Mechanics in the Eighteenth Century

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Abstract and Keywords

This article focuses on mechanics in the eighteenth century. The publication in 1687 of Isaac Newton's Mathematical Principles of Natural Philosophy has long been regarded as the event that ushered in the modern period in mathematical physics. The success and scope of the Principia heralded the arrival of mechanics as the model for the mathematical investigation of nature. This subject would be at the cutting edge of science for the next two centuries. This article first provides an overview of the fundamental principles and theorems of mechanics, including the principles of inertia and relativity, before discussing the dynamics of rigid bodies. It also considers the formulation of mechanics by Jean-Baptiste le Rond d'Alembert and Joseph-Louis Lagrange, the statics and dynamics of elastic bodies, and the mechanics of fluids. Finally, it describes major developments in celestial mechanics.

Keywords: mechanics, Isaac Newton, Mathematical Principles of Natural Philosophy, mathematical physics, rigid bodies, Jean-Baptiste le Rond d'Alembert, Joseph-Louis Lagrange, elastic bodies, fluids, celestial mechanics

12.1 Introduction

The publication in 1687 of Isaac Newton's *Mathematical Principles of Natural Philosophy* has long been regarded as the event that ushered in the modern period in mathematical physics. Newton developed a set of techniques and methods based on a geometric form of the differential and integral calculus for dealing with the motion of a single mass-point, and further showed how the results obtained could be applied to the motion of the solar system. Other topics studied in the *Principia* included the motion of bodies in resisting fluids and the propagation of disturbances through continuous media. The success and scope of the *Principia* heralded the arrival of mechanics as the model for the mathematical investigation of nature. This subject would be at the cutting edge of science for the next two centuries.

The view that the entire modern edifice of classical mechanics can be traced back to the Principia was promulgated by Ernst Mach in his famous book *The Science of Mechanics A Critical and Historical Account of Its Development*, first published in 1883. In introducing the period in history following Newton, Mach wrote:

> The principles of Newton suffice by themselves, without the introduction of any new laws, to explore thoroughly every mechanical phenomenon practically occurring, whether it belongs to statics or to dynamics. If difficulties arise in any such consideration, they are invariably of a mathematical, or formal, character, and in no respect concerned with questions of principle. (Mach 1883, Eng. trans., p. 256)

(p. 359) Mach's opinion has influenced scholars for over a century. However, while *The Science of Mechanics* is deservedly regarded as a classic of the history of science, it was mainly intended as a contribution to the epistemology of physics, and in this respect it was a work of fundamental importance. Nevertheless, most of its historical comments were taken from secondary sources and it displayed a limited sense for what was achieved in the eighteenth century. Mach was also mistaken in his positivistic conviction that it is possible to neatly divide science into the physical investigation of the phenomena on the one hand, and the development of a mathematical theory on the other. In fact, the appearance in the eighteenth century of new physical principles and modes of description was organically linked to the mathematical elaboration of a coherent theory of mechanics. Since Mach composed his book, historians Emile Jouguet, René Dugas, Stephen Timoshenko, Clifford Truesdell and István Szabó, as well as many others, have drawn a different picture of mechanics in the age of the Enlightenment.

There is now a consensus among historians of science that the generation after Newton was left with a problematic legacy. The Newtonian theory had not been developed enough to deal with systems of interacting bodies subjected to constraints, nor to solve most of the problems in continuum mechanics. In addition, the infinitesimal calculus—the mathematical language of the new mechanics—was then of a relatively recent invention and lacked a well-established and generally accepted foundation. To overcome these difficulties required the efforts of a group of exceptionally gifted scientists: Gottfried Leibniz, Pierre Varignon, the brothers Jacob and Johann Bernoulli and the latter's son Daniel, Jacob Hermann, Leonhard Euler, Brook Taylor, Alexis Clairaut, Jean d'Alembert. and Joseph-Louis Lagrange. Their highly creative approach to a wide range of problems shaped the classical theory handed down to us today.

In the light of these developments, the old image of the period from Newton's *Principia* (1687) to Lagrange's *Analytical Mechanics* (1788) as uniformly dominated by a purely formal revision of the principles must be discarded in favour of a more complex pattern of change. There was at first a period of translation of Newton's mechanics into the analytic language of Leibniz's differential calculus, culminating with the publication of Euler's *Mechanics* (1736). During the 1740s, new principles and mathematical methods were created at an almost unbelievable pace by a handful of mathematicians—Johann and Daniel Bernoulli, Euler, d'Alembert and Clairaut—working in more or less friendly rivalry. The collection of special results that they produced was revised, completed, and unified in the 1750s by Euler. From the 1760s onward the resultant theories were refined and formalized by a new generation of scientists, among whom Lagrange and Laplace figure prominently.

The crucial event of this history was the generalization and completion of pre-existing partial theories, a project to which Euler was the leading contributor. The magnitude of this transformation can be compared in terms of its impact on physical theory with the advent of quantum mechanics in the twentieth century. While the concept of revolution in science has been overused and criticized, it is not an exaggeration to call this period of profound change in mathematical physics the analytic revolution. It was characterized by the creation of theories for systems of mass-points, (p. 360) celestial objects, rigid bodies, fluids and linearly elastic bodies; they were based on a highly effective formulation of the principles of mechanics and expressed by means of sets of ordinary or partial differential equations. It is the ensemble of these theories that forms the core of classical mechanics, usually called ‘Newtonian’.

