delta \int (P\, dp + Q\, dq) = P\, \delta p + Q\, \delta q + \int (\delta P\, dp - dP\, \delta p + \delta Q\, dq - dQ\, \delta q).$$

Now $dP = (\partial P/\partial p)\, dp + (\partial P/\partial q)\, dq$ and $\delta P = (\partial P/\partial p)\, \delta p + (\partial P/\partial q)\, \delta q$, with similar expressions for dQ, δQ. By substituting these values into the expression under the integral sign in the right side of the above equation and collecting terms we obtain:

$$\delta P\, dp - dP\, \delta p + \delta Q\, dq - dQ\, \delta q = (\partial P/\partial q - \partial Q/\partial p)\, dp\, \delta q + (\partial Q/\partial p - \partial P/\partial q)\, \delta p\, dq.$$

If $\partial P/\partial q - \partial Q/\partial p = 0$, this quantity will be zero. Q.e.d.

If this is not the case, LAGRANGE adds, his procedure will not lead to the true equations of motion and the variational method is inapplicable.

This difficulty is mentioned by LAGRANGE in several places and he appears to be bothered by it. In modern terminology we see that he is aware that the success of his approach depends on the existence of a potential function for the system. Thus it will not always be possible to derive the equations of motion from the condition that the variation of a definite integral is zero.[31] Though this restriction on the generality of his method is clearly undesirable, LAGRANGE emphasizes that even when the method fails we can still recover some of the results presented in the course of the derivation. He may be hinting here at the future direction of his own approach to the foundations of mechanics. We discuss this point in more detail in Part Two.

LAGRANGE concludes his study of the motion of a particle with an interesting illustration of the power of his method. He shows how it may be applied to derive the equations of motion when the particle's position is specified in cylindrical co-ordinates r, ϕ, z. The variables r and ϕ furnish the polar representation of the point (x, y) in the x-y plane and z is as usual the perpendicular distance to this plane. Rather than describe LAGRANGE's derivation I simply present the final three equations he obtains:

$$
\begin{aligned}
&d(ur^2\,d\phi/ds) + \omega\,dt = 0,\\
&d(u\,dr/ds) - ur\,d\phi^2/ds + \Pi\,dt = 0,\\
&d(u\,dz/ds) + \Psi\,dt = 0.
\end{aligned}
\tag{13}
$$

This result may be rewritten in the more familiar form

$$
\begin{aligned}
&d(r^2\dot{\phi})/dt + \partial V/\partial\phi = 0,\\
&d(\dot{r})/dt - r\dot{\phi}^2 + \partial V/\partial r = 0,\\
&d(\dot{z})/dt + \partial V/\partial z = 0,
\end{aligned}
$$

where V is the potential expressed in terms of the variables ϕ, r and z. Note that (13) are simply the 'Lagrangian' equations of motion corresponding to these variables. LAGRANGE will use these equations later in deriving the so-called "law of areas".

LAGRANGE subsequently remarks that an advantage of his method is the freedom it permits in choosing co-ordinates to describe the motion. He further shows how the method may be modified to handle cases in which the particle is constrained to move on a surface. Since these two features of his approach—ability to deal with constraints and flexibility in the choice of co-ordinates—arise again later I postpone discussion of them. LAGRANGE ends his account with some comments on how he has improved upon EULER's analysis in the *Methodus Inveniendi* (EULER had only considered motion in a plane and had used special co-ordinate systems).

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[31] It is important to note that this difficulty would remain even if we were to follow our earlier suggestion and interpret LAGRANGE's procedure in the sense of 'HAMILTON's principle'.

iii) Problem II: The General Dynamical System

With the presentation of Problem I LAGRANGE had established the basic outline of his variational method. The purpose of Problem II is first to show what modifications must be introduced into this method in the general case and second to demonstrate how the principle of least action may be used to derive other general laws of mechanics. The two laws LAGRANGE chooses to discuss were known during his time as the theorem on the center of gravity and the law of areas. These results later became known as the principles of linear and angular momentum.

LAGRANGE begins Problem II with the following characterization of an arbitrary dynamical system: it consists of "several bodies", each acted upon by forces to fixed centers, which may also be subject to "any mutual forces of attraction." In the subsequent analysis it becomes clear that there are some restrictions on these mutual forces; they must, for example, be functions of the distances which separate the bodies. In addition, it is only later that LAGRANGE explains the important role the notion of constraint plays in his conception of a dynamical system.

LAGRANGE continues by showing how his variational method leads in the general case to the equations of motion. Most of the steps in the derivation are a straightforward generalization of Problem I. LAGRANGE begins with the principle of least action

$$\delta \int \Sigma m_i u_i \, ds_i = 0, \tag{14}$$

and interchanges the δ and $\int$ to obtain

$$\int [\Sigma m_i u_i \, \delta ds_i + \Sigma m_i u_i \, \delta u_i \, dt] = 0. \tag{15}$$

The quantity $\Sigma m_i u_i \, \delta ds_i$ is expressed in terms of Cartesian co-ordinates and substituted into (15) to yield

$$\int [\Sigma m_i(\ddot{x}_i \, \delta x_i + \ddot{y}_i \, \delta y_i + \ddot{z}_i \, \delta z_i) + \Sigma m_i u_i \, \delta u_i] = 0, \tag{16}$$

where it is assumed the variations of the co-ordinates are zero at the initial and final configurations. The expression $\Sigma m_i u_i \, \delta u_i$ is now related to the forces in the system by means of the law of conservation of vis viva. It is here that the derivation differs from LAGRANGE's earlier analysis of a single particle. Thus, in addition to the external forces to fixed centers, we must also consider the mutual actions of the particles in the system. I shall describe LAGRANGE's procedure for the case of two masses M and M'. Conservation of vis viva is laid down in the form

$$\begin{aligned} Mu^2 + M'u'^2 = \text{const} - 2M \int (P \, dp + Q \, dq + \ldots) \\ - 2M' \int (P' \, dp' + Q' \, dq' + \ldots) - 2MM' \int F \, df. \end{aligned} \tag{17}$$

The appearance of the last term on the right side of (17) is a consequence of the interaction of M and M'. $MM'F$ denotes the magnitude of the force arising from this interaction (LAGRANGE is tactily invoking the equality of action and reaction) and f denotes the distance between M and M'. Let us take the variation of each side of (17):

$$(18)\qquad Mu\,\delta u + M'u'\,\delta u' = -PM\,\delta p - MQ\,\delta q - \ldots - M'P'\,\delta p' - M'Q'\,\delta q' - \ldots - 2MM'F\,\delta f.$$

In the general case, where more than two particles are present, there will be additional terms in (17) and (18) corresponding to the additional particles. These equations will hereafter be understood to refer to this case.

By combining (16) and (18) we would obtain a single equation relating the accelerations and the forces. LAGRANGE does not do this, apparently because of the notational difficulty in expressing (18) in a suitable general form. Instead he describes verbally the remaining steps of the analysis. If the particles are supposed completely free, we express the right side of (18) in terms of the variations δx_i, δy_i, δz_i, substitute into (16), and in the equation thus formed set equal to zero the coefficients of the variations. If there are constraints in the system (rigid rods, inextensible strings, fixed surfaces, *etc.*) we use the relations that describe these constraints to reduce the variations δx_i, δy_i, δz_i to a smaller set that may be independently varied. The coefficients of the reduced variations are then equated to zero to yield the equations of motion. The advantage of this approach is of course that it obviates the need to consider the forces of constraint.[32]

LAGRANGE proceeds to derive the theorem on the center of gravity for the case where the system is completely free.[33] He assumes that three external forces parallel to the x, y and z axes act on the system. If P, Q and R denote the accelerations per unit mass arising from these forces, then the external force acting on a mass m will equal (mP, mQ, mR). LAGRANGE now chooses a given particle, call it $m_1 = (x_1, y_1, z_1)$, and expresses the co-ordinates of each particle m_i $(i > 1)$ as follows:

$$x_i = x_1 + x_i',\ y_i = y_1 + y_i',\ z_i = z_1 + z_i'.$$

The purpose of this device is to obtain a description of the system which depends only on the relative positions of the particles. In particular, we may assume that

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[32] It is important to note that the constraints may be of two types. First, there will be connections among the particles (rigid rods *etc.*) which give rise to mutual forces. We then have the choice of i) retaining these forces in the analysis, in which case the variations are independent, or ii) using the conditions imposed by the connections to eliminate some of the co-ordinates and thereby reduce the number of independent variations, in which case the forces of constraint are eliminated from the analysis. There is, however, another type of constraint, one that is exterior to the system; an example would be when some of the particles are constrained to move on fixed surfaces. The forces arising from these constraints have no place in LAGRANGE's conception of a dynamical system. Thus it would be necessary to follow the second course described above, in which these forces do not appear.

[33] The words "completely free" are intended here to exclude the second type of constraint described in the preceding note. Mutual forces of constraint are permitted. However, they must be retained in the analysis and the variations δx_i, δy_i, δz_i will be independent.

their mutual actions depend on x_i', y_i', z_i' and not on x_1, y_1, z_1. We now take the variations of the previous decompositions:

$$\delta x_i = \delta x_1 + \delta x_i', \quad \delta y_i = \delta y_1 + \delta y_i', \quad \delta z_i = \delta z_1 + \delta z_i'.$$

Because the system is free we are permitted to assume the variations δx_1, δy_1, δz_1 are independent of $\delta x_i'$, $\delta y_i'$, $\delta z_i'$. Consider the terms of equation (18) corresponding to the mutual forces. Clearly these terms will involve only the variations $\delta x_i', \delta y_i', \delta z_i'$. The external forces in turn give rise to the following expressions:

$$m_i P\, \delta x_i + m_i Q\, \delta y_i + m_i R\, \delta z_i$$

or

$$m_1 P\, \delta x_1 + m_1 Q\, \delta y_1 + m_1 R\, \delta z_1 + m_i P\, \delta x_i' + m_i Q\, \delta y_i' + m_i R\, \delta z_i'.$$

We proceed to substitute these expressions into (16) and form the coefficients of δx_1, δy_1, δz_1. Because the latter are independent their coefficients are equal to zero:

$$\begin{aligned} &\Sigma m_i \ddot{x}_i - (\Sigma m_i)\, P = 0, \\ (19) \qquad &\Sigma m_i \ddot{y}_i - (\Sigma m_i)\, Q = 0, \\ &\Sigma m_i \ddot{z}_i - (\Sigma m_i)\, R = 0. \end{aligned}$$

Equations (19) express the desired theorem: the center of gravity of the system moves as would a single (unit) mass acted upon by the three forces P, Q and R. This result is one form of the principle of linear momentum.

LAGRANGE turns now to a demonstration of the law of areas. He first specifies the position of each particle using cylindrical co-ordinates ϕ_i, r_i, z_i. By beginning with equation (15) above and expressing the quantity $\Sigma m_i u_i\, \delta ds_i$ in terms of these co-ordinates we obtain

$$(16') \qquad \int [\Sigma m_i \{(d(r_i^2 \dot{\phi}_i)/dt)\, \delta\phi_i + (\ddot{r}_i - r_i \dot{\phi}_i^2)\, \delta r + \ddot{z}_i\, \delta z_i\} + \Sigma m_i u_i\, \delta u_i] = 0,$$

where it is assumed the variations $\delta\phi_i$, δr_i, δz_i are zero at the initial and final configurations. The right side of equation (18) must now be expressed in terms of ϕ_i, r_i, z_i and their variations; by substituting for $\Sigma m_i u_i\, \delta u_i$ into (16′) we would obtain as before a single equation.

