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CHAPTER 12

LEONHARD EULER, BOOK ON THE CALCULUS OF VARIATIONS (1744)

Craig G. Fraser

In this book Euler extended known methods of the calculus of variations to form and solve differential equations for the general problem of optimizing single-integral variational quantities. He also showed how these equations could be used to represent the positions of equilibrium of elastic and flexible lines, and formulated the first rigorous dynamical variational principle.

First publication. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti, Lausanne and Geneva: Bousquet, 1744. 320 pages.

Later edition. As Euler, *Opera omnia*, series 1, vol. 24 (ed. Constantin Carathéodory), Zurich: Orell Fussli, 1952.

Partial German translations. 1) Chs. 1, 2, 5 and 6 in *Abhandlungen über Variationsrechnung* (ed. Paul Stäckel), Leipzig: Engelsmann, 1894 (*Ostwald's Klassiker der exakten Wissenschaften*, no. 46). 2) App. 1 in *Abhandlungen über das Gleichgewicht* ... (ed. H. Linsenbarth), Leipzig: Engelsmann, 1910 (*Ostwald's Klassiker*, no. 175).

Related articles: Newton (§5), Leibniz, Euler and Lagrange on the calculus (§4, §14, §19).

1 INTRODUCTION

Euler's *Methodus inveniendi* was the first of a series of books that he wrote on calculus in the 1740s and the years that followed; notable later works were the *Introductio* of 1748 on infinite series and the *Institutiones* of 1755 and 1768–1774 on the differential and integral calculus (§13, §14). Although the *Methodus inveniendi* was published in 1744, it was completed by 1741, and was written when Euler was a young man in his late twenties and early thirties at the Academy of Sciences in Saint Petersburg. Born in 1707 to a pastor in Basel in

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Switzerland, he had quickly showed his mathematical abilities, especially under the tutelage of Johann Bernoulli (1667–1748). His career fell into three parts, all served under some kind of monarchical support. The first and third parts were passed at the (new) Academy in Saint Petersburg: from 1727 to 1741 (when he wrote the *Methodus inveniendi*), and from 1766 to his death in 1783. In between he worked at the Academy in Berlin, where he wrote the other two writings that feature in this book. Apart from this trio, he was extraordinarily prolific, contributing importantly to virtually all areas of mathematics of his day [Thiele, 1982].

The *Methodus inveniendi* is of two-fold interest for historians of mathematics. First, it was a highly successful synthesis of what was then known about problems of optimization in the calculus, and presented general equational forms that became standard in the calculus of variations. Euler's method was taken up by Joseph Louis Lagrange (1736–1813) 20 years later and brilliantly adapted to produce a novel technique for solving variational problems (§16). The two appendices to Euler's book applied variational ideas to problems in statics and dynamics, and these too became the basis for Lagrange's later researches. Second, in Euler's book some of his distinctive contributions to analysis appear for the first time or very nearly the first time: the function concept, the definition of higher-order derivatives as differential coefficients; and the recognition that the calculus is fundamentally about abstract relations between variable quantities, and only secondarily about geometrical curves. The *Methodus inveniendi* is an important statement of Euler's mathematical philosophy as it had matured in the formative years of the 1730s.