12.2 General Principles

The history of the fundamental principles and theorems of mechanics is still partially uncharted territory. Much work on the eighteenth century has been done by historians in the last fifty years, but the general picture is not yet fully understood. The following account is only a summary, clearly not exhaustive. (In order not to confound the reader with outdated notations, formulae will be given in a slightly modernized form.)

Systems of units. Since in the *Principia* mechanics was formulated in geometric fashion, Newton did not need any rational system of units for physical quantities. In the following decades, Varignon and Johann Bernoulli began the process of transforming mechanics into a purely analytical science. Some of their formulas look strange to a modern student, for they are, in effect, a shorthand for Euclidean proportions: physical constants are missing, numerical factors disappear, and results are expressed in the form of proportionalities. Beginning in the late 1730s, Daniel Bernoulli and Euler used a system of units based on length and weight. In Euler’s ‘Dissertation on the best way to construct a pulley’ (1745) we find an early example of the dimensional checking of a formula. Euler was well aware of the importance of the question of physical units, which is mentioned time and again in his memoirs. He systematized these ideas in his *Theory of the Motion of Solid or Rigid Bodies* (1765a), which remained the most advanced presentation of the problem of dimensions until the appearance of Fourier’s *Analytical Theory of Heat* (1822). (See Ravetz (1961B) and Roche (1998).)

The principle of inertia. While Newton regarded the principle of inertia as an axiom, some mathematicians of the eighteenth century hoped to demonstrate it. The argument given by Euler in his *Mechanics* (1736) consists essentially in the remark that, in the absence of external forces, there is no reason for a point moving with a given velocity to change its speed or the direction of its motion. Similar ideas were later put into a more elaborate mathematical form by d'Alembert, leading to a non-trivial functional equation (1768a). However, Euler reverted to inertia as an axiom in the *Theory of the Motion of Solid or Rigid Bodies* (1765a), in which he analysed the question at length. Other remarks are scattered throughout his works: in the 'Investigations on the origin of forces' (1752b), for example, he criticized the locution 'vis inertiae' as denoting something which is not a force.

Euler's most substantial contribution to the subject is his 'Reflections on space and time' (1750), in which he discussed Newton's absolute space. Although modern (p. 361) writers beginning with Mach have criticized the concept of absolute space, Newton and Euler found it to be a useful and meaningful notion. Euler wrote that 'anyone who wishes to deny absolute space will fall into the most serious inconvenience.' Indeed modern mathematical presentations of mechanics define the intrinsic geometry of classical space-time before stating the laws of motion. Newton, with his famous rotating-bucket experiment, had related absolute space to centrifugal forces; in the 'Reflections,' Euler instead emphasized its connection with inertial motion. In short, for Euler, the existence of inertial motion independent of the distribution of matter in the universe demonstrated the existence of a structure of space itself. Going beyond Newton, he also remarked that the uniform motion of free mass-points defines absolute time. This way of looking at the nature of space anticipated some of the ideas of general relativity.

The principle of relativity. The classical principle of relativity was well known to Galileo Galilei, Christiaan Huygens and Newton. In the eighteenth century Euler systematized its use. He first discussed the matter in his *Mechanics* (1736), where he noted that observers at rest in the absolute space, or uniformly moving along a constant direction with respect to it, experience the same physical laws. Therefore, he remarked, 'we will not be very concerned with absolute motion, since relative motion is ruled by the same laws.' Euler gave a more detailed account of the whole question in the introductory chapters of *Theory of the Motion of Solid or Rigid Bodies* (1765a), in which he showed that 'the same differential equations are obtained both for absolute and relative motion; the distinction is discerned in the integration, for each case must be duly adapted to the initial conditions.' Moreover, he calculated the corrections for a non-inertial observer. A discussion of these matters also occurs in Euler's *Guide to Natural Science* (1862, probably written shortly after 1755) in relation to the problem of establishing a unitary theory of physics. The foundations laid by Newton and Euler for the problem of absolute and relative motion were strong enough to sustain physics until the advent of Maxwellian electromagnetism. (See Bertoloni Meli (1993) and Maltese (2000).)

Vectors. While the use of the word 'vector' to describe physical entities of the eighteenth century is slightly anachronistic, at that time there was already a clear distinction between directed quantities, represented by line segments in diagrams, and pure numbers. In the first decades of the century, the parallelogram rule was applied routinely to the composition of forces and velocities. Euler, in his *Mechanics* (1736), resolved acceleration along the tangent and the normal to the trajectory. (See Dugas (1957) and Radelet-de Grave (1996).)

However, with the discovery of the general laws of rigid body motion around 1750, more complicated vector quantities appeared. In 1759 the Italian mathematician Paolo Frisi demonstrated that two infinitesimal rotations about concurrent axes 'can be composed into one exactly in the same way that two forces, represented by the two sides of a parallelogram, are composed into a third force represented by the diagonal.' The corresponding result for moments of forces came much later. In 1780 Euler found the analytical expression for the moment of a force about any axis through the origin, and immediately recognized its similarity to the well-known formula (p. 362) for the projection of a force along a given direction; this meant that the moment of a force was a line-segment, not just a number (1793). In the last lines of his paper, Euler remarked that 'this marvellous harmony deserves to be considered with the greatest attention, for in general mechanics it can deliver no small development.'