LAGRANGE now supposes the system is completely free or contains a fixed point about which it is free to turn. If completely free, we take any point in space as the origin of our co-ordinate system; if there is a fixed point, we take the origin to coincide with this point. In either case LAGRANGE assumes the external forces "converge" at the origin, *i.e.* have directions which pass through it. This given, he isolates for special attention the angle ϕ_1 corresponding to the mass m_1 and expresses the remaining angles ϕ_i $(i > 1)$ as follows:

$$\phi_i = \phi_1 + \phi_i',$$

so that

$$\delta\phi_i = \delta\phi_1 + \delta\phi_i'.$$

We may assume: i) ϕ_1 and ϕ_i' are independent; ii) the terms of equation (18) arising from the mutual forces do not contain ϕ_1 or $\delta\phi_1$. Furthermore, though this is neither mentioned nor explained by LAGRANGE, because the external forces pass through the origin the terms in (18) corresponding to these forces involve only the variations δr_i and δz_i. Thus there are no terms in (18) containing $\delta\phi_1$. We may therefore take the coefficient in (16′) of this variation and set it equal to zero:

$$\Sigma m_i(d(r_i^2\dot{\phi}_i)/dt) = 0$$

or

$$d(\Sigma m_i r_i^2 \dot{\phi}_i)/dt = 0. \tag{20}$$

Equation (20) is integrated twice with respect to time to yield the final result, the so-called law of areas:

$$\int \Sigma m_i r_i^2\, d\phi = Ht, \quad H = \text{constant}. \tag{21}$$

LAGRANGE describes this result verbally as follows: the sum of the masses m_i multiplied by the areas swept out in a given time by the radii r_i is directly proportional to time. He refers to earlier versions of the theorem which had appeared in the writings of D'ARCY, DANIEL BERNOULLI and EULER. While today we see that (21), or more directly (20), is equivalent to the principle of angular momentum,[34] LAGRANGE himself was apparently unaware of its full generality. Thus he had unnecessarily restricted its applications to the case where each of the external forces passes through the origin.[35]

Despite the fact that LAGRANGE devotes some space to demonstrating the theorem on the center of gravity and the law of areas he nevertheless makes no use of these results in any of the subsequent problems of the memoir. The presentation of these laws here is apparently intended to provide evidence for the deductive power of his variational method, as further support for his claim to have established in the principle of least action a general foundation of dynamics.

Section 3: Some Applications

The remainder of the memoir consists of an application of the theory developed in Problems I and II to a wide range of particular dynamical systems. LAGRANGE investigates the interaction through gravity of freely moving particles, particles joined by strings and strings loaded with arbitrarily many masses. In the later sections he examines the behavior of continuous media, flexible bodies and questions in fluid mechanics. In each case he begins with the principle of least action and applies his variational method to obtain the differential equations of motion which describe the system.

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[34] The quantity $\Sigma m_i r_i^2 \dot{\phi}_i$ is (except for a change of sign) the z component of the angular momentum vector. Because the external forces pass through the origin their torque is zero. Equation (20) states that in this situation $\Sigma m_i r_i^2 \dot{\phi}_i$ is constant in time. By symmetry we would obtain similar results for the x and y components of the angular momentum vector.

[35] This and other criticisms of LAGRANGE'S formulation appear in C. TRUESDELL'S "Whence the law of moment of momentum?" *Essays in the History of Mechanics* (Springer-Verlag, 1968) pp. 239–271.

LAGRANGE seldom refers to the work of his contemporaries and he only rarely comments on the importance of his own research. However, many of the problems presented here were developed again later in the *Méchanique Analitique* of 1788. At the conclusion of Section Five, Part Two of that treatise LAGRANGE describes the significance of what he has accomplished:

> These different examples comprise nearly all the problems on the motion of a body or a system of bodies that the Geometers have solved; we have chosen them on purpose so that one may better judge the advantages of our method, by comparing our solutions with those found in the works of Messr. Euler, Clairaut, d'Alembert etc., in which one arrives at the differential equations only by reasonings, constructions and analyses often rather long and complicated. The uniformity and the swiftness of the course of [our] method are what should principally distinguish it from all the others, and what we wished especially to show in these applications.[36]

To be sure, LAGRANGE is referring here, not to the principle of least action, but rather to a later method, one that we shall examine in Part Two. Nevertheless, the passage cited is a quite accurate summary of the results presented in the memoir of 1760. Thus the variational method of this memoir also has the virtue of being "uniform and swift" and leads for problems of constrained mass point dynamics to the same 'Lagrangian' equations of motion. The account which follows will document this assertion in greater detail as well as focus on some aspects of LAGRANGE'S research that are of particular interest for his future work on foundations.

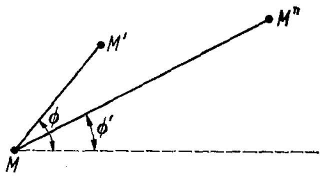
In Problems IV and V LAGRANGE takes up examples which had appeared in JEAN D'ALEMBERT'S *Traité de Dynamique* of 1743; the problems are even numbered in the same way in the two treatises.[37] I shall concentrate on Problem IV, which concerns a system consisting of a body joined in a plane to two other bodies by inextensible strings. It is required to find the motion of the system when forces act on the first body. This example is typical of those which appear both in D'ALEMBERT'S *Traité* as well as in a similar treatise composed by CLAIRAUT in 1742.[38] The analysis of such problems was presented by the leading geometers of the period not so much to display new results (though this did happen) as to illustrate the application to dynamics of new principles and methods. In D'ALEMBERT'S case the solution to Problem IV was quite intricate, involving a complicated

[36] *Mécanique Analytique* (1788) pp. 336–337.

[37] D'ALEMBERT *Traité de Dynamique* (1743) pp. 86–119; (1758) pp. 126–183.

[38] CLAIRAUT "Sur quelques principes qui donnent la solution d'un grand nombre de problèmes de dynamique" *Mémoires de Mathématique et de Physique de l'Académie Royale des Sciences de Paris de l'Année* 1742 (1745). Although there is considerable overlap between this memoir and D'ALEMBERT'S *Traité*, CLAIRAUT does not actually treat Problem IV. The latter is nevertheless representative of the problens which do appear in the memoir. CLAIRAUT states that these problems "have nearly all been proposed by the savants Messr. Bernoulli and Euler" (p. 213).

application of his difficult and famous principle.[39] In contrast, LAGRANGE'S solution is a model of simplicity. He first specifies the position of the three bodies M, M', M'' using four parameters: the x and y co-ordinates of M and two variables ϕ, ϕ' that fix the angles between a given reference line and the strings joining M to M' and M to M'' (see diagram). He then applies his variational method and derives four equations of motion corresponding to the four parameters. I shall not present these relations; they are in fact what one would obtain by writing down the 'Lagrangian' equations for the the system.[40] Of greater interest than this particular result is the fact that his method would lead to such equations in any other problem involving the constrained motion of particles.[41] Although LAGRANGE does not compare his solution with those of his contemporaries, he must have been impressed by the superiority of his own method in dealing with this type of problem.



$M = (x, y)$ (Note: No Diagram in the Original.)

Fig. 1

LAGRANGE continues in Problem IV by examining several modifications in the system: M is constrained to move on a fixed curve, M is replaced by a ring through which the string is free to slide, the strings are replaced by rigid rods, *etc.* He then turns to Problem V and generalizes D'ALEMBERT'S earlier analysis in the *Traité* of the small oscillations of a heavy hanging cord. D'ALEMBERT had arrived at important results in his investigation of this problem; LAGRANGE

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[39] I discuss D'ALEMBERT'S principle in Part Two Section 1.

[40] LAGRANGE starts with the following relation:

$$\int \{\Sigma m_i v_i \, \delta ds_i + \Sigma m_i v_i \, \delta v_i \, dt\} = 0. \tag{15}$$

He then calculates ds_i in terms of the four variables x, y, ϕ, ϕ' and transforms the quantity $\Sigma m_i v_i \, \delta ds_i$ using his variational method. This step in the derivation corresponds in modern analysis to the calculation of $d(\partial T/\partial \dot{q}_i)/dt - \partial T/\partial q_i$. The quantity $\Sigma m_i v_i \, \delta v_i \, dt$ is now related to the work or potential function by means of the law of conservation of vis viva. This step corresponds to the calculation of $\partial V/\partial q_i$. The two steps are then combined to yield the final equations of motion. The entire procedure is similar to the derivation of the Lagrangian equations from HAMILTON'S principle, with the following difference: LAGRANGE does not begin with the integral $\int (T - V) \, dt$ and therefore needs the equation of energy to bring the variation of the potential into the integrand.

[41] Thus Problem IV illustrates the two major strengths of LAGRANGE'S variational method: its ability to deal with constraints and the freedom it permits in the choice of co-ordinates to desribe the system.

in turn is successfully able to extend these results in ways worthy of note.[42] Though he makes no reference to D'ALEMBERT, his treatment is once again from the viewpoint of clarity and elegance a considerable advance.

We move now to LAGRANGE'S analysis in Problem VIII of the general motion of a rigid body. This subject would occupy a significant place in his future research and it is worthwhile to summarize briefly the context of these early researches. Interest in rigid bodies during this period stemmed first from their intrinsic value as an idealized description of an important class of physical objects. More specifically, a theory of rigid bodies was becoming increasingly necessary in celestial mechanics to analyze effects (precession, nutation) that could not be treated by the methods of traditional mass point dynamics. Although D'ALEMBERT had used results about rigid bodies in his work in theoretical astronomy, the true foundations of the subject were laid by EULER in three important memoirs submitted to the Berlin Academy between 1750 and 1758.[43] In the first of these memoirs EULER lays down the principle of linear momentum as his fundamental dynamical axiom and derives for a general rigid body the 'EULER' equations of motion relative to space fixed axes. In the later memoirs he introduces the concepts and terminology that have since become standard, the notions of principal axis, moment of inertia, Eulerian angle, *etc.*, and obtains the equations of motion with respect to reference axes fixed in the rigid body.

In Problem VIII LAGRANGE develops using his variational method much of the theory that had appeared in the first of EULER'S three memoirs. I shall outline his solution for the case where the body is free to move about a point O located at its center of gravity. Given a Cartesian reference frame fixed in space with origin at O, the variation of the position of each mass element dm is expressed in terms of its Cartesian co-ordinates and three infinitesimal angles representing

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[42] We outline LAGRANGE'S solution and its relation to D'ALEMBERT'S earlier research. LAGRANGE begins by setting up the differential equations for the motion of n bodies attached to a string fixed at one end. He then turns to the case in which the string is vertical, the bodies are acted upon by gravity, and the oscillations are small and smooth. The equations he obtains, when there are only two bodies and the motion is planar, are equivalent to those contained in § 98 of D'ALEMBERT'S *Traité* (1743, p. 97). LAGRANGE proceeds to the passage to the limit and derives the partial differential equations of the heavy hanging cord. The two dimensional form of the latter, which reduce to a single equation, appears in § 110 of the *Traité* (1743, p. 117); its derivation by D'ALEMBERT had constituted one of his most important achievements. Indeed, C. TRUESDELL has stated that the appearance of this equation marks "a turning point in the whole history of mechanics: *the first general statement of the law of motion of a continuous medium*." (See TRUESDELL'S "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788" in EULER'S *Opera Omnia* II **11**^{2} p. 187.) LAGRANGE, unlike D'ALEMBERT, extends his analysis to the case in which the cord is elastic. TRUESDELL comments on the significance of this last result in the previously cited study (p. 373).