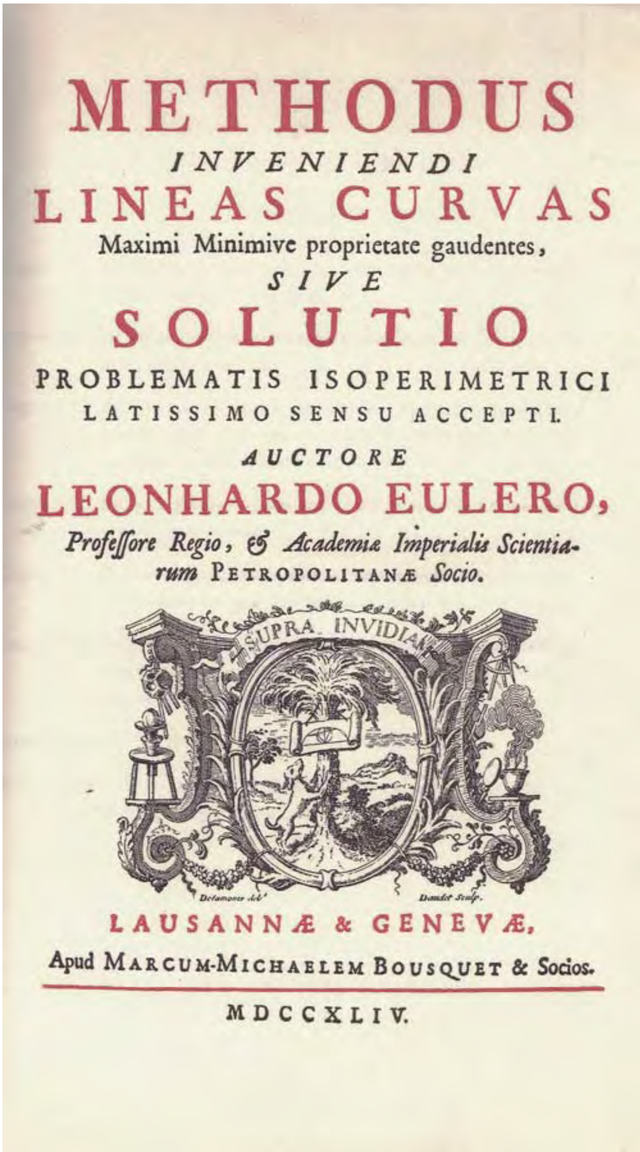
2 ORIGINS AND BASIC RESULTS

The early Leibnizian calculus consisted of a sort of geometrical analysis in which differential algebra was employed in the study of 'fine' geometry (§4.2). The curve was analysed in the infinitesimal neighbourhood of a point and related by means of an equation to its overall shape and behaviour. An important curve that was the solution of several variational problems was the cycloid, the path traced by a point on the perimeter of a circle as it rolls without slipping on a straight line. This curve appeared on the frontispiece of Euler's *Methodus inveniendi* (Figure 1) and was a kind of icon of the early calculus. The cycloid possessed a simple description in terms of the infinitesimal calculus. Let the generating circle of radius r roll along the x-axis and let the vertical distance be measured downward from the origin along the y-axis (Figure 2). An elementary geometrical argument revealed that the equation of the cycloid is

$$\left(\frac{ds}{dy}\right)^2 = \frac{2r}{y}, \tag{1}$$

where $ds = \sqrt{(dx^2 + dy^2)}$ is the differential element of path length.

The cycloid was most notably the solution to the brachistochrone problem. Consider a curve joining two points in a vertical plane and consider a particle constrained to descend along this curve. It is necessary to find the curve for which the time of descent is a minimum. Let us take the origin as the first point and let the coordinates of the second be $x = a$ and $y = b$. We assume the particle begins from rest. By Galileo's law the speed of a particle



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

Figure 1.

in constrained fall when it has fallen a distance y is $\sqrt{(2gy)}$, where g is an accelerative constant. We have the relations

$$\frac{ds}{dt} = \sqrt{2gy} \quad \text{or} \quad dt = \frac{1}{\sqrt{2gy}}\,ds = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}. \tag{2}$$

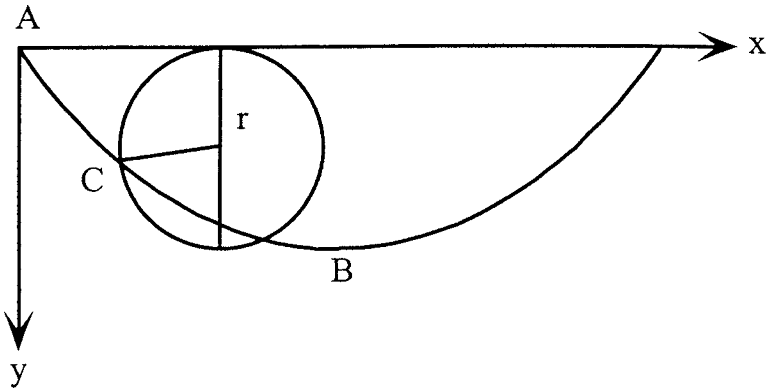


Figure 2.

Hence the total time of descent is given by the integral

$$T = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx. \tag{3}$$

The problem of the brachistochrone is to find the particular curve $y = y(x)$ that minimizes this integral.