Unfortunately, these two important discoveries lay dormant for years. Euler's papers were read only at the beginning of the next century; their content was then reworked by Laplace, Prony, and Poisson. Independently of Euler, in 1803, Louis Poinsot developed a purely geometric vector theory of moments. Frisi's theorem was rediscovered or appropriated by Lagrange in the first volume of the second edition of his *Analytical Mechanics*, published in 1811. At the beginning of the nineteenth century a new structure of mechanics emerged from these discoveries—one in which most of the fundamental entities were vectors. These ideas played an important part in the creation of vector calculus. (See Caparrini (2002, 2003, 2007).)

Composition of forces. The paradigmatic example of the parallelogram rule for the sum of vectors is the composition of forces. Curiously enough, from the time of Newton to the end of the nineteenth century it was essentially considered a theorem, obtainable from first principles. Every mathematician of some repute, and several of the less famous, tested his ability by looking for a proof. It was for decades the most popular theorem of applied mathematics. Significant proofs were given by Daniel Bernoulli (1728), who considered it a theorem in pure geometry and formulated a demonstration in rigorous Euclidean fashion, and by Daviet de Foncenex (1762), who translated the geometrical problem into a functional equation.

While in the end the search proved illusory, the critique of the different proofs resulted in a deepening of the criteria of rigour in physics. Thus, for example, mathematicians began to wonder if statics had priority over kinematics, or were led to explore the connections between the parallelogram of forces, the law of the lever, and the principle of virtual work. After the discovery of non-Euclidean geometry it was shown that some of the proofs retained part of their validity in spaces of constant curvature. (See Bonola (1912), Benvenuto (1991) and Radelet-de Grave (1987, 1998).)

Newton's second law of motion. Our textbooks usually attribute the law $F = ma$ to Newton, but this formula is nowhere to be found in the *Principia*. In fact, rather than an equation, Newton had a principl—roughly equivalent to the assertion that the force acting on a particle is proportional to the variation of momentum—which he applied to a single, unconstrained mass-point. Newton's principle covered both the case of impulsive forces (force proportional to $\Delta(mv)$) and of continuous motion (force proportional to the acceleration).

After the publication of the *Principia*, it took decades for Newton's *lex secunda* to reach the modern formulation. The analytic form $f = dv/dt$ for a particle under the action of gravity was used by Varignon (1703), while the more general expression(12.1)

$$f = m\frac{dv}{dt}$$

first appeared in Hermann's *Phoronomia* (1716). Euler (1738) discussed the impact of hard bodies as a continuous deformation, thus attempting to subsume impulsive (p. 363) forces under the case of a continuous acceleration. By the end of the 1720s the differential form of the second law was well-known to everyone working in dynamics. However, the points of view concerning its significance differed greatly. In his *Mechanics* (1736) Euler tried unconvincingly to prove the formula from the kinematics of accelerated motion. On the contrary, Daniel Bernoulli remarked that this was a clear example of knowledge derived from experience; one could easily imagine, for example, acceleration to be proportional to the square or the cube of the force (1728). D'Alembert held instead that 'force' was simply a formalism for the exchange of momentum, and the second law its mathematical definition (1743).

The final step in this long process was induced by the latest discoveries in continuum mechanics. In the 1740s d'Alembert and Johann Bernoulli had determined respectively the equations of the vibrating string and of perfect fluids by applying the relation between force and acceleration. Their successes led Euler to postulate that the 'true and genuine method' in dynamics consisted in applying the formulas(12.2)

$$F_x = ma_x, F_y = ma_y, F_z = ma_z,$$

expressed in orthogonal Cartesian coordinates, to every particle of the system, taking into account both the applied forces and the constraints (1752a). This apparently simple idea yielded a harvest of results in a matter of few years. (See Hankins (1967), Truesdell (1968), Cohen (1971), Blay (1992) and Guicciardini (2009).)

Momentum and moment of momentum. While the intuitive ideas that underlie these two concepts—respectively, the impetus of a moving body and the law of the lever—go back to antiquity, the analytic formulation for systems of bodies was obtained only in the eighteenth century. In 1740 Euler, in his *Naval Science* (1749), demonstrated two particular but important cases of the general principles. The first is the formula(12.3)

$$\mathbf{R} = \mathbf{M}\mathbf{a}_{\mathbf{G}}$$

where **R** is the resultant of the applied forces, M is the total mass and $\mathbf{a_G}$ is the acceleration of the centre of mass; the second is the law of moment of momentum in the special case of a rigid body turning about a fixed axis (see the next section on rigid bodies). Shortly afterwards, both Euler (1746) and Daniel Bernoulli (1746) gave the proof of the conservation of angular momentum for a mass-point sliding along a smooth tube rotating in a horizontal plane. (This is an early example of a first integral of the equations of motion.) By that time, several theoreticians were working on the same problems; in a letter to Euler of 23 April 1743, Clairaut remarked as follows: 'I was charmed by your theorem on the conservation of rotational moments, but what struck me is that if I had reflected a little on my equations, I also would have found it.' The conservation of angular momentum was also enunciated by the Chevalier d'Arcy (1752) in the form of the law of areas—a generalization of Kepler's second law of planetary motion.

The general principle that the rate of change of the total angular momentum is equal to the sum of moments of the applied forces appeared—somehow hidden under the formalism—in d'Alembert's *Researches on the Precession of the Equinoxes* (1749). (p. 364) Euler (1752A) immediately gave a clearer formulation of it. Both d'Alembert and Euler apparently thought that the internal forces do not influence the motion of the system, an assumption that was discussed by subsequent authors. The relation between first principles and general theorems in dynamics was somewhat clarified when Lagrange, (1779) derived the main integrals of motion for a system of mutually gravitating mass-points from the Newtonian laws.