[43] EULER "Découverte d'un nouveau principe de mécanique" *Mémoires de l'académie des sciences de Berlin* **6** 1750 (1752) pp. 185–217 = *Opera Omnia* II **5** pp. 81–108.

"Recherches sur la connoissance mécanique des corps" *Mémoires de ... Berlin* **14** 1758 (1765) pp. 131–153 = *Opera Omnia* II **8** pp. 178–199.

"Du mouvement de rotation des corps solides autour d'un axe variable" *Mémoires de ... Berlin* **14** (1765) pp. 154–193 = *Opera Omnia* II **8** pp. 200–235.

infinitesimal rotations about the x, y and z axes. LAGRANGE uses the symbol "S" to denote summation over the mass elements that make up this body and eventually obtains by his method three equations (the 'EULER' equations) corresponding to the three angles of rotation. The values for the moments and product of inertia in these relations appear as unnamed analytical quantities; furthermore, because the reference co-ordinate system is fixed in space these quantities are functions of time.

LAGRANGE proceeds to prove the following result. If we assume the rotation of the body occurs only about one of the axes, say the x-axis, and if the moment of the external forces is zero, then: i) the angular velocity of rotation is constant; ii) the products of inertia $Sxy\,dm$ and $Sxz\,dm$ vanish. Having demonstrated this result, he adds that conversely we may use the conditions $Sxy\,dm = 0$, $Sxz\,dm = 0$ to determine an "axis of rotation", *i.e.* one about which the body will turn with constant velocity in the absence of any external moment. Unfortunately, no details are provided on the steps necessary to arrive at such a determination. Indeed, the remaining theory developed by LAGRANGE is rather incomplete and (as we shall see later) would undergo considerable elaboration in his future treatises.

With this survey of LAGRANGE's research on rigid body analysis I conclude our discussion of the memoir. It would be in a treatise on celestial mechanics devoted to a problem in this subject that he would initiate the next stage in his approach to the principles and methods of mechanics. Before turning to an examination of these later developments I briefly summarize the discussion thus far. The young LAGRANGE, influenced by EULER and motivated by his own successful research in the calculus of variations, had attempted to establish in the principle of least action a general foundation for dynamics. Though his treatment of the principle is susceptible to criticism and though he himself was aware of certain of its limitations, it nevertheless led him to a powerful method for solving problems of concern to the geometers of the period. Despite this promising beginning LAGRANGE would abandon least action as the basis of his approach. For details on why this change occurred and discussion of his new method we move now to Part Two.

Part Two: The General Principle of Virtual Velocities (1764)

Section 1: Background

In 1762 the Paris Academy of Sciences set as its prize competition for the year 1764 the following subject:

> If one can explain by some physical reason why the Moon always presents to us nearly the same face; and how one can determine by observation and theory if the axis of this planet is subject to some proper motion, similar to that which we distinguish in the axis of the Earth, and which produces the precession of the equinoxes and nutation.[44]

[44] This statement is taken from the introduction to LAGRANGE's winning submission. See note 46.

By choosing this subject the Academy was demanding a physical analysis of the astronomical phenomenon of lunar libration: the apparent and real irregularities in the motion of the Moon which cause it to reveal to us over the course of time slightly more than one half of its surface. The theory of libration had attracted the attention of NEWTON, CASSINI and D'ALEMBERT, with no conclusive results forthcoming, a state of affairs which led to the Paris competition.[45] LAGRANGE composed a memoir on the subject, submitted it to the Academy in August of 1763, and was subsequently awarded the prize. His winning submission was titled "Recherches sur la libration de la Lune dans lesquelles on tâche de résoudre la Question proposée par l'Académie royale des Sciences, pour le Prix de l'année 1764."[46] It marks one of the first in an extensive series of investigations in celestial mechanics which would occupy LAGRANGE for the remainder of his life.

The basis of LAGRANGE's memoir consists of a joining together of two mechanical principles: the statical rule of virtual velocities and the dynamical axiom known as 'D'ALEMBERT'S principle'. I shall hereafter refer to the union of these two laws as the 'general principle of virtual velocities'. This principle would become the fundamental axiom of LAGRANGE'S *Méchanique Analitique* of 1788; its appearance in the piece of 1764 on libration marks a turning point in his approach to the foundations of mechanics.

LAGRANGE does not in any of his published treatises discuss the circumstances which led him to adopt a new mechanical method to analyze lunar libration. However, one clue to his changing direction appears in a letter to EULER, dated 24 November 1759, where he writes:

> I have myself composed some elements of mechanics and of the differential and integral calculus for the use of my students, and I believe I have developed the true metaphysics of their principles, as much as this is possible.[47]

In earlier correspondence LAGRANGE had informed EULER that he had substantially completed his memoir on least action. The announcement contained in the passage quoted would therefore appear to refer to something new, something developed in connection with his teaching duties at the Royal Artillery School of Turin. Certainly by 1763 LAGRANGE had thoroughly worked out the basis of his new approach and he took the opportunity provided by the Paris competition to display his latest results to the French scientific community. Thus although his memoir is devoted to a problem in celestial mechanics it also contains much additional detail on the theoretical workings of his new method. To give one example, he provides in outline form the procedure required to obtain the 'Lagrangian' equations of motion, thereby anticipating by some twenty-five years his presentation in the *Méchanique Analitique*.

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[45] For a survey of 18th century research on lunar libration see ROBERT GRANT'S *History of Physical Astronomy* (London, 1852) pp. 72–76.

[46] The memoir was received by the Paris Academy on 9 August 1763 and was published in 1777 in *Prix de l'Académie royale des Sciences de Paris tome IX* 1764 = *Oeuvres de Lagrange* **6** (1878) pp. 5–61. The delay in publication has obvious implications for the dissemination of LAGRANGE's ideas, a topic which we do not consider in this article.

[47] *Oeuvres de Lagrange* **14** (1892) p. 173.

In Section 3 I discuss in more detail the reasons for LAGRANGE's shift from least action to the general principle of virtual velocities. Of greater significance for the immediate background to the memoir is the emergence of JEAN D'ALEMBERT as a new and important influence on LAGRANGE. In his earliest work on the principles and methods of mechanics LAGRANGE appears to have been guided almost exclusively by the writings of EULER. The only ones during this period who could be said to rival the latter as innovators in dynamical theory were DANIEL BERNOULLI, CLAIRAUT and D'ALEMBERT; it was to this last savant that LAGRANGE would turn for inspiration in setting down the foundations presented in the prize memoir. Although D'ALEMBERT would later assume a personal role in assisting the career of his younger contemporary, his influence was by all accounts at this time of a general intellectual character.[48]

The importance of the French geometer will become evident in the emphasis of the following account, where we examine the earlier history of the principle of virtual velocities and of 'D'ALEMBERT'S principle'. Rather than attempt a detailed study of this subject I shall simply describe the relevant sections of those treatises that LAGRANGE actually cites in his prize memoir. Consideration of these writings will therefore serve both as an introduction to the place of the two principles in the 18th century mechanics as well as a brief survey of the immediate theoretical background to LAGRANGE'S own research.

The concluding sections of EULER'S "Harmonie entre les principes généraux de repos et de mouvement" (1751) contain a clear and succinct presentation of many of the ideas involved in virtual work arguments.[49] EULER wishes to show that the equilibrium properties of simple machines are a consequence of MAUPERTUIS's law of rest. In each case he uses the fact that the potential or "effort" is an extremum to obtain a condition for equilibrium. This condition is then expressed in terms of the infinitesimal displacements experienced by the bodies of the the system when the latter is subjected to a small disturbance about its equilibrium state. I shall illustrate EULER'S procedure by describing his analysis of the inclined plane. In the diagram the body O lies on the inclined plane EFG and is supported by a force at B. Let B denote the magnitude of this force and A the weight of O. $OA = x$ designates the vertical distance from O to the base FG; $OB = y$ is its distance to B. In this situation EULER states that

[48] The correspondence between LAGRANGE and D'ALEMBERT presented in the former's *Oeuvres 13* did not begin in earnest until 1764. The letters prior to this date are few and formal and contain nothing of interest for our subject. At the end of his life LAGRANGE reportedly stated that D'ALEMBERT and EULER were (in that order) the most important early influences on him. However, because of the difficulty of D'ALEMBERT'S treatises, he strongly advised anyone then taking up the study of mathematics to turn to EULER: "quand on voulait être géomètre, il fallait étudier Euler." (These remarks are attributed to LAGRANGE in a letter that appeared in the *Moniteur Universalis* (1814) pp. 226–228. The letter, which is identified only by the initials L.B.M.D.G., is a commentary on DELAMBRE'S éloge of LAGRANGE. I have relied on excerpts from it reprinted in an Essay Review by I. GRATTAN-GUINESS: "Recent researches in French mathematical physics of the early 19th century", *Annals of Science* **38** (1981) pp. 663–690, p. 679.)

[49] EULER *Opera Omnia* II **5** (1957) pp. 168–172.

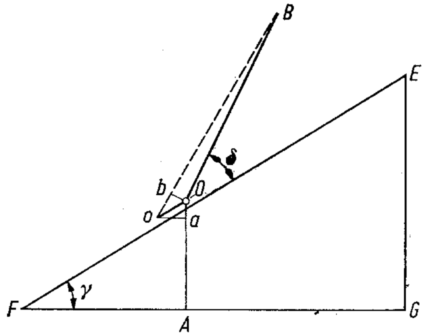

Fig. 2. EULER's Fig. 10 (1751)

the "effort" will equal $Ax + By$. The condition that it be an extremum is therefore $A\,dx + B\,dy = 0$. His analysis continues as follows:

> Let the body O change infinitely little its position on the inclined plane and let it arrive at o, having advanced the space $Oo = ds$. Draw from the point o onto OA the perpendicular oa, and from O onto Bo the perpendicular Ob; after this change it is clear that $Oa = -dx$ and $ob = dy$. Now because the angle $Ooa = \gamma$ we will have $Oa = ds\,(\sin\gamma)$, and the angle $Oob = EOB = \delta$ will give $ob = ds\,(\cos\delta)$, so that $dx = -ds\,(\sin\gamma)$ and $dy = ds\,(\cos\delta)$. Then in the state of equilibrium it is necessary that $-A\,ds\,(\sin\gamma) + B\,ds\,(\cos\delta) = 0$ or $A\,(\sin\gamma) = B\,(\cos\delta)$; or that the force Ob will be to the weight O as the sine of the elevation of the inclined plane is to the cosine of the angle EOB which the direction of the force OB makes with the inclined plane, and this same proposition is derivable from the ordinary principles of statics.[50]

From the viewpoint of the history of the principle of virtual velocities EULER's approach is somewhat atypical in that he begins with an expression for the potential and uses as his starting point the condition that this function be an extremum. A more traditional statement of the principle appears in the last chapter (Chapter IV) of D'ALEMBERT'S *Traité de Dynamique* (1743, 1758). D'ALEMBERT is attempting here to furnish a demonstration of the law of conservation of vis viva. In a general scholium at the end of the chapter he summarizes his main result:

> It follows from all we have said until now that in general the conservation of vis viva depends on this principle, that when the powers [puissances] are in equilibrium, the velocities of the points where they are applied estimated in the directions of these powers are in inverse ratio to these same powers. This principle has long been recognized by geometers as the fundamental principle of equilibrium; but no one that I know has yet demonstrated the principle or shown that the conservation of vis viva necessarily results from it.[51]

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[50] *Ibid.* pp. 170–171.