Following Johann Bernoulli's public challenge in 1696 solutions to this problem were devised by his elder brother Jakob, by Johann himself and by Isaac Newton and G.W. Leibniz. They all showed that the condition that the time of descent is a minimum leads to (1) and, with the exception of Leibniz, concluded that the given curve is a cycloid. Johann's solution was based on an optical–mechanical analogy that is well-known today from its description by Ernst Mach in his *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt* (1883). Although of interest, his solution did not provide a suitable basis for further work in the subject.

Jakob Bernoulli's solution on the other hand was illustrative of the ideas that would develop into the calculus of variations. He considered any three points C, G and D on the hypothetical minimizing curve, where the points are assumed to be infinitesimally close to each other. He constructed a second neighbouring curve identical to the first except that the arc *CGD* was replaced by *CLD* (Figure 3). Because the curve minimizes the time of descent it is clear that the time to traverse *CGD* is equal to time to traverse *CLD*. Using this condition and the dynamical relation $ds/dt \propto \sqrt{y}$ Bernoulli was able to derive (1).

Jakob Bernoulli also investigated problems in which the minimizing or maximizing curve satisfied an auxiliary integral condition. The classical isoperimetric problem was the prototype for this class of examples. His idea was to vary the curve at two successive ordinates, thereby obtaining an additional degree of freedom, and use the side constraint to derive a differential equation. Although Jakob died in 1705, some of his ideas were taken up by Brook Taylor in his *Methodus incrementorum* of 1715. Taylor skillfully developed and refined Jakob's conception, introducing some important analytical innovations of his own. Stimulated by Taylor's research, and concerned to establish his brother's priority,

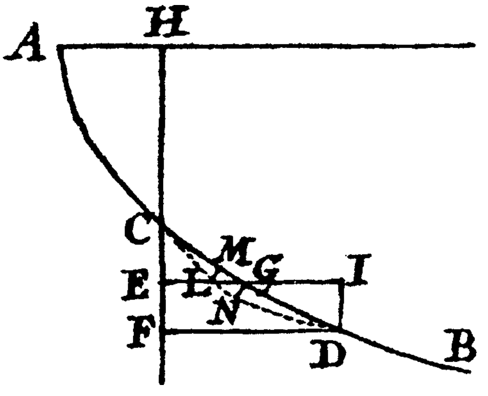


Figure 3.

Johann, then thirty-eight, also adopted Jakob's methods and developed them along more geometric lines in a paper that was published in 1719.

In two memoirs published in the St. Petersburg Academy of Sciences in 1738 and 1741, Euler extracted from the various solutions of Jakob and Johann Bernoulli, as well as the researches of Taylor, a general approach to single-integral variational problems. These investigations were further developed and became the subject of the *Methodus inveniendi*, of which the contents is summarised in Table 1. Its title may be translated 'The method of finding plane curves that show some property of maximum or minimum, or the solution of isoperimetric problems in the widest accepted sense'.

Euler realized that the different integrals in the earlier problems were all instances of the single form

$$\int_a^b Z\big(x, y, y', \ldots, y^{(n)}\big)\, dx, \tag{4}$$

where Z is a function of x, y and the first n derivatives of y with respect to x. He derived a differential equation, known today as the Euler or Euler–Lagrange equation, as a fundamental condition that must be satisfied by a solution of the variational problem.

Table 1. Contents by Chapters of Euler's book.

Part	**Page**	**Content**
Ch. 1	1	'Method of maximum and minimum' in general.
Ch. 2	32	Differential equations for the optimizing curve.
Ch. 3	83	Side conditions in the form of differential equations.
Ch. 4	130	Resolution of various problems.
Chs. 5–6	171	Isoperimetric problems.
App. 1	245	Elastic curves.
App. 2	311	Principle of least action. [End 320.]

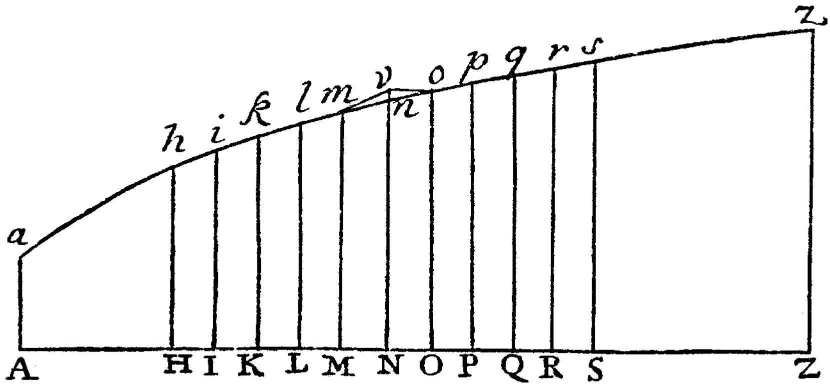


Figure 4.