Towards the end of his life, Euler (1776B) wrote down explicitly the two relations(12.4)

$$\mathbf{R} = \mathbf{M}, \mathbf{L_O} = \dot{\mathbf{H}},$$

where **R** is the resultant of the external forces, **M** the total momentum, **L** the sum of the moments of the external forces about a fixed point O and **H** the total moment of momentum about O, as expressing the laws of motion of a mechanical system; it is not clear if he considered them axioms or theorems. With the publication of Lagrange's *Analytical Mechanics* (1788), the conservation of momentum and of moment of momentum became consequences of more general principles. (See Truesdell (1968) and Caparrini (1999).)

Energy. The eighteenth-century conservation of *vis viva* (live force) mv^2 may be interpreted as an early form of our conservation of mechanical energy. There were several formulations of this principle; the best known is the equality between 'actual descent' (that is, kinetic energy) and 'potential ascent' (that is, the height of the centre of mass). Apart from its significance in philosophy (Leibnizians vs. Cartesians), the *vis viva* controversy, which opposed partisans of mv and mv^2, was important in discussions of the fundamental axioms of mechanics during the first half of the century.

The most active propagandist of *vis viva* was Johann Bernoulli, who took the idea from Huygens and Leibniz and elevated it to the rank of general principle for the dynamics of systems. With its help, he was able to solve several tricky problems (1729, 1735). This role of the *vis viva* in early eighteenth-century mechanics can be seen, for example, in the correspondence between Johann Bernoulli and Euler. In a letter of 2 April 1737 to Euler, Bernoulli expressed the hope that in the *Mechanics* (1736), then just out of the press, Euler would make use of *vis viva*; Euler replied that he did not need it, because 'these volumes are not the place to present the theory of living forces. It will however appear in the following volumes, where the motion of bodies of limited extent will be thoroughly considered.'

The proponents of *vis viva* knew that there was something that could be dissipated into microscopic motion. This was in effect the main criticism of Johann Bernoulli to his son Daniel's use of the conservation of *vis viva* in the *Hydrodynamics* (1738): at discontinuities of a channel, a finite part of the *vis viva* of flowing water went into the formation of vortices.

During the 1740s both d'Alembert (1743) and Daniel Bernoulli (1750) demonstrated the conservation of *vis viva* from the relations between force and motion. From that point on, the principle became a first integral of motion; the formulation given in Lagrange's *Analytical Mechanics* (1788) is formally the same as the modern one. In Lazare Carnot's *Essay on Machines in General* (1782) there is a description of the (p. 365) conservation of *vis viva* in more physical terms that might be seen to resemble the modern principle of conservation of mechanical energy as it is presented in textbooks today. (For more on Carnot's essay see (Fraser 1983).)

The concept of mechanical work has a different history. It appeared implicitly at the end of the seventeenth century in the solution of problems in technical mechanics (How can we measure the work of several men? How can we compare it with the work of a horse?), and was incorporated into higher dynamics by Euler and Daniel Bernoulli. The formal relation(12.5)

$$2\int_1^2 f ds = mv_1^2 - mv_2^2$$

was derived from the second law of motion by Varignon (1703).

During the first decades of the nineteenth century the conservation of *vis viva* was discussed in every textbook of theoretical or applied mechanics. However, while in books on abstract mechanics this was just a corollary of the basic laws, in texts on applied mechanics it was employed as the basic principle. The two traditions appeared back to back in the second edition of Poisson's *Treatise of Mechanics* (1833). English texts of the period even made frequent use of the word 'energy' to denote the intensity of a force, and in books on mathematical mechanics the terms *vis viva* and kinetic energy were used interchangeably well into the twentieth century. Historian Thomas Kuhn (1959) has argued that the French practice in the early nineteenth century of denoting *vis viva* with a *1/2* factor (with each side of eq. (12.5) being divided by *2*) was significant as an explicit recognition of the conceptual priority of work in the equations of mechanics. However, in many of the actual writings of the period, the issue of the *1/2* factor did not arise or amounted to a secondary question, more related to typographical problems than to scientific issues.

It could be argued that several of the ingredients of the general principle of the conservation of energy were already present in mathematical physics at the end of the eighteenth century. It is therefore not a surprise, as Kuhn (1959) has observed, that between 1830 and 1850 twelve scientists, independently, announced that heat and work are manifestations of a single ‘force’ of nature. Kuhn remarks that ‘the history of science offers no more striking instance of the phenomenon known as simultaneous discovery.’ Part of the explanation is, obviously, that all of these discoverers shared a common scientific background, rooted in a well-established tradition of engineering textbooks. When faced with the same problem, they reacted in similar ways. The connection between the old conservation of *vis viva* in mechanics and its extension to all of physics is especially apparent in Hermann von Helmholtz’s *On the Conservation of Force* (1847), which takes as its starting point Lagrange’s mechanics. These few examples suffice to show that in considering the history of the conservation of energy during the nineteenth century, special attention should be paid to eighteenth-century sources. (See Kuhn (1959), Hiebert (1962), Hankins (1965), Pacey and Fisher (1967), Truesdell (1968), Mikhailov (2002), Villaggio (2008), and Fonteneau (2008, 2009).)

Potential. The concept of potential emerged in several widely different contexts. The unifying background for these apparently unrelated developments was (p. 366) the application of partial differential equations to mathematical physics from the 1740s onward.

