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Osserman, Robert (1-STF)

Mathematics of the gateway arch.

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The expository article under review is one of the last publications of Robert Osserman, a distinguished American mathematician who died in November of 2011 at age 85. The inspiration for the article is the famous Gateway Arch in St. Louis, Missouri. The basic premise is that there is a connection between the problem of the shape of a hanging chain in equilibrium and the shape of an ideal stable arch. Both curves are described in terms of hyperbolic trigonometric functions. At any point of a hanging chain the weight of the chain from this point to the bottom of the chain must equal the vertical component of the tension in the chain at that point. The horizontal component is constant and equal to the tension at the lowest point. The curve that the chain assumes is called a catenary (the name comes from the Latin word "catena" for "chain"). In an ideal stable arch the forces of compression at each point of the arch act along the tangent to the centroid curve of the arch at the point. The total force of compression at any point of the arch results from the weight of the arch from that point to the highest point of the arch. The arch for which these facts are true is an inverted catenary. While much of the article is devoted to the hanging chain, the associated theory is understood to apply to arched structures.

The article opens by examining the original statement of the catenary problem by Galileo. Galileo believed that the shape of the chain is parabolic. Using methods of the calculus of variations, researchers around 1700 were able to derive a differential equation to describe the catenary. They obtained this result from the assumption that the centre of gravity of the chain is at its lowest point in equilibrium. In later mathematics the functions that arose in the integration of the differential equation were called hyperbolic trigonometric functions.

In his account, the author eschews a variational approach in favour of a purely mathematical analysis of the hanging chain. A general mathematical description is set out that is taken to be consistent with the underlying physics of the hanging chain. The resulting approach will probably appeal more to mathematicians than to physicists and engineers.

A section in the middle of the article is a kind of extended parenthesis and describes some results in differential geometry, focussing on tubes through n-dimensional space. The material is abstract, and, while very interesting, is somewhat detached from the primary subject of the article. Only in a general sense are the theorems here applicable to engineering structures in the world around us.

In the last part of the article the author discusses the geometry of the Gateway Arch. The Arch is not a simple catenary but rather a modified or weighted catenary. Consideration is given to the values of the constants entering into its equation involving the cosh function. The catenary curve and variants of this curve are analysed in terms of their curvature properties at different points.

Both the hanging chain and the Gateway Arch are described by hyperbolic trigonometric functions. However, the Arch is not governed by a variational law similar to the one that describes the hanging chain. The two structures are in fact very different: the hanging chain is completely flexible, while the Arch is a rigid body. The hanging chain naturally takes the shape of a catenary in equilibrium, whereas the catenary arch has the form assigned to it by its maker. The article concludes by raising the possibility that the catenary arch might also be characterized using a variational principle. This possibility is presented as a subject for further investigation. *Craig G. Fraser*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.