In Chapter 2 Euler developed his derivation of this equation (for the case $n = 1$) with reference to Figure 4, in which the line *anz* is the hypothetical extremizing curve. The letters M, N, O designate three points of the x-axis *AZ* infinitely close together. The letters m, n, o designate corresponding points on the curve given by the ordinates Mm, Nn, Oo. Let $AM = x$, $AN = x'$, $AO = x''$ and $Mm = y$, $Nn = y'$, $Oo = y'$. The differential coefficient p is defined by the relation $dy = p\,dx$; hence $p = dy/dx$. We have the following relations

$$p = \frac{y' - y}{dx}, \qquad p' = \frac{y'' - y'}{dx}. \tag{5}$$

The integral $\int_a^b Z\,dx$ was regarded by Euler as an infinite sum of the form $\cdots + Z,$ $dx + Z\,dx + Z'\,dx + \cdots$, where $Z,$ is the value of Z at $x - dx$, Z its value at x and Z' its value at $x + dx$, and where the summation begins at $x = a$ and ends at $x = b$. It is important to note that Euler did not employ limiting processes or finite approximations. Let us increase the ordinate y' by the infinitesimal 'particle' nv, obtaining in this way a comparison curve $amvoz$. Consider the value of $\int_a^b Z\,dx$ along this curve. By hypothesis the difference between this value and the value of $\int_a^b Z\,dx$ along the actual curve will be zero. The only part of the integral that is affected by varying y' is $Z\,dx + Z'\,dx = (Z + Z')\,dx$. Euler wrote:

$$dZ = M\,dx + N\,dy + P\,dp, \qquad dZ' = M'\,dx + N'\,dy' + P'\,dp'. \tag{6}$$

He proceeded to interpret the differentials in (6) as the infinitesimal changes in Z, Z', x, y, y', p, p' that result when y' is increased by nv. From (5) we see that dp and dp' equal nv/dx and $-nv/dx$. (These changes were presented by Euler in the form of a table, with the variables in the left column and their corresponding increments in the right column.) Hence (6) becomes

$$dZ = P \cdot \frac{nv}{dx}, \qquad dZ' = N' \cdot nv - P' \cdot \frac{nv}{dx}. \tag{7}$$

Thus the total change in $\int_a^b Z\,dx$ equals $(dZ + dZ')\,dx$ or $nv \cdot (P + N'\,dx - P')$. This expression must be equated to zero. Euler set $P' - P = dP$ and replaced N' by N. He therefore obtained $0 = N\,dx - dP$ or

$$N - \frac{dP}{dx} = 0, \tag{8}$$

as the final equation of the problem.

Equation (8) is the simplest instance of the Euler differential equation, giving a condition that must be satisfied by the minimizing or maximizing arc. Noting that N and P are the partial derivatives of Z with respect to y and y' respectively, we may write (8) in modern notation as

$$\frac{\partial Z}{\partial y} - \frac{d}{dx}\frac{\partial Z}{\partial y'} = 0. \tag{9}$$

He also derived the corresponding equation when higher-order derivatives of y with respect to x appear in the variational integral. This derivation was a major theoretical achievement, representing the synthesis in one equational form of the many special cases and examples that had appeared in the work of earlier researchers.