[51] *Traité de Dynamique* (1743, pp. 182–183), (1758, p. 267).

(It is worth emphasizing here that D'ALEMBERT attaches no specific name to the principle and that the phrase "virtual velocity" only received widespread acceptance through the later work of LAGRANGE.) The formulation of the principle contained in this passage finds a simple illustration in the previous example of the inclined plane. Assume the small displacement ds experienced by O results from a hypothetical velocity $u = ds/dt$ imparted to O. EULER's result

$$-A\,ds\,(\sin\gamma) + B\,ds\,(\cos\delta) = 0$$

may be rewritten

$$u\,(\sin\gamma) : u\,(\cos\delta) = B : A,$$

which for this particular example is the analytical formulation of D'ALEMBERT'S statement of the principle. In the general case, a system of particles in equilibrium is assumed subjected to a small disturbance, so that each mass m_i of the system experiences an infinitesimal displacement ds_i. If $\vec{u}_i$ is the velocity corresponding to this displacement and $\vec{F}_i$ is the sum of the forces acting on m_i, the general statement of the principle becomes

$$\Sigma \vec{F}_i \cdot \vec{u}_i = 0.$$

Note that the system will normally be constrained (as is the case in all the examples D'ALEMBERT considers) and that the 'virtual velocities' $\vec{u}_i$ must then be compatible with the constraints; in this situation $\vec{F}_i$ refers only to the applied (non-constraint) forces acting in m_i.

D'ALEMBERT'S use of the principle of virtual velocities to demonstrate the conservation of vis viva is of direct interest for LAGRANGE'S memoir of 1764. Indeed, as we shall see in the next section, LAGRANGE takes up this demonstration and gives it a simple analytical formulation. Let us now, however, turn to the dynamical axiom known as 'D'ALEMBERT'S principle'. Interpretation of this principle is, in contrast to the previous statical law, surprisingly difficult.[52] Our discussion, while remaining faithful to D'ALEMBERT'S original formulation, shall emphasize only those aspects of his treatment that are necessary for an appreciation of LAGRANGE'S later work.

In the *Traité de Dynamique* D'ALEMBERT presents the following general rule for determining the motion of a system of bodies. At a given instant we assume the "motions" $a, b, c, \ldots$ are "impressed" on the bodies $A, B, C, \ldots$. Because of the mutual actions in the system, $A, B, C, \ldots$ actually follow the motions $\bar{a}, \bar{b}, \bar{c}, \ldots$. We proceed to form the decompositions $a = \bar{a} + \alpha$, $b = \bar{b} + \beta$,

[52] As a result of the work in recent decades of C. TRUESDELL and I. SZABÒ much of the confusion surrounding D'ALEMBERT'S principle has been removed (see TRUESDELL'S "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788" in EULER'S *Opera Omnia* II **11**^{2} (1960) pp. 186–192, and SZABÒ'S *Geschichte der mechanischen Prinzipien* (Basel, 1977) pp. 31–43). However, even the account of these authors is, in the opinion of this writer, in need of qualification with respect to D'ALEMBERT'S original formulation and use of his principle in the *Traité*. Further discussion of this point may be found in my Ph. D. dissertation *The Approach of Jean d'Alembert and Lazare Carnot to the Theory of a Constrained Dynamical System* (University of Toronto, 1981), Chapters One and Two.

$c = \bar{c} + \gamma, \ldots,$ in which $\alpha, \beta, \gamma, \ldots$ designate the "lost motions". "D'ALEMBERT'S principle" asserts that the lost motions, if applied alone, produce equilibrium in the system. His general rule for solving all problems of dynamics consists in this:

> Decompose the motions $a, b, c, \ldots$ impressed on each body into two others $\bar{a}, \alpha; \bar{b}, \beta; \bar{c}, \gamma, \ldots,$ which are such that if the motions $\bar{a}, \bar{b}, \bar{c}, \ldots$ were impressed alone on the bodies they would retain these motions without interfering with each other; and that if the motions $\alpha, \beta, \gamma, \ldots$ were impressed alone the system would remain at rest; it is clear that $\bar{a}, \bar{b}, \bar{c}, \ldots$ will be the motions the bodies will take by virtue of their action.[53]

Such is the original statement of D'ALEMBERT'S celebrated principle. I now describe how it is interpreted in the large collection of problems that make up Chapter III of the *Traité*. First, the term "body" must be understood to mean point mass or corpuscle and "mutual action" refers only to changes in motion arising from the constraints in the system. This given, let v denote the velocity of the body A at time t. In the next instant dt the forces acting on A would, if this body were free and unconstrained, impart to it the increment of velocity $dv^{(I)}$. The quantity $v + dv^{(I)}$ then represents the "impressed motion": $a = v + dv^{(I)}$. Because A is connected to the other bodies, it actually acquires during dt the increment dv and the "actual velocity" at the end of this instant equals $v + dv$: $\bar{a} = v + dv$. Hence the decomposition $a = \bar{a} + \alpha$ becomes

$$(v + dv^{(I)}) = (v + dv) + dv^{(L)}, \tag{*}$$

where $\alpha = dv^{(L)}$ designates the motion "lost" to the constraints. Similar decompositions will obtain for the rest of the bodies in the system. D'ALEMBERT'S principle asserts that the quantities $dv^{(L)}$, if applied alone to each of these bodies, produce equilibrium. In any given problem the principle is used to derive differential equations of motion. We begin with an appropriate statical law and obtain a relation among the $m\,dv^{(L)}$ (m = mass). This fact combined with decomposition (*) eventually leads to a relation among $m\,dv^{(I)}$ and $m\,dv$. A knowledge of the forces is assumed given and $m\,dv^{(I)}$ will therefore be a determinate function of whatever co-ordinates are used to describe the system. By expressing dv in terms of the differentials of these co-ordinates we obtain as the final result the equations of motion.

By its nature the previous account cannot convey the intricate character of D'ALEMBERT'S solutions to the individual problems presented in the *Traité*.[54] That he himself was aware of difficulties in the principle's original statement is evidenced by the fact that in his later treatises on hydromechanics and theoretical

[53] *Traité* (1743, p. 51) (1758, pp. 74–75).

[54] Two circumstances contribute to the intricacy of these solutions. First, the entities appearing in decomposition (*) are represented geometrically by line segments. Second, in many problems D'ALEMBERT does not hold dt constant (that is, time is not the independent variable); thus, for example, the line segments which designate dv do not represent increments of velocity acquired in equal increments of time. A more detailed discussion of this subject is contained in the reference cited at the end of note 52.

astronomy he provides a somewhat simplified version of it.[55] Assume once again that A is an arbitrary body in a given system. Let v and v^+ denote the velocities of A at times t and $t + dt$ respectively. D'ALEMBERT forms the relation

$$v = v^+ + u,$$

which we shall write as

$$v = v^+ - dv.$$

The velocity of A at time $t + dt$, if this body were free and unconstrained, would be $v + dv^{(I)}$ (the "impressed motion"), or, using the previous relation, $v^+ - dv + dv^{(I)}$. By assumption the actual velocity of A at time $t + dt$ is v^+. Hence $-dv + dv^{(I)}$ applied respectively to each body in the system must be such as to produce equilibrium. (D'ALEMBERT regards this argument as a "demonstration" of his principle.[56]) In any given problem we now invoke as before an appropriate statical law and obtain a relation among $-m\,dv$ and $m\,dv^{(I)}$. By focusing directly on the latter two quantities D'ALEMBERT has avoided the explicit decomposition that so complicated the original statement of his principle.[57] Although its application to particular problems remains difficult, the formulation described here is clearer and is the one LAGRANGE would later develop and present as "D'ALEMBERT'S principle".

Section 2: The Basic Theory

LAGRANGE'S prize memoir may be divided into two parts. The first part is devoted to establishing the equations which describe the motion of the Moon about its center of gravity when subject to the gravitational action of the Earth and the Sun. It is here that he outlines in a series of scholia and remarks the basis of his new method. The second part is devoted exclusively to the astronomical problem of lunar libration. Given certain assumptions about the figure of the Moon LAGRANGE is able to provide a satisfactory explanation for why the same side of the Moon always faces the Earth.

In this section we focus on the theory developed in the first part of the memoir. LAGRANGE uses his formulation of the astronomical problem to illustrate his method and it will be necessary as background to consider some of these astronomical details. An understanding of his explanation of libration is not required in what follows; for completeness and because of its considerable intrinsic interest, I summarize LAGRANGE'S main result in an appendix.

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[55] *Traité de l'Équilibre et du Mouvement des Fluides* (Paris, 1744) pp. 70–71.

Recherches sur la Précession des Equinoxes, et sur la Nutation de l'Axe de la Terre dans le Systême Newtonien (Paris, 1749) pp. 35–36.

Essai d'une Nouvelle Théorie de la Résistance des Fluides (Paris, 1752) pp. 1–3.

[56] For a critique of the idea underlying this sort of "proof" see G. HAMEL'S *Theoretische Mechanik* (Berlin, 1949) pp. 217–225.

[57] Unfortunately, this gain is mitigated by the fact that D'ALEMBERT'S procedure still requires the decomposition $v = v^+ + u$ to arrive at a value for $-dv$. This awkward calculation of the acceleration through the consideration of geometrically represented finite velocities is avoided in LAGRANGE'S later treatment.