Gravitational potential appeared implicitly in the form of the integral of force in Johann Bernoulli’s formula for the motion of a point under the action of a central force (1712). Bernoulli did not connect his result with the conservation of *vis viva*; this step was taken in 1738 by his son Daniel, who also explicitly gave the form of the potential in the case of Newtonian attraction (1747). After an interval of more than twenty years, Lagrange began to make extensive use of the potential, at least as a formal mathematical entity in the equations of mechanics. He first applied it to the study of astronomical perturbations (1776), then to the general motion of mass-points subjected to mutual gravitational interactions (1779), and, finally, he inserted the potential in the equations of mechanics that bear his name (1782) (see eq. (12.17)).

The potential as stored *vis viva* in an elastic system was first mentioned by Daniel Bernoulli in a letter to Euler of 20 October 1743: ‘For a naturally elastic band, I express the potential live force of the curved band as $\int ds/r^2$ […] Since no one has perfected the isoperimetric method as much as you, you will easily solve this problem of making $\int ds/r^2$ a minimum.’ Euler proceeded immediately to develop this suggestion in his ‘On the elastic curve’ (1744). (This essay is discussed in Section 12.5.) A few years later Euler introduced the potential in fluid dynamics; believing (erroneously) that every stationary flux of an incompressible homogeneous fluid is irrotational, he applied the first elements of differential forms and arrived at ‘Laplace’s equation’(12.6)

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} = 0,$$

where S is the velocity potential (1761a). Moreover, following d'Alembert's lead in the use of complex numbers in two-dimensional fluid dynamics, Euler (1757C) also introduced what we would now call a complex potential. (See Wilson (1995B) and Truesdell (1954).)

Laplace's name is associated with the famous partial differential equation for the potential in a region free of sources. He first wrote the equation in two variables (1779), then in three variables in spherical coordinates (1785) and lastly in three variables in Cartesian coordinates (1789). Finally, he made it central in his *Treatise on Celestial Mechanics* (1798–1825).

12.3 Rigid Bodies

The dynamics of rigid bodies—as opposed to their statics—is a creation of the eighteenth century. In fact, before the end of the 1730s there were few results worthy of notice. In 1673 Huygens had determined the motion of a compound pendulum using an indirect principle equivalent to the conservation of mechanical energy, and in 1703 Jacob Bernoulli had addressed the same problem by the equilibrium of the (p. 367) moments of lost forces. For more complicated systems, nothing was known. Newton had left only a few remarks on finite-sized rigid bodies, most of them erroneous (see Dobson (1998)). As late as the early 1730s, Euler was still unable to write a chapter on rigid bodies for his *Mechanics* (1736).

During the first decades of the eighteenth century the question of finding at least *some* rules for rigid body dynamics surfaced in a number of special problems: oscillations of pendula, naval science, the impact of hard bodies, the rolling of round objects down inclined planes, and the precession of the equinoxes. *Ad hoc* hypotheses had to be put forward in each case, and the theory was usually limited to two-dimensional systems.

The first successful attack on the general problem appeared at the beginning of Euler's *Naval Science*, completed in 1741 but published only in 1749. Euler observed that the motion of the centre of mass and the motion relative to it are quite independent of each other. The simple case of a body with a fixed axis could be treated by considering an equivalent mass-point (that is. having the same moment of inertia about the axis as the body) acted upon by an equivalent force (that is, the same moment about the axis as the totality of forces), leading to the relation(12.7)

$$\frac{d^2\theta}{dt^2} = \frac{L}{I},$$

where θ is the angular rotation of the body, L is the moment of the applied forces and I is the moment of inertia of the body, both about the axis of rotation. (It is here that the locution 'moment of inertia' originated.) This equation, as Euler remarked, is formally similar to Newton's second law. Euler also calculated the moments of inertia of several common shapes and demonstrated the theorem of parallel axes (today usually called ‘Steiner’s theorem’).

Lacking a treatment of the general case, the best that Euler could do at this stage of his investigation was to conjecture that in every rigid body there are three orthogonal axes through the centre of mass, about which the body can oscillate without wobbling. This hypothesis made it possible to study the small motions of a rigid body as a superposition of independent pendular oscillations. A few years later (1745) he discovered that the rotation about an axis is stable only if the two appropriate products of inertia are both equal to zero. (For a thorough analysis of *Naval Science*, see the essay by Habicht (1978).)

In the same years, the aged Johann Bernoulli, following an idea proposed by his son Daniel, made a fundamental contribution to two-dimensional rigid-body mechanics (1742b). Bernoulli considered a planar rigid body at rest, set in motion by a force acting on an arbitrary point, and tried to determine the position of the centre of the initial rotation. To solve the problem he applied ‘the principle of the lever’—that is, the equilibrium of moments—and obtained the correct solution. He termed the point in question *the centre of spontaneous rotation*.

However innovative, Bernoulli’s work also contained significant errors. He believed that a three-dimensional rigid system starting from rest under the action of an impulsive force would begin to rotate about an axis orthogonal to the plane (p. 368) passing through the line of action of the force and the centre of mass. Moreover, in considering the oscillations of a compound pendulum, he tried to reduce the problem to the motion of an equivalent particle, and was thus led to use moments of inertia about the wrong axis. These errors clearly demonstrate that the main properties of rotational inertia were still very imperfectly known. (On Johann Bernoulli’s mechanics, see the extensive essay by Villaggio (2007).)