3 FOUNDATIONS OF ANALYSIS

Near the beginning of his book Euler noted that a purely analytical interpretation of the theory is possible. Instead of seeking the curve which makes W an extremum one seeks that 'equation' between x and y which among all such equations when introduced into (1) makes the quantity W a maximum or minimum (p. 13). He wrote:

> Corollary 8. In this way questions in the doctrine of curved lines may be referred back to pure analysis. Conversely, if questions of this type in pure analysis be proposed, they may be referred to and solved by means of the doctrine of curved lines.
> Scholium 2. Although questions of this kind may be reduced to pure analysis, nevertheless it is useful to consider them as part of the doctrine of curved lines. For though indeed we may abstract from curved lines and consider absolute quantities alone, so these questions at once become abstruse and inelegant and appear to us less useful and worthwhile. For indeed methods of resolving these sorts of questions, if they are formulated in terms of abstract quantities alone, are very abstruse and troublesome, just as they become wonderfully practical and become simple to the understanding by the inspection of figures and the linear representation of quantities. So although questions of this kind may be referred to either abstract or concrete quantities it is most convenient to formulate and solve them by means of curved lines. Thus if a formula composed of x and y is given, and that equation between x and y is sought such that, the expression for y in terms of x given by the equation being substituted, there is a maximum or minimum; then we can always transform this question to the determination of the curved line, whose abscissa is x and ordinate is y, for

> which the formula W is a maximum or minimum, if the abscissa x is assumed to have a given magnitude.

Euler's view seems to have been that while it is possible in principle to approach the calculus of variations purely analytically it is more effective in practice to refer problems to the study of curves. This conclusion could hardly have seemed surprising. Each of the various examples and problems which historically made up the subject had as its explicit goal the determination of a curve; the selection of such objects was part of the defining character of this part of mathematics. What is perhaps noteworthy about Euler's discussion is that he should have considered the possibility at all of a purely analytical treatment.

The basic variational problem of maximizing or minimizing (4) involves the selection of a curve from among a class of curves. In the derivation of (8) the variables x and y are regarded as the orthogonal Cartesian coordinates of a curve. Each of the steps in this derivation involves reference to the geometrical diagram in Figure 4 above. In Chapter 4, however, Euler returned to the point of view that he had indicated at the beginning of the treatise. In the opening proposition the variational problem is formulated as one of determining that 'equation' connecting two variables x and y for which a magnitude of the form (4) (given for the general case where higher-order derivatives and auxiliary quantities are contained in Z) is a maximum or minimum. In his solution he noted that such variables can always be regarded as orthogonal coordinates and so determine a curve. The solution then follows from the theory developed in the preceding chapters. In the first corollary he wrote:

> Thus the method presented earlier may be applied widely to the determination of equations between the coordinates of a curve which makes any given expression $\int Z\,dx$ a maximum or a minimum. Indeed it may be extended to any two variables, whether they involve an arbitrary curve, or are considered purely in analytical abstraction.

Euler illustrated this claim by solving several examples using variables other than the usual rectangular Cartesian coordinates. In the first example he employed polar coordinates to find the curve of shortest length between two points (Figure 5). We are given the points A and M and a centre C; it is necessary to find the shortest curve AM joining A and M. Let x be the pole angle ACM and y the radius CM. Because the differential element of path-length is equal to $\surd(dy^2 + y^2\,dx^2)$ the formula for the total path-length is $\int dx\,\surd(yy + pp)$, where $p\,dx = dy$ and the integral is taken from $x = 0$ to $x = \angle ACM$. Here x does not appear in the integrand Z of the variational integral, so that $dZ = N\,dy + P\,dp$. The equation (8) gives $N = dP/dx$ so that we have $dZ = dP\,p + P\,dp$ and a first integral is $Z + C = Pp$, where C is a constant. Since $Z = \surd(yy + pp)$ we have

$$C + \sqrt{(yy + pp)} = \frac{pp}{\sqrt{(yy + pp)}}, \quad \text{i.e.,} \quad \frac{yy}{\sqrt{(yy + pp)}} = Const. = b. \tag{10}$$

Let PM be the tangent to the curve at M and CP the perpendicular from C to this tangent. By comparing similar triangles in Figure 5 we see that $Mm : Mn = MC : CP$. Since $Mm = dx/\sqrt{y^2 + p^2}$, $Mn = y\,dx$ and $MC = y$ it follows that $CP = y^2/\sqrt{y^2 + p^2}$. Hence CP is a constant. Euler concluded from this property that the given curve AM is a straight line.