To analyze the Moon's rotational motion LAGRANGE first chooses a Cartesian reference frame $X - Y - Z$ fixed in space with origin at the Moon's center of gravity. The $X - Y$ plane is taken parallel to the plane of the ecliptic; the X-axis points to the first point of Aries on the celestial sphere. In this way each mass element α of the Moon may be specified by co-ordinates X, Y, Z. The element α is acted upon by the gravitational forces of the Earth and the Sun, which, because of the choice of reference system, now revolve about the Moon. If R and R' denote the respective distances from α to these bodies then the "forces" exerted on α are given in magnitude by

$$\frac{E}{R^2}, \frac{S}{R'^2} \quad (E = \text{mass of the Earth}, \quad S = \text{mass of the Sun})$$

and are directed along the respective lines joining α to the Earth and Sun. Note that the gravitational constant is here taken to be unity. The quantities E/R^2, S/R'^2 must be assumed multiplied by this unit and therefore actually represent accelerations. LAGRANGE further supposes that the quantities d^2X/dt^2, d^2Y/dt^2, d^2Z/dt^2 may be regarded as "accelerative forces" acting on α in the directions of the X, Y and Z axes. He continues as follows:

> ... it is necessary, by the general principle of Dynamics, that these last forces taken in the opposite direction and combined with the forces E/R^2, S/R'^2 hold the system of all the points α, that is the entire mass of the Moon, in equilibrium about its center of gravity supposed fixed.[58]

LAGRANGE'S next step is to invoke a "principle generally true in statics" to obtain a suitable equilibrium condition. Assume we "vary infinitely little the position of the Moon about its center" so that the quantities X, Y, Z, R and R' become

$$X + \delta X,\ Y + \delta Y,\ Z + \delta Z,\ R + \delta R, \text{ and } R' + \delta R'.$$

The statical principle applied to the "powers" $\alpha\, d^2X/dt^2$, $\alpha\, d^2Y/dt^2$, $\alpha\, d^2Z/dt^2$, $\alpha E/R^2$ and $\alpha S/R'^2$ leads to the following general equation, designated (A) in the original:

$$\text{(1)} \quad \text{(A)} \quad (1/dt^2) \int \alpha(d^2X\, \delta X + d^2Y\, \delta Y + d^2Z\, \delta Z) + E \int (\alpha\, \delta R/R^2) + S \int (\alpha\, \delta R'/R'^2) = 0,$$

where the symbol "$\int$" here denotes a summation over the mass elements α. Equation (1) forms the basis for LAGRANGE'S analysis of the Moon's rotational motion.

LAGRANGE subsequently observes that the statical law being invoked here is none other than the principle of virtual velocities:

> if any system of as many bodies or points as one wishes, each acted upon by any powers whatever, is in equilibrium and an arbitrary small motion is given

[58] *Oeuvres* **6** (1873) p. 8.

> to this system, by virtue of which each point traverses an infinitely small space, the sum of the powers, each multiplied by the space that its point of application traverses following the direction of this same power, will always be equal to zero.[59]

In a scholium following the presentation of (1) he embarks upon a more detailed discussion of the history and significance of this and the previous dynamical law; here appear the references to EULER and D'ALEMBERT which we examined in the previous section. In particular, LAGRANGE shows how D'ALEMBERT'S demonstration of the conservation of viv viva may be given a simple analytical formulation. Assume the bodies $m, m', \ldots$ are attracted to centers by "forces quelconques" $P, Q, \ldots, P', Q', \ldots$; let $p, q, \ldots, p', q', \ldots$ denote their respective distances to these centers. (LAGRANGE does not elaborate further on the nature of these forces but it is at least clear that they do *not* arise from the constraints in the system.) By the "general principle of dynamics" these forces are in equilibrium with the "forces"

$$-m\,(dv/dt),\ -m'(dv'/dt), \ldots,$$

where $v = ds/dt$, $v' = ds'/dt, \ldots$ are the directed speeds of $m, m', \ldots$. The quantities $v, v', \ldots$ are now regarded as "virtual velocities" and the statical principle gives

$$\left.\begin{array}{l} -m\,(dv/dt)\,ds + mP(-dp) + mQ(-dq) + \ldots \\ \quad -m'(dv'/dt)\,ds' + m'P'(-dp') + m'Q'(-dq') + \ldots \\ \quad - \ldots \end{array}\right\} = 0,$$

which, when integrated, yields the desired result:

$$(2)\qquad mv^2 + m'v'^2 + \ldots = mV^2 + m'V'^2 + \ldots - 2m \int (P\,dp + Q\,dq + \ldots) - 2m' \int (P'\,dp' + Q'\,dq' + \ldots) -, \ldots,$$

$V, V', \ldots$ here being the initial speeds of $m, m', \ldots$.

Of greater interest than this result itself is the comparison its demonstration affords with D'ALEMBERT's earlier analysis. D'ALEMBERT had first isolated for special attention the quantities $m\,dv^{(L)}$, where $dv^{(L)}$ represents the incremental speed or velocity lost by m to the constraints. The collection $\{m\,dv^{(L)}\}$ was then regarded as a new set of external "powers", which, when applied to the system, produces equilibrium. The statical rule in conjunction with the decompositions furnished by his dynamical principle led in the particular cases he examined to the desired result.

D'ALEMBERT'S idea of combining these two principles is of course at the base of LAGRANGE'S procedure. D'ALEMBERT, however, never succeeded in avoiding the cumbersome decompositions required by his principle, though, as we saw in the previous section, he did in his later work arrive at an improved formulation.

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59 *Ibid.* pp. 8–9; p. 10.

LAGRANGE'S fundamental achievement was to free the principle from all reliance on geometric considerations, involving the awkward representation of finite and infinitesimal quantities, and to derive from it a simple and general analytical method for the solution of constrained dynamical problems. At the conclusion of the scholium he provides a concise summary of his new method:

> Moreover the principle of Statics which I have just described, being combined with the principle of Dynamics given by M. d'Alembert, constitutes a type of general formula which includes the solution of all problems concerning the motion of bodies. For we will always have an equation similar to (A) [equation (1) above], and the whole difficulty will further consist only in finding the analytical expression of the forces which we suppose act on the bodies and of the lines along which these forces act, by employing in these expressions only the smallest possible number of indeterminate variables, so that their differentials designated by the δ be one and the other entirely independent; after which, equating to zero separately the terms which are each multiplied by the differentials of which I speak, we will have at last as many particular equations as is necessary for the solution of the Problem, as we shall see in the Articles which follow.[60]

We return to LAGRANGE'S formulation of the astronomical problem and examine how he uses it to illustrate his new method. In LAGRANGE'S investigation the problem of the Moon's rotational motion is reduced to a question in rigid body dynamics: given a rigid body free to rotate about its center of gravity and acted upon by external central forces (in this case the gravitational attractions of the Earth and the Sun) we must use relation (1) to derive differential equations which describe its motion. To do this, LAGRANGE introduces three angles π, ε and ω that specify the orientation of the Moon relative to the Cartesian reference frame. The positions of the mass elements of the Moon are given internally by a second, spherical, reference frame which remains fixed in that body. LAGRANGE then presents formulas that express the Cartesian co-ordinates X, Y, Z of each mass element α in terms of the parameters π, ε, ω and its spherical co-ordinates. Notice that the latter remain constant in time; in addition, they do not enter into the variations involved in the principle of virtual velocities. The variation of the Moon's position required by this principle will therefore be completely determined by the variations $\delta\pi$, $\delta\varepsilon$, $\delta\omega$. The parameters π, ε, ω thus represent for this particular problem the "smallest number of indeterminate variables" needed in the application of his method. In modern parlance they are simply a set of 'Eulerian angles' serving here as 'generalized co-ordinates' to describe the system.

LAGRANGE proceeds to calculate the values of $d^2X\,\delta X + d^2Y\,\delta Y + d^2Z\,\delta Z$ and $E\,\delta R/R^2 + S\,\delta R'/R'^2$ in terms of the angles π, ε, ω and their variations. After a lengthy series of computations he obtains an expression for $d^2X\,\delta X + d^2Y\,\delta Y + d^2Z\,\delta Z$ of the form

$$f\,\delta\pi + g\,\delta\varepsilon + h\,\delta\omega,$$

[60] *Ibid.* p. 12.

where f, g, h are functions of π, ε, ω and their first and second derivatives. A separate computation is applied to the quantity $E\,\delta R/R^2 + S\,\delta R'/'R^2$ to obtain a corresponding expression of the form

$$f^*\,\delta\pi + g^*\,\delta\varepsilon + h^*\,\delta\omega.$$

By substituting these values into relation (1), simplifying and setting the coefficients of the variations of the angles equal to zero, LAGRANGE arrives at three second order differential equations in the variables π, ε and ω. These equations constitute the basis for his subsequent investigation of the Moon's oscillatory motion.

From our point of view the most interesting step in LAGRANGE's analysis concerns his calculation of the quantity $d^2X\,\delta X + d^2Y\,\delta Y + d^2Z\,\delta Z$. To obtain a value for this quantity he proceeds directly, using the relations connecting X, Y, Z to π, ε, ω and performing the necessary differentiations; the desired expression is ultimately arrived at through a very involved computation. In an extended "Remark" LAGRANGE describes an alternate method which leads to the same result, a method which "though indirect is nevertheless preferable by its simplicity and generality."[61] Indeed, the method in question would lead for any particular problem to the 'Lagrangian' equations of motion, or, more accurately, to the part of these equations involving the kinetic energy function. Since LAGRANGE wishes only to give "an idea of the method" he restricts his discussion to the example at hand. To simplify the description of his procedure and to emphasize its generality I shall in what follows employ modern notation.

Assume the system is completely specified by the independent variables $q_1, q_2, \ldots$. LAGRANGE shows that to arrive at an expression for $\ddot{\vec{r}} \cdot \delta\vec{r}$ (*i.e.*, $(d^2X\,\delta X + d^2Y\,\delta Y + d^2Z\,\delta Z)/dt^2$) in terms of these variables, their variations and their derivatives it is sufficient to calculate the value of $\frac{1}{2}v^2$ (v = speed of α):

$$\frac{1}{2}v^2 = f(\ldots, q_i, \ldots, \ldots, \dot{q}, \ldots). \tag{3}$$

Given (3) the desired expression becomes

$$\ddot{\vec{r}} \cdot \delta\vec{r} = \Sigma\{d(\partial f/\partial\dot{q}_i)/dt - \partial f/\partial q_i\}\,\delta q_i. \tag{4}$$

LAGRANGE does not employ partial differentials in his analysis; instead, as was customary during the period, the quantities $\partial f/\partial q_i$ and $\partial f/\partial\dot{q}_i$ appear as the coefficients of δq and $\delta\dot{q}$ respectively in the expanded expression for $\delta(\frac{1}{2}v^2) = \delta f$. This, however, is only a matter of notation, and it is clear that the next step in the analysis—a summation of (4) over α—would lead to the usual expression involving the kinetic energy in the Lagrangian equations of motion. LAGRANGE himself emphasizes that the above device is applicable in "the solution of all Problems of Dynamics which one would wish to treat by our method."[62]

The justification presented by LAGRANGE for this new device is of interest in connection with his earlier and later work on the foundations of mechanics.

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[61] *Ibid.* pp. 16–18.
[62] *Ibid.* p. 18.