The next giant step forward was made by d’Alembert in his *Researches on the Precession of the Equinoxes* (1749), which contained a correct description of the motion of an ellipsoid of rotation about its centre of mass under the action of the gravitational forces exerted by two distant mass-points. In doing so, d’Alembert obtained the general equations of equilibrium of a mechanical system, applied the principle of moment of momentum to every infinitesimal part of the body and, most important of all, demonstrated the existence of the *instantaneous axis of rotation*, the key concept in the kinematics of a rigid body. However, these results were presented in a very difficult and confused way, making d’Alembert’s book a flawed masterpiece. With respect to rigid-body dynamics, the near misses are as relevant as the positive achievements: the role of the moments of inertia was buried under the geometrical symmetry of the body, the instantaneous axis of rotation played a secondary role with respect to the axis of figure and, more importantly, there were no general dynamical equations. (For an analysis of d’Alembert’s book, see (Wilson 1987, 1995A), (Nakata 2000) and (Chapront-Touzé and Jean Souchay 2006).)

Stimulated by d'Alembert's discovery of the instantaneous axis of rotation, which made the general kinematics of a rigid body analogous to the simple case of the rotation about a fixed axis, Euler wrote his famous memoir 'Discovery of a new principle of mechanics', published in 1752. The 'new principle' referred to in the title is the set of three equations relating forces and motion for a rigid body turning about its centre of mass. These equations, he wrote, are 'the subject of this memoir, which I finally arrived at after several useless attempts undertaken over a long time.' What Euler achieved here, expressed with his usual clarity, surpassed by far every previous effort. (For an appraisal of Euler's theory of rigid bodies, see (Wilson 1987, 1995) and (Langton 2007). Euler's theory has also been described by Blanc (1948, 1968).)

The core of the memoir was about pure kinematics. Referring every point to three fixed orthogonal axes meeting in the centre of mass, and making use only of the hypothesis of rigidity, Euler determined the formulae for the velocity,(12.8)

$$\frac{dx}{dt} = \lambda x - \mu z, \frac{dy}{dt} = \nu z - \lambda x, \frac{dz}{dt} = \mu x - \nu y,$$

where x, y, z are the coordinates of the point and the coefficients λ, μ, ν depend only on time. From these expressions, the axis of instantaneous rotation could be found easily. A modern reader may interpret immediately λ, μ, ν as the components of the angular velocity vector, but the vectorial character of angular velocity was then unknown. (For an account of the genesis and development of the vector idea in mechanics in the eighteenth and nineteenth centuries, see (Caparrini 2002, 2003, 2007).)

(p. 369) The dynamics followed naturally from the fundamental Newtonian principle that force is proportional to acceleration. Differentiating the velocities, Euler obtained the accelerations, and from the accelerations the forces, and from the forces their moments about the coordinate axes. The final result was a set of formulae expressing the moments of the forces in terms of the moments and products of inertia about the three coordinate axes and of the components of the angular velocity.

But the general problem was far from settled, for these equations were extremely difficult to solve. In effect, if the rotating body is referred to fixed axes, the moments of inertia depend on time and must be determined from the equations themselves. To overcome this difficulty, in 1751 Euler launched a fresh attack on the problem. (This pattern of reiterated assaults over the years was typically Eulerian.) The memoir 'On the motion of a rotating solid body about a mobile axis' was published only in 1767, by which time it had been superseded by more sophisticated works, but it is of interest to historians for reconstructing Euler's chain of thought.

At the very beginning, Euler introduced two different frames of reference with a common origin in the centre of mass—one fixed in space, the other rotating with the body. The expressions for the basic kinematic quantities were first determined with respect to the fixed frame, then transformed into the rotating frame. It was a method based on calculations rather than principles; no use was made of any general principle on the relations between mutually rotating frames. After a *tour de force* of algebra and spherical trigonometry, Euler finally reached the formulae for the accelerations in the moving frame. By the same process he had utilized in his previous memoir, he then obtained a simplified form of the fundamental equations, though still encumbered by the products of inertia.

With complicated but manageable equations at his disposal, Euler was able to get some definite results. He studied the conditions which allow an axis of rotation to be fixed, determined the motion when the resultant moments are zero (leading to an elliptic integral) and, in particular, examined the case in which the body is a solid of revolution.

At the end of the memoir, Euler wrote that all that remained to be done was to apply the ‘new principle’ to specific cases. As a matter of fact, there was still a long way to go. In 1755 his friend Johann Andreas von Segner, professor of mathematics and physics at the University of Halle, published the booklet *Essay on the theory of rotations*, in which he demonstrated that every rigid body has at least three axes of permanent rotation. This result obviously could lead to further simplifications, as Euler proceeded to show. His memoir ‘Research on the mechanical knowledge of bodies’ (published 1765) was a fairly complete exposition of the properties of the centre of mass and of the moments of inertia, differing from standard nineteenth-century textbooks mainly in the absence of the ellipsoid of inertia (introduced by Augustin Cauchy in 1827) and in its great attention to foundational issues. It established the existence of rotational inertia, as distinguished from linear inertia.

Several classic results made their first appearance in this essay. Euler remarked that the centre of gravity (the common name of the centre of mass at the time) might (p. 370) be more aptly called centre of inertia, and demonstrated that its position does not depend on the choice of the axes. Turning to the problem of calculating the moments of inertia, he argued that the set of all moments of inertia through the centre of mass should have a maximum and a minimum. Euler showed that the two axes enjoying this property are also axes of permanent rotation, that they are perpendicular to each other, and that there is a third such axis perpendicular to the other two. He gave them their current name: principal axes.