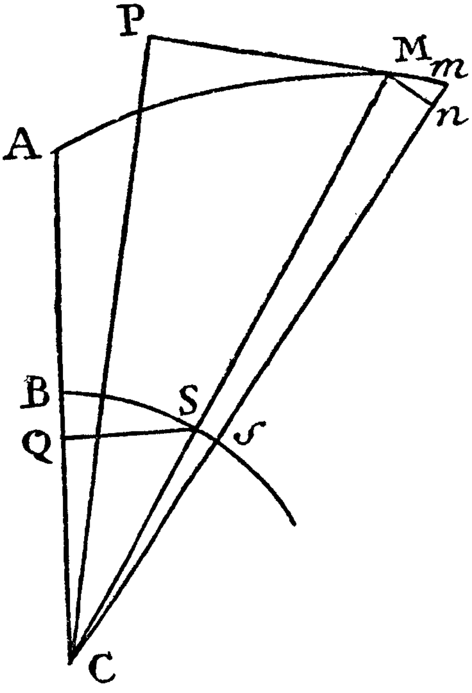


Figure 5.

In the second example Euler displayed a further level of abstraction in his choice of variables. Here we are given the axis *AC* with the points *A* and *P*, the perpendicular line *PM* and a curve *ABM* joining *A* and *M* (Figure 6). Given that the area *ABMP* is some given constant value we must find that curve *ABM* which is of the shortest length. Euler set the

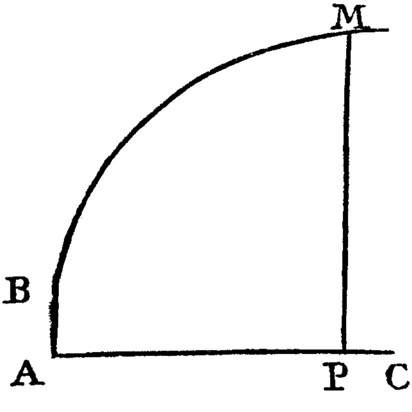


Figure 6.

abscissa $AP = t$, the ordinate $PM = y$ and let x equal the area under the curve from A to P. We have $dx = y\,dt$ and the variational integral becomes $\int \surd(dy^2 + dx^2/yy)\,dx$. Because x does not appear in the integrand we obtain as before the first integral $Z = C + pP$. Substituting the expressions for Z and P into this integral we obtain

$$\frac{\surd(1+yypp)}{y} = C + \frac{ypp}{\surd(1+yypp)}. \tag{11}$$

Letting $dx = y\,dt$, we obtain after some further reductions the final equation $t = c \pm \surd(bb - yy)$. Hence the desired curve is the arc of a circle with its centre on the axis *AP* at the foot P of the ordinate corresponding to M.

A range of non-Cartesian coordinate systems had been employed in earlier mathematics but never with the same theoretical import as in Euler's variational analysis. Here one had a fully developed mathematical process, centred on the consideration of a given analytically-expressed magnitude, in which a general equational form was seen to be valid independent of the geometric interpretation given to the variables of the problem. Thus it is not at all essential in the reasoning employed in the derivation of (9) that the line *AZ* be perpendicular to *Mm* (Figure 4); indeed it is clear that the variable x need not be a length nor even a coordinate variable in the usual sense. As Euler observed in the first corollary, the variables of the problem are abstract quantities, and Figure 4 is simply a convenient geometrical visualization of an underlying analytical process.

Euler and later 18th-century analysts broke with the geometrical tradition, but they did not thereby adopt the point of view of modern real analysis. Euler's understanding was very different from our outlook today, in which the expression Z that is to be optimized is any quantity whatsoever formulated in terms of the function $y = y(x)$ and its derivatives. For Euler, the quantities and relations of analysis are always 'given': they arise from definite problems in geometry, mechanics or some other area of mathematical science. He developed an abstract interpretation of the variational formalism—the fundamental objects of study were relations between variables 'given in analytical abstraction'—but his point of view was structured as well by tacit assumptions concerning the logical status of the problems of the subject as things that were given from without. The notion that at the outset one could consider any expression Z defined according to logically prior and autonomous criteria was quite beyond Euler's conceptual horizons and was foreign to the outlook of 18th-century analysis.