Beginning with (3) above he obtains

$$(5)\qquad v\,\delta v = \Sigma(\partial f/\partial q_i)\,\delta q_i + \Sigma(\partial f/\partial \dot{q})\,\delta \dot{q}_i.$$

Using the relation $\dot{\vec{r}}\cdot\delta\dot{\vec{r}} = v\,\delta v$ and the fact that the d and δ are interchangeable he integrates (5) by parts:

$$(6)\qquad \dot{\vec{r}}\cdot\delta\vec{r} - \int(\ddot{\vec{r}}\cdot\delta\vec{r})\,dt = \Sigma(\partial f/\partial\dot{q}_i)\,\delta q_i - \int\Sigma\{d(\partial f/\partial\dot{q}_i)/dt - \partial f/\partial q_i\}\,\delta q_i\,dt.$$

LAGRANGE proceeds to argue that "this equation must be identical and consequently it is necessary that the algebraic part of the first member be equal to the algebraic part of the second, and the integral part be equal to the integral part."[63] Hence "removing the integral sign" he concludes that

$$(4)\qquad \ddot{\vec{r}}\cdot\delta\vec{r} = \Sigma\{d(\partial f/\partial\dot{q}_i)/dt - \partial f/\partial q_i\}\,\delta q_i.$$

LAGRANGE'S appeal here to ideas from the calculus of variations in what is essentially an analysis formulated in terms of *differential* principles is not wholly justified ("it is easily understood that ..."). Indeed, the basis of the demonstration, the inference from (6) to (4), seems more an inappropriate carry over from his earlier treatment of the *integral* principle of least action. In a memoir of 1780 LAGRANGE would return to the method outlined above and develop it in greater detail.[64] Although much of this later material consists merely of formal elaboration of ideas already present here, it does contain one substantive advance. LAGRANGE replaces the argument described in the preceding paragraph with a demonstration based on the following relation:

$$(7)\qquad \ddot{\vec{r}}\cdot\delta\vec{r} = d(\dot{\vec{r}}\cdot\delta\vec{r})/dt - \frac{1}{2}\,\delta v^2.$$[65]

This equation is obtained from the interchangeability of the d and the δ, a fact LAGRANGE interestingly refers to as "the fundamental principle of the calculus

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[63] *Ibid.* p. 17.

[64] "Théorie de la libration de la lune, et des autres phénomènes qui dependent de la figure non sphérique de cette planète" *Nouveaux Mémoires de l'Académie royale des Sciences et de Belles Lettres de Berlin 1780* = *Oeuvres* **5** (1870) pp. 5–122.

[65] This relation is considered very important by the German physicist GEORG HAMEL. Multiplying it by α and summing over the mass elements of the system we obtain what HAMEL calls the "Zentralgleichung" or central equation. HAMEL regards this equation as important because it establishes a relation between an invariant of the second order and two invariants of the first order. HAMEL'S work on Lagrangian mechanics appears in

"Die Lagrange-Eulerschen Gleichungen der Mechanik" *Zeitschrift für Mathematik und Physique* **50** (1904) pp. 1–57.

"Über die virtuellen Verschiebungen in der Mechanik" *Mathematischen Annalen* **59** (1904) pp. 416–434.

HAMEL'S remarks on the invariance properties of the central equation appear in *Theoretische Mechanik* (Springer-Verlag, 1949) p. 223.

of variations."[66] The demonstration itself is often presented today and it would be unnecessary to describe it.[67] We shall, however, return in the next section to the question of the changing role of the calculus of variations in LAGRANGE'S approach to the foundations of mechanics.

The previous discussion focused solely on LAGRANGE'S treatment of the quantity $(d^2X\,\delta X + d^2Y\,\delta Y + d^2Z\,\delta Z)/dt^2$, involving the 'kinetic reaction' or 'force of inertia'. The other major step in the analysis, the calculation of $E\,\delta R/R^2 + S\,\delta R'/R'^2$, though of direct importance for his solution to the astronomical problem of lunar libration, is of less interest from the viewpoint of the principles and methods of mechanics. It is, however, possible to make some general observations on the treatment accorded to the forces in LAGRANGE'S new method. Recall first his statement of the conservation of vis viva:

$$(2)\quad mv^2 + m'v'^2 + \ldots = mV^2 + m'V'^2 + \ldots - 2m \int (P\,dp + Q\,dq + \ldots) - 2m' \int (P'\,dp' + Q'\,dq' + \ldots) - \ldots .$$

In presenting this result he had commented that it "comprises ... the conservation of vis viva taken in all its extension." This claim combined with his reticence about specifying the nature of the forces would suggest LAGRANGE understood (2) to include cases in which the forces are not integrable, *i.e.* in which the quantities $\int (P\,dp + Q\,dq + \ldots), \ldots$ are not finite functions of $p, q, \ldots$. Now it was precisely the existence of such cases which had compelled him in the memoir of 1760 to restrict the application of the principle of least action. This principle had required the validity of (2) in the strong sense, in which the forces are integrable. While it is indeed true that (2) would hardly be of interest otherwise, the important point is that his new method no longer requires this equation for its application; it will be valid regardless of the nature of the forces. In addition, conservation in the strong sense when it does hold will be a consequence of the general principle of virtual velocities. These considerations would appear to be of evident importance for LAGRANGE'S decision to abandon least action as the basis of his approach and will be discussed again in the following section.

The remainder of the memoir is with one exception devoted to the astronomical problem of lunar libration. The exception concerns another long scholium consisting this time of a discussion of rigid body theory. LAGRANGE had previously shown that the study of the Moon's motion is conserably simplified if one assumes this body possesses an axis about which it would turn with constant angular velocity

[66] *Oeuvres* **5** (1870) p. 21. The derivation of LAGRANGE'S equations contained in the memoir of 1780 is reproduced in the *Méchanique Analitique* (1788). Following the presentation of this derivation LAGRANGE writes: "What has just been found in a particular manner could have been found as simply and more generally by the method of variations." He proceeds to present an expanded version of the argument which had appeared in the memoir of 1764. The passage quoted might suggest that LAGRANGE now considered the derivation of 1780 to be non-variational. However, in presenting it he again uses the interchangeability of the d and δ and the latter is once again referred to as a "fundamental principle of the calculus of variations." In addition, the alternate argument seems to involve a no more genuine application of the calculus of variations; it strikes this reader as being as inappropriate in 1788 as it was in 1764.

[67] See for instance H. GOLDSTEIN'S *Classical Mechanics* (Addison-Wesley, 1950) p. 16–18.

in the absence of external forces. He deduced that a necessary condition for this to be the case is

$$S=0, \quad T=0,$$

where S, T denote the products of inertia with respect to the axis of rotation, expressed in terms of a spherical co-ordinate system in which this axis coincides with the polar axis. (It may be recalled that this result had been derived in a different setting in the memoir of 1760.) In the scholium LAGRANGE shows it is unnecessary to postulate the existence of an axis of constant rotation in order to conclude the products of inertia vanish. That is, he furnishes a general algebraic proof that every rigid body possesses such an axis, and, moreover, that there exist two additional mutually perpendicular axes with the same property. This result had in fact been established by EULER in a memoir presented in 1758 to the Berlin Academy.[68] It is, however, important to note that EULER's analysis was motivated by topological considerations; in contrast, LAGRANGE'S treatment in the scholium is exclusively algebraic. Although LAGRANGE does not develop his study of rigid bodies any further here, he would return to this subject ten years later in an important paper that laid the foundations for the modern approach to rigid body analysis.[69] Interestingly, THOMAS HAWKINS has argued that this later treatise, which is also completely algebraic in character, led, through CAUCHY in the 19th century, to the creation of modern spectral theory.[70]

[68] In "Recherches sur la connoisance mécanique des corps" (1758) (*op. cit.* n. 43) EULER introduces the concept of 'moment of inertia' of a rigid body about a given line. He defines a 'principal axis' to be that line through the center of gravity for which the value of the moment of inertia is an extremum. This condition leads to two equations in the angles defining the position of the axis. EULER shows these equations also follow from the condition that the rigid body be free to rotate about the given axis with constant angular velocity in the absence of external forces. Through an analysis of the two equations he is able to establish the location of one principal axis. He proceeds to show that there are two other such such axes and that all three are mutually perpendicular.

LAGRANGE'S demonstration, though analagous to EULER'S, contains a difference in emphasis: the purpose of the scholium is to show that the existence of a 'principal axis' (this terminology is not employed) is independent of dynamical considerations. LAGRANGE takes the algebraic condition that the products of inertia vanish to be the defining property of such an axis and his subsequent discussion is devoted solely to a study of the equations which arise from this condition.

[69] "Nouvelle Solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accelératrice", *Nouveaux Mémoires de l'Académie royale des Sciences de Berlin 1773* (1775) = *Oeuvres* **3** (1869) pp. 579–616.

[70] THOMAS HAWKINS "Cauchy and the Origins of Spectral Theory" *Historia Mathematica* **2** (1975) pp. 1–29. HAWKINS describes how LAGRANGE in 1773 introduced orthogonal transformations and solved the problem of reducing a general quadratic form to the sum of square terms by solving the corresponding eigenvalue problem. This approach completely superseded his earlier analysis. It is therefore ironic that the approach to quadratic forms which became standard in the first decades of the 19th century (due to POISSON and HACHETTE and described by HAWKINS p. 19 (*ibid.*)) is essentially the same as LAGRANGE's earlier method of 1764. (Although LAGRANGE was dealing with the determination of the principal axes of an arbitrary rigid body, it is not difficult to show that his solution is mathematically the same as the reduction of a general (positive definite) quadratic form in three variables.)

Section 3: Reflections on Lagrange's Shift

Perhaps the most interesting fact to emerge from the previous study is LAGRANGE's remarkable shift from the integral principle of least action to the differential principle of virtual velocities. Although he never discusses the reasons for this shift it is possible on the basis of circumstantial and technical evidence to arrive at a plausible explanation for it. It is first clear that LAGRANGE had completed the transition to the general principle of virtual velocities by 1763. From 1756, when he first submitted a memoir on least action to the Berlin Academy, until this year, LAGRANGE was active in the formation and subsequent deliberations of the Academy of Sciences at Turin. During this period he was engaged in research on a wide range of subjects: the propagation of sound, the integral and differential calculus, the method of variation of arbitrary constants in theoretical astronomy as well of course as the calculus of variations and the foundations of mechanics. In addition, he had since 1755 been involved in the teaching of courses on mathematics and mechanics at Turin's Royal Artillery School.[71] Earlier I cited a letter LAGRANGE wrote to EULER in 1759 in which he announced that he had in connection with these teaching duties arrived at the "true metaphysics" of the principles of mechanics. I suggested that he may have been referring to the foundations later set down in the memoir of 1764. There is in fact considerable evidence to support this conjecture. A course in mechanics at an elementary level would almost certainly have begun with an exposition of statics. In contrast, LAGRANGE's research on the principle of least action was devoted to advanced problems in dynamics. Neither in the memoir of 1760 nor in the *Mécanique Analytique* does he attempt to develop a theory of statics from this principle. From a pedagogical viewpoint it would have been natural to begin with a study of equilibrium—and the principle of virtual velocities was during this period the most widely accepted law of statics—and then proceed to a study of motion.

Any presentation of the principle of least action to students would also undoubtedly have necessitated as motivation some mention of its background, including MAUPERTUIS's and EULER's claims for its metaphysical significance. Throughout his career LAGRANGE exhibited a marked reluctance to discuss philosophical claims of any sort. It is an interesting fact that the term "least action" never actually appears in the memoir of 1760, despite his use of this phrase in earlier correspondence with EULER. In the *Méchanique Analitique* LAGRANGE would sarcastically dismiss those who employ such terms, stating that they act

> as if these vague and arbitrary denominations comprise the essence of the laws of mechanics and can by some secret virtue establish in final causes the simple results of the known laws of mechanics.[72]

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[71] The lecture notes for this course have been lost. For the suggestion that they may have constituted a sort of 'first draft' of the *Mécanique Analytique* I am indebted to a conversation with Professor RENÉ TATON.

[72] *Méchanique Analitique* (1788) p. 187. LAGRANGE is referring in particular to D'ARCY and MAUPERTUIS.