Euler now had at his disposal all the ingredients for a successful theory. The final version was given in the memoir ‘On the motion of rotation of solid bodies About a variable axis’, (1765c). The main problem consisted in finding the motion of a generic point of the body with respect to a fixed frame. The usual formulae for the velocities followed, then the accelerations. Surprisingly, at this point Euler remarked that the expressions for the accelerations found in the fixed frame were also valid in the moving frame at the given moment, if the fixed axes were instantaneously coincident with the moving ones. While this might at first seems unlikely (in a rotating frame, inertial accelerations must be taken into account), it is absolutely correct. Euler’s justification of this assertion amounted in effect to a verbal description of the relation(12.9)

$$\left(\frac{d\varpi}{dt}\right)_{fixed\ frame} = \left(\frac{d\varpi}{dt}\right)_{rotating\ frame},$$

which apparently he could somehow visualize. In addition, he took as moving axes the principal axes of inertia. The fundamental equations were then found in their definitive form,(12.10)

$$\frac{dp}{dt} + \frac{c^2 - b^2}{a^2}qr = \frac{P}{Ma^2}, \frac{dq}{dt} + \frac{a^2 - c^2}{b^2}rp = \frac{Q}{Mb^2}, \frac{dr}{dt} + \frac{b^2 - a^2}{c^2}pq = \frac{R}{Mc^2},$$

where p, q, r are the components of the angular velocity, M is the mass of the body, Ma^2, Mb^2, Mc^2 and P, Q, R are respectively the moments of inertia and the moments of the external forces about the coordinate axes.

Having finally arrived a correct theory, in 1760 Euler wove together all the different strands of his research on rigid bodies in his *Theory of the Motion of Solid or Rigid Bodies* (1765), a milestone in the history of mechanics. A modern reader is likely to be surprised by the abundance of special cases Euler explored before he arrived at the fundamental equations. He took this approach not only for didactic reasons, but also because of the difficulty of applying the general formulae to particular problems. Every previous work was made obsolete by the appearance of this fascinating treatise; Johann Bernoulli's problem of the centre of spontaneous rotation, for example, became a simple corollary of a theorem on plane motion.

Before the publication of Euler's book the young Lagrange had made his entrance into the field with a section of his long memoir 'Application of the method presented in the preceding memoir to the solution of different problems of dynamics' (1762). (The previous method referred to in the title was his new δ-method in the calculus of variations, a subject we examine in the next section.) Lagrange's aim was to derive **(p. 371)** the main properties of rigid motion using a variational principle and to subsume the physics to differential algebra. While he was successful in these respects, the tone of the work is very formal, and there is no critical discussion of the concepts. Thus, for example, the proposition that every infinitesimal displacement of a rigid body about a fixed point can be considered as the superposition of three infinitesimal rotations about three orthogonal axes meeting at the point is considered 'easy to see' and is not proved at all. Some years later, Lagrange returned to the subject with the memoir 'New solution of the problem of the motion of a body of any shape that is not subject to any external accelerative force' (1775), in which he tried to find 'a solution completely direct and purely analytical of the question under consideration.' The approach taken involved a minimum of physical assumptions. Lagrange's point of view on the formulation of mechanics, obviously very different from Euler's, marks the beginning of a period of formalization of the discoveries made in the preceding decades.

By the time that Lagrange's 'New solution' was published, Euler was 68 years old and almost completely blind. The task of reading his younger contemporary's memoir proved to be nearly impossible. Not being able to understand what Lagrange had done, Euler felt that his own solution of the problem had been criticized for being too complicated. He therefore went back through his old work, believing that simplification could only be achieved by a deeper analysis of the geometry. He studied the finite motions of a rigid body, thus initiating the analytic representation of displacements. He also demonstrated by synthetic geometry that any rigid motion with a fixed point is equivalent to a rotation. Euler went on to express the position of a generic point of a free rigid body as a function of its initial position and of the time. Putting these formulae into the principles of momentum and of moment of momentum, he obtained a different form of the fundamental equations, in which the instantaneous rotation is expressed by the rate of change of what we would now call the elements of the orthogonal transformation matrix.

Unbeknownst to Euler, a more elegant theorem on the finite motions of a rigid body had been obtained about ten years before by the Italian poet, mathematician and politician Giulio Mozzi in his short book *A Mathematical Discourse on the Instantaneous Rotation of Bodies* (1763). Mozzi's theory is mainly a generalization to three dimensions of Johann Bernoulli's work on rigid bodies set in motion by impulsive forces. His most important result was the demonstration that any finite displacement of a rigid body is equivalent to a screw motion along a fixed line. This theorem remained unnoticed, and was rediscovered only in the early 1830s. (For more information on Mozzi, see (Marcolongo 1905) and (Ceccarelli 2007).)

The theory of rigid bodies was discussed by Lagrange in the *Analytical Mechanics* (1788) and by Laplace in the first volume of the *Treatise on Celestial Mechanics* (1798). Both accounts drew heavily on Euler. A fuller exposition along Eulerian lines was given by Simé on-Denis Poisson in the *Treatise of Mechanics* (1811). It is mainly from these sources, directly or indirectly, that the scientists of the nineteenth century learned the subject.