4 LATER DEVELOPMENTS: LAGRANGE, EULER AND THE CALCULUS OF VARIATIONS

In his book Euler had noted the somewhat complicated character of his variational process and called for the development of a simpler method or algorithm to obtain the variational equations. Lagrange's first important contribution to mathematics, carried out when he was 19 years old, consisted of his invention of the δ-algorithm to solve the problems of Euler's *Methodus inveniendi*. He announced his new method in a letter of 1755 to Euler, and published it as [Lagrange, 1762] in the Proceedings of the Turin Society. His algorithm permitted the systematic derivation of the variational equations and facilitated the treatment of conditions at the endpoints. His innovation was immediately adopted by Euler, who introduced the name 'calculus of variations' to describe the subject founded on the new method. Lagrange's new approach originated in his (tacit) recognition that the symbol d was being used in two distinct ways in Euler's derivation of (8). In (8) and the final step by which it is obtained, d was used to denote the differential as it was customarily used and understand in Continental analysis of the period. The differential dx was held constant; the differential of any other variable equalled the difference of its value at x and its value at an abscissa a distance dx from x. By contrast, the differentials dx, dy, etc. that appear in (6) were interpreted by Euler as the changes in x, y, etc. that result when the single ordinate y is increased by the 'particle' nv. Thus the 'differentials' dy' dp, dp' equal nv, nv/dx, $-nv/dx$; the 'differentials' dx, dy, dp'', etc. are zero.

The young Lagrange had the perspicacity to recognize this dual usage and invented the symbol 'δ' to denote the second type of differential change. Using it he devised a new analytical process to investigate problems of maxima and minima. Although the purpose of his method was to compare curves in the plane, it was nonetheless introduced in a very formal manner. The symbol δ has properties analogous to the usual d of the differential calculus. Thus $\delta(x+y)=\delta x+\delta y$ and $\delta(xy)=x\delta y+y\delta x$. In addition, d and δ are interchangeable ($d\delta=\delta d$) as are d and the integral operation $\int$.

The δ-process led to a new and very simple derivation of the Euler equation (8). It is necessary to determine $y=y(x)$ so that

$$\delta\int_a^b Z\,dx=0, \tag{12}$$

where $Z=Z(x,y,p)$ and $p=dy/dx$. Applying the δ operation to the expression Z we obtain

$$\delta Z=N\delta y+P\delta p. \tag{13}$$

Note that here all of the ordinates are simultaneously being varied, and not just one, as had been the case in Euler's analysis. Because the δ and $\int$ are interchangeable we have

$$\delta\int_a^b Z\,dx=\int_a^b \delta Z\,dx=\int_a^b (N\delta y+P\delta p)\,dx \tag{14}$$

and also $\delta p=\delta(dy/dx)=d(\delta y)/dx$. An integration by parts gives rise to the identity

$$\int_a^b P\delta p\,dx=\int_a^b P\frac{d(\delta y)}{dx}\,dx=P\delta y|_a^b-\int_a^b\frac{dP}{dx}\delta y\,dx. \tag{15}$$

Hence the condition $\delta\int_a^b Z\,dx=0$ becomes

$$P\delta y|_a^b-\int_a^b\left(N-\frac{dP}{dx}\right)\delta y\,dx=0. \tag{16}$$

We suppose that δy is zero at the end values $x = a, b$. (16) then reduces to

$$\int_a^b \left(N - \frac{dP}{dx}\right)\delta y \, dx = 0. \tag{17}$$

From (17) we are able to infer the Euler equation

$$N - \frac{dP}{dx} = 0. \tag{18}$$

Euler took up Lagrange's new method in his writings of the 1760s and 1770s. In a paper published in 1772 he presented what would become the standard interpretation of the δ-process as a means for comparing classes of curves or functions. We assume that y is a function of x and a parameter t, $y = y(x, t)$, where the given curve $y = y(x)$ is given by the value of $y(x, t)$ at $t = 0$. We define δy to be $\frac{\partial y}{\partial t}|_{t=0}\, dt$. (It would be logically more consistent to define $\delta y = \frac{\partial y}{\partial t}|_{t=0} t$, and require that t be small. Euler apparently used dt rather than t so as to indicate explicitly that the multiplicative factor is small.) One way of doing this, Euler explained, is to set $y(x, t) = X(x) + tV(x)$, where $y(x) = X(x)$ is the given curve and $V(x)$ is a comparison or increment function; hence we have $\delta y = dt\, V(x)$. In this conception the variation of a more complicated expression made up of $y(x, t)$ and its derivatives with respect to x is obtained by taking the partial derivative with respect to t, setting $t = 0$ and introducing the multiplicative factor dt. In later variational mathematics the parameter 'ε' would often be used instead of 't'.

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