Lagrange's disdain for teleological speculation was shared by his older contemporary d'Alembert. In the second edition of the *Traité de Dynamique* (1758) d'Alembert cautions against the use of principles based on "final causes", warning that they may be "only simple mathematical consequences of a few general formulas."[73] This apparent negative reference to the principle of least action coupled with his own alternate approach to the foundations of mechanics must surely have influenced Lagrange's decision to take up the principle of virtual velocities. Furthermore, the occasion for his presentation of the new method based on this principle—a prize competition of the Paris Academy—would certainly have been felicitous at a time when d'Alembert was one of its leading members.[74]

There are, finally, some technical considerations of direct relevance for Lagrange's shift. I noted in the preceding section that the principle of least action is less general than the principle of virtual velocities; the former had required for its application the condition that the forces be integrable. Furthermore, Lagrange was aware of this fact, even bothered by it. To delve into this matter more deeply recall that his derivation of equations of motion from the integral principle had consisted of two steps: the first led to a suitable expression involving the acceleration; the second led by means of a potential function to an expression involving the forces. The restriction on Lagrange's method arose from the second step, which required the integrability of these forces. It is clear that a shift to the differential principle would remove this restriction; the remaining difficulty would then consist of adapting the treatment contained in the first step to the new setting. Lagrange had claimed in his treatise on the principle of least action that certain of the steps remain valid when the principle itself fails. The theory developed in the memoir of 1764 may be viewed as the successful translation of this claim into the general method there presented. Thus the principle of virtual velocities besides possessing the advantages described in the previous paragraphs, would also lead in practice to the resolution of any problem susceptible to solution by the integral method.

I turn now to another issue related to the preceding one: the changing role of the calculus of variations in Lagrange's approach to the foundations of mechanics. His transition to the principle of virtual velocities clearly depended on his ability to apply to this principle techniques originally developed in the calculus of variations. The fundamental problem of this branch of mathematics is the determination of a function for which the variation of a certain integral is equal to zero. Lagrange's later method, however, was based on a differential

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[73] *Traité* (1758) p. xxx.

[74] D'Alembert, however, was not on the committee set up to judge the competition. He would later communicate to Lagrange his opinion of the latter's winning submission in the following remarkable passage: "I read your piece on the libration of the Moon and I said as did John the Baptist: «Opportet illum crescere, me autem minui.»" See *Oeuvres de Lagrange* **13** (1882) p. 10. (The Latin quotation is a reference to John 3:30, in which John the Baptist sets forth his judgement of Jesus: "He must increase but I must decrease.")

principle; the few traces of the integral approach which are present in the memoir of 1764 would be replaced by more direct procedures in his subsequent treatises. The question therefore arises as to precisely in what sense his later treatment is variational. The answer to this question would appear to lie in LAGRANGE's own conception of the nature of mathematics. That is, though it is indeed possible to interpret his later method non-variationally, it is nevertheless true that he himself regarded the techniques he employs as ones taken from the calculus of variations and governed in application by the principles of this science.[75] LAGRANGE more than anyone else during the 18th century viewed mathematics as an internally consistent formalism; any application of the formalism was, *ipso facto*, an application of the mathematics itself. This formalistic tendency became more pronounced as his career progressed and received its most famous illustration in his celebrated attempt to found the differential calculus on a theory of formal power series.[76] It is also interesting that LAGRANGE's last work in science, an important contribution to the theory of perturbations in celestial mechanics, was developed using techniques and methods based on the algebraic procedures of the calculus of variations.[77]

Appendix: Lagrange's Analysis of Lunar Libration (1764)

The subject set by the Paris Academy for the prize competition of 1764 consisted of two parts: the first demanded a physical explanation for the equality of the Moon's period of rotation and its sidereal period of revolution about the Earth; the second demanded an investigation of the precession and nutation of

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[75] LAZARE CARNOT, the French statesman, mathematician, and father of SADI CARNOT, had also based mechanics on the general principle of virtual velocities in his treatise *Essai sur les machines en général* (1783). CARNOT was influenced strongly by D'ALEMBERT's *Traité de Dynamique* and appears to have worked independently of LAGRANGE. Although the *Essai* contains some variational results, in genesis and substance the approach of this treatise is non-variational. Unlike LAGRANGE, who always worked with displacements, CARNOT takes the notion of virtual velocity (what he calls "geometric" velocity) quite seriously and attempts to develop his analysis using this concept. In his later treatise *Principes fondamentaux de l'équilibre et du mouvement* (1803) CARNOT would situate his own work in relation to LAGRANGE's better known system. Further details may be found in my Ph. D. dissertation, *op. cit.* note 52.

[76] *Théorie des fonctions analytiques contenant les principes du calcul differentiel, dégagés de toute considération d'infiniment petits, d'évanouissants, de limites et de fluxions, et réduit a l'analyse algébrique des quantities finies* (Paris, 1797) = *Oeuvres* **9** (1881). LAGRANGE's approach to the foundations of the calculus was of course abandoned by subsequent researchers. However, JUDITH V. GRABINER has recently argued that his theory was of direct importance for CAUCHY's later successful rigorization of the calculus. See GRABINER's *The Origins of Cauchy's Rigorous Calculus* (M.I.T. Press, 1981).

[77] This research involved the method of variation of arbitrary constants and appears in the second (1811, 1816) and later editions of the *Mécanique Analytique* (V. 1 Sect. 5, and V. 2 Sect. 6 (Chap II), Sect. 7 (Chap. I)). LAGRANGE's reference to the calculus of variations is contained in V. 1 Sect. 5 § 1.

the Moon's axis of rotation. LAGRANGE in his winning submission provided what came to be regarded as the definitive answer to the first of these parts. His analysis of the motion of the Moon's axis of rotation was, by his own later admission, less successful.[78] The following account will focus solely on his explanation of the equality of the Moon's periods of rotation and revolution.[79]

Reasoning from hydrostatical considerations LAGRANGE concludes that the Moon in its original fluid state had acquired through the action of the Earth and its own rotational motion the shape of an almost spherical ellipsoid with three unequal axes. This ellipsoidal solid represents the figure acquired by the Moon when it solidified. Two of the axes lie in the elliptical section formed by the Moon's equatorial plane; the longer of the two is directed to the prime meridian, the latter being that which is constantly pointed very nearly to the Earth. Given these assumptions and the relations furnished by his variational method LAGRANGE deduces a single equation which describes the oscillations of the prime meridian about its mean position. These oscillations are of two types. The first is a consequence of the variable velocity of the Moon in its orbit about the Earth and produces the so-called libration in longitude. This libration is an optical phenomenon independent of any variation in the Moon's rotational velocity. The second is a consequence of the Moon's non-spherical shape; it produces the physical libration and reflects actual changes in this rotational velocity. LAGRANGE shows that both the optical and physical libration given by his equation are small and periodic. In addition, he shows it is no longer necessary to assume that in the beginning the velocities of rotation and revolution were *exactly* equal; any original difference, if small, would remain small. LAGRANGE regards this last fact to be the crux of his explanation. Thus he has avoided the "paradox", against all laws of chance and suggestive of divine design, involved in the assumption of original exact equality.

Before continuing with a more detailed account of LAGRANGE'S result I comment briefly on D'ALEMBERT'S curious role in the background to his research. The suggestion that libration could be explained by assuming the Moon possesses a long axis pointed to the Earth had originated in NEWTON'S *Principia Mathematica.*[80] D'ALEMBERT had considered this suggestion and rejected it, aparently

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[78] "Théorie de la libration de la lune" *Nouveaux Mémoires de l'Académie royale des Sciences de Berlin* 1780 (1782) = *Oeuvres* **6** (1870) p. 9.

[79] To facilitate the description of LAGRANGE's main result I depart from his analysis in several non-essential ways. First, I use such terms as moment and product of inertia to designate what appear in the original text as unnamed analytical entities. Second, I assume (as does LAGRANGE eventually) that the action of the Sun on the Moon is negligible. Finally, I present his result for the special case in which the Moon is assumed to be an ellipsoid of uniform density. (This case is actually treated in a series of scholia.) At the end I show how his analysis would apply under less restrictive conditions.

[80] Book III Proposition xxxviii. LAGRANGE remarks in a scholium that his explanation of libration (based on the assumption that the Moon in its original formation had acquired the shape of a triaxial ellipsoid) may be regarded as a commentary on this proposition. See *Oeuvres* **6** p. 47.

on the basis of a surprising oversight.[81] Meanwhile, he became interested in investigating the rotational motion of the Earth under the assumption that this body is an ellipsoid with three unequal axes. (The more customary assumption was that it is an oblate ellipsoid of revolution.) The results of his investigation, presented in a memoir to the Paris Academy in 1756, were based on three differential equations.[82] No one to my knowledge followed D'ALEMBERT'S proposal concerning the Earth. Nevertheless, as LAGRANGE explicitly observes in the introduction to his piece on libration, the equations which form the basis of his investigation are formally identical to those which had appeared in D'ALEMBERT'S memoir of 1756. To be sure, LAGRANGE'S method for obtaining these equations differs from D'ALEMBERT'S and his application of them to the Moon violated the

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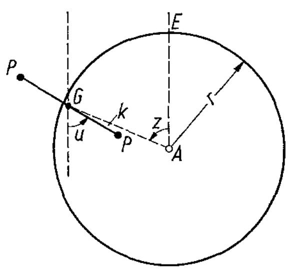
[81] D'ALEMBERT'S discussion of lunar libration appears in his *Recherches sur Différens Points Importans du Systême du Monde Seconde Partie* (Paris, 1754) pp. 243–261. On page 256 he states that certain geometers have attempted "to explain why the Moon always presents to us the same face by supposing this planet is elongated in the direction of the line which goes from the Earth to the Moon." He proceeds to argue against this explanation. He considers a rigid rod with equal masses P and p attached to each end. The rod rotates about its center G, revolving at the same time in a circular path of radius r under the action of gravity about a heavy body A. The system thus defined is taken to represent the motion of the Moon about the Earth according to the Newtonian hypothesis.



D'ALEMBERT'S *Figure 46* (*1756*)

The variables z and u denote the angles made by the lines GA and PGp with a fixed reference line AE. The variable $k = z - u$ therefore represents the 'libration' experienced by the rod. D'ALEMBERT derives the following differential equation to describe this system:

$$(*) \qquad d^2k/dt^2 = -(3A/2r^3)\sin(2k).$$

He proceeds to argue that the solution to (*) (involving elliptic integrals) is inconsistent with the assumption that k remain small. His argument, however, is based on a misguided choice of initial conditions. He assumes the rod has no initial angular velocity about its center. Thus when $t = 0$, $k = 0$ and $\dot{u} = 0$; consequently $\dot{k} = \dot{z}$. The latter quantity is not small and it is indeed clear that equation (*) will not lead in this situation to small stable values for k. LAGRANGE'S later account of libration would assume that $\dot{z} - \dot{u} = \dot{k}$ is initially small (but not zero!).

[82] "Recherches sur la précession des équinoxes et sur la terre dans l'hypothèse de la dissimilitude des Méridiens". Presented to the Academy in December 1756 and published in *Histoire de l'académie royale des Sciences de Paris. Avec les mémoires de mathématique et de physique 1754* (Paris, 1759) pp. 413–428.

latter's own views on libration.[83] He was, however, apparently able to derive some benefit from the work of his older contemporary in devising the explanation set forth in the prize memoir.

We move now to a closer examination of LAGRANGE'S theory. With reference to the discussion in Part Two Section 2, the three angles π, ε and ω which specify the Moon's position relative to the ecliptic Cartesian co-ordinate system are defined as follows: π is the inclination of the lunar equator to the ecliptic; ε is the angular distance measured along the ecliptic from the first point of Aries to the equatorial descending node; ω is the angular measured along the lunar equator from the descending node to the prime meridian. The quantity $\varepsilon + \omega$ therefore equals the celestial longitude of the prime meridian.[84] LAGRANGE assumes that the quantity π is small (thus its square may be neglected) and that the polar axis is a principal axis for the Moon viewed as a rigid body. Of the three general equations which correspond to the parameters π, ε and ω he makes use in his explanation of libration only of the last one, corresponding to ω. He derives the following expression for the kinetic part of this equation:

(1) $$\Omega = (d(d\omega + d\varepsilon)/dt^2)\, C,$$

where

$C =$ the moment of inertia of the Moon about its polar axis.

The part arising from the gravitational action of the Earth is shown to be given by the expression:

(2) $$-(3E/r^3)\,\Omega' = -(3E/2r^3)\,(B - A)\sin\,[2(\nu - (\varepsilon + \omega))],$$

where

$E =$ mass of the Earth

$r =$ distance from the Moon to the Earth

$A =$ moment of inertia of the Moon about the equatorial axis which is directed to the prime meridian

$B =$ moment of inertia of the Moon about the equatorial axis which is perpendicular to the direction of the prime meridian

$\nu =$ true celestial longitude of the Earth as seen from the Moon.

[83] Unlike LAGRANGE, D'ALEMBERT does not use the principle of virtual velocities in his analysis. He relies instead on other equilibrium properties of rigid bodies. Although his theory is based on his dynamical principle, the latter had appeared in many of his treatises, and there is no evidence to suggest the memoir of 1756 was of any particular significance in influencing LAGRANGE'S approach to the foundations of mechanics.

[84] LAGRANGE'S decision to measure ε from Aries to the equatorial *descending* node requires some explanation. From observation it was known that the following three planes constantly intersect in a line: i) the plane of the Earth's orbit about the Moon; ii) the plane parallel to the ecliptic passing through the Moon's center; iii) the plane of the Moon's equator. In addition, it was known that ii) lies intermediate between i) and iii). (It is these observational facts that LAGRANGE unsuccessfully attempts to explain in the concluding parts of the memoir.) Hence the distance from Aries to the ascending node of i) is equal to ε, the distance from Aries to the descending node of iii). Since LAGRANGE assumes the prime meridian is the one which is always pointed very nearly to the Earth, the quantity $\varepsilon + \omega$ will be approximately equal to the Earth's true longitude as seen from the Moon.

By adding (1) and (2) we obtain the desired equation corresponding to ω:

(3) $$(d(d\varepsilon + d\omega)/dt^2)\, C - (3E/2r^3)\,(B - A) \sin [2(\nu - (\varepsilon + \omega))] = 0.$$

LAGRANGE now introduces several simplifications into (3). Let n denote the mean motion of Moon in its orbit about the Earth. We have by KEPLER's third law

$$n^2 = \frac{E}{r^3},$$

where we have assumed i) the Moon's mass is negligible in comparison to the Earth's, and ii) the Moon's orbit is approximatey circular in the sense that a'^3/r^3 (a' = length of semi-major axis of the Moon's orbit) is a small quantity of the second order. In addition, because the prime meridian always points very nearly to the Earth, the quantity $\theta = \nu - (\varepsilon + \omega)$ is small. Hence $\sin [2(\nu - (\varepsilon + \omega))] = \sin 2\theta = 2\theta$ and $d\omega + d\varepsilon = d\nu - d\theta$. Equation (3) then becomes

$$((d^2\nu - d^2\theta)/dt^2)\, C - 3n^2(B - A)\,\theta = 0,$$

or, dividing by C,

(4) $$\frac{d^2\nu}{dt^2} - \frac{d^2\theta}{dt^2} - (3n^2(B - A)/C)\,\theta = 0.$$ [85]

Recall that ν designates the true longitude of the Earth as seen from the Moon; ν therefore equals $L + 180°$, where L is the true longitude of the Moon in its orbit about the Earth. Using the equation of the center LAGRANGE expresses L in terms of the Moon's mean motion n:

(5) $$L = l - a \sin (nmt),$$

where l is the lunar mean longitude, a is a constant numerically equal to twice the eccentricity of the lunar orbit, m (a number which differs "very little from unity") is the ratio of the motion of the lunar mean anomaly to the lunar mean motion and where we have retained only the first term ($-a \sin (nmt)$) in the expansion given by the equation of the center. We therefore have

$$\frac{d^2\nu}{dt^2} = \frac{d^2L}{dt^2} = an^2m^2 \sin (nmt).$$

Substituting this value for $d^2\nu/dt^2$ into (4) yields

(6) $$d^2\theta/dt^2 + (3n^2\ B - A)/C)\,\theta - an^2m^2 \sin (nmt) = 0.$$

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[85] In passing from (3) to (4) LAGRANGE replaces dt by dV, where V in his analysis equals nt. Thus (4) appears in the original text as

$$d^2\nu - d^2\theta - (3M/H)\, dV^2 = 0.$$

(I have expressed LAGRANGE's M and H in the language of moments of inertia as $B - A$ and C.) The subsequent relations are presented as differential equations in the variable V. I have not followed LAGRANGE in this convention.

We now assume that the Moon is an ellipsoid of uniform density and that the major equatorial axis is the one directed to the prime meridian. The quantities A and B therefore represent the values of the Moon's moments of inertia about its major and minor equatorial axes. LAGRANGE had earlier shown in this situation that B is greater than A. Hence $3n^2(B-A)/C > 0$, and (6) may therefore be integrated "by known methods" to yield

$$\theta = Q \sin\left(n\sqrt{3(B-A)/C}\,t\right) - (am^2/(m^2 - 3(B-A)/C)) \sin(nmt),$$

where Q is a constant and we have assumed $\theta = 0$ when $t = 0$. By letting $h = 3(B-A)/C$ and $m^2/(m^2-h) = h/(m^2-h)+1$ we express this result as follows:

$$(7) \qquad \theta = Q \sin(n\sqrt{h}t) - (ah/(m^2-h)) \sin(nmt) - a\sin(nmt).$$

Equation (7) furnishes the oscillations of the prime meridian about its mean position. These oscillations are of two types. The first, given by the term $-a\sin(nmt)$, is a consequence of the Moon's variable velocity in its orbit about the Earth; it produces the libration in longitude and is a purely optical effect. The second, given by the terms $Q\sin(n\sqrt{h}t)$ and $-(ah/(m^2-h))\sin(nmt)$, is a physical consequence of the non-spherical shape of the Moon; it produces the physical libration and reflects actual variations in the Moon's rotational motion. Note that the Moon, though not spherical, is very nearly so. Hence $h = 3(B-A)/C$ and therefore $-(ah/(m^2-h))\sin(nmt)$ are small quantities. The term $Q\sin(n\sqrt{h}t)$ is periodic with period $360°/n\sqrt{h}$ and maximum amplitude Q. Although observation would be required to determine the precise value of Q, it is clear from the fact that the total physical libration remains small that Q itself is also small.[86]

LAGRANGE proceeds to calculate $\omega + \varepsilon$ as a function of time. By definition we have $\omega + \varepsilon = \nu - \theta$. Using $\nu = L + 180°$ and (5) we obtain

$$\omega + \varepsilon = l + 180° - a\sin(nmt) - \theta.$$

Substituting the value for θ given by (7) into this relation yields

$$(8) \qquad \omega + \varepsilon = l + 180° - Q\sin(n\sqrt{h}t) + (ah/(m^2-h))\sin(nmt).$$

We now differentiate (8) with respect to time:

$$(9) \qquad d(\omega+\varepsilon)/dt = n - Q\,n\sqrt{h}\cos(n\sqrt{h}t) + (ahnm/(m^2-h))\cos(nmt),$$

where we have used $dl/dt = n$. By setting $t = 0$ we arrive at the initial value for $d(\omega+\varepsilon)/dt$:

$$(10) \qquad d(\omega+\varepsilon)/dt\,|_{t=0} = n(1 - Q\sqrt{h} + ahm/(m^2-h)).$$

Equation (10) furnishes the initial velocity of the Moon's rotation relative to the fixed stars. Since Q and h are small it is clear that this velocity is almost equal to

[86] LAGRANGE tacitly assumes Q is small, providing no further discussion of the matter. In fact, the Moon's physical libration was never actually observed during the 18th century (GRANT *A History of Physical Astronomy* (London, 1852) p. 75). Observational astronomers of the period were unable to distinguish this libration from the more noticeable libration in longitude. Q and h would therefore have to be very small.

n, the magnitude of the Moon's mean motion about the Earth. However, it need not be exactly equal to n; Q, though small, is an arbitrary constant whose precise value is a matter for observation. Thus $(d(\omega + \varepsilon)/dt)|_{t=0}$ could in principle take on any value whatever so long as this value is close to n. LAGRANGE regards this fact to be the crux of his explanation. He notes that until now astronomers have assumed the Moon's rotational motion is uniform. To explain the fact that the Moon always reveals the same face to the Earth these astronomers have therefore needed to suppose the Moon's initial rotational velocity was *exactly* equal to n, a supposition LAGRANGE considers "very difficult to understand". He summarizes this point, and his analysis as a whole, as follows:

> It seems to me that the preceding Theory furnishes a completely simple solution to this paradox, or, better said, this paradox does not arise in the Theory I have just given for the rotation of the Moon. Thus I may, in this regard, flatter myself in having fully satisfied the first part of the question proposed by the Academy.[87]

I present a final observation on LAGRANGE's analysis. In the derivation of equation (7) the assumption that the Moon is an ellipsoid of uniform density was used only at the end, to ensure that $B - A$ is greater than zero. In making this assumption I have departed slightly from LAGRANGE'S own presentation; he actually considers this case in a series of scholia. In the more general case we need only suppose that the axes which lie in the equatorial plane are principal axes and that the physical constitution of the Moon is such that $B - A > 0$.[88] This last point is important and is not fully articulated by LAGRANGE. If $B - A < 0$, the integration of (6) will not lead to small periodic values for θ. If $B - A = 0$, equation (7) with $h = 0$ would be a possible solution; however, we would then be faced in equations (9) and (10) with the "paradox" LAGRANGE wishes to avoid. The crucial condition $B - A > 0$ and its relation to the physical constitution of the Moon emerge clearly in LAGRANGE'S analysis only when he turns to the case where this body is assumed to be an ellipsoid of uniform density.

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[87] *Oeuvres* **6** (1878) p. 45–46.

[88] In the general case LAGRANGE actually assumes these axes are almost principal in the sense that the product of inertia $\int \alpha xy$ is a very small quantity. This assumption leads to an additional small term in (7). Nothing of consequence hinges upon this assumption and I have thought it unnecessary to describe it.

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