(p. 372) 12.4 The d'Alembert–Lagrange Formulation of Mechanics

Jean-Baptiste le Rond d'Alembert was a remarkable savant—almost a man born out of his time, who possessed a deep understanding of technical and foundational issues. In subjects as diverse as the fundamental theorem of algebra, the metaphysics of the calculus, the nature of functions and the principles of mechanics he displayed an acute critical sense, grasping issues that would only become the focus of study much later. These gifts were evident early in his career in his seminal *Treatise on Dynamics* (1743). This book was an investigation of the constrained interaction of bodies: the collision of spheres, the motion of pendula, the movement of bodies as they slide past each other, and various other connected systems. Many of the problems would today be studied as part of engineering mechanics. His basic conception was that of a 'hard body.' Such a body is impenetrable and non-deformable. Assume a small hard sphere hits a wall with a velocity that is perpendicular to the wall. When the sphere hits the wall all motion ceases. The closest modern approximation to d'Alembert's conception is that of a perfectly inelastic body, although it must be emphasized that d'Alembert's point of view was different from the modern one. D'Alembert thought in a Cartesian way of hard bodies as geometrical solids in motion, whose laws of interaction could be determined by deductive reasoning from *a priori* postulates or principles. (Hankins (1970) documents the importance of Cartesian philosophy in d'Alembert's scientific thought.) In this conception dynamics is very similar to geometry, where the properties of the objects under study are derivable from a few postulates that are believed to be necessarily true.

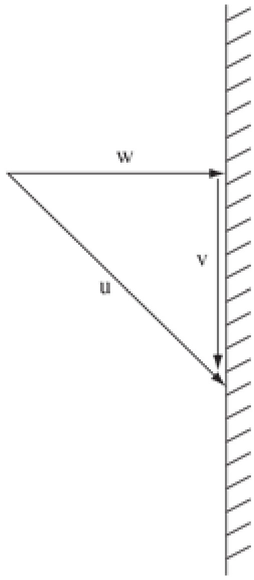

Fig. 12.1 . D'Alembert's principle.

Central to d'Alembert's dynamics was a principle that he enunciated at the beginning of the *Treatise* and which in various later forms became known as 'd'Alembert's principle.' (The account which follows is based on (Fraser 1985).) In its original and most basic form the principle may be illustrated by the example of a hard particle that strikes a wall obliquely with velocity *u* (Fig. 12.1). We must determine the velocity of the particle following impact. Decompose *u* into two components *v* and *w*, *v* being the post-impact velocity and *w* being the velocity that is 'lost' in the collision. D'Alembert's principle asserts that if the particle were animated by the lost velocity alone then equilibrium would subsist. From this condition it follows that *w* must be the component of *u* that is perpendicular to the wall. Hence *v* is the component of *u* that is parallel to the wall, and the collision problem is solved.

Assume now that two hard bodies *m* and *M* approach each other with velocities *u* and *U* along the line joining their centres. It is necessary to find the velocities after impact. We write $u = v + (u - v)$ and $U = V + (U - V)$, where *v* and *V* are the post-impact velocities of *m* and *M*. The quantities *u*, *v* and $u - v$ are the impressed, actual and 'lost' motions of the body *m*; a similar decomposition holds for *M*. Because *v* and *V* are followed unchanged *v* must equal *V*. In addition, the application of the lost velocities $u - v$ and $U - V$ to *m* and *M* must produce equilibrium. D'Alembert (p. 373) reasoned from the very concept of hard body itself that for this to happen we must have $m(u - v) + M(U - V) = 0$. Hence *v* or *V* is equal to $(mu + MU)/(M + m)$.

In modern dynamics we would analyse this collision using what are known as impulsive forces. It is assumed that in the collision very large forces act for a very short period of time. Integrating over these forces and using Newton's second law we are able to calculate the changes of velocity that result in the collision. In modern dynamics d'Alembert's principle involves a decomposition of forces or accelerations and is basically a statement combining Newton's second and third laws. By using impulsive forces in the modern principle and integrating, we can produce a decomposition of velocities that looks somewhat (and misleadingly) like the one d'Alembert originally presented. However, d'Alembert's point of view was very different. From the outset in his analysis of the collision of the two bodies he used a decomposition involving finite velocities. There are no forces, and the entire interaction is analysed using the conception of a hard body and the assumption—supported by *a priori* reasoning—that equilibrium would subsist if the bodies were animated by the motions they lose in the collision.

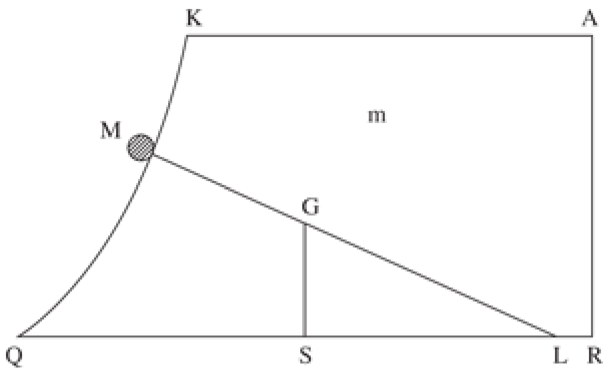

Fig. 12.2 . D'Alembert's problem X.

In examples involving forces that act continuously and in which the motion is continuous, d'Alembert analysed the system in a way that has some similarities to the model of instantaneous impulses set out by Newton in the opening proposition of Book One of the *Principia*. The motion is understood to consist of a succession of discrete impulses in which each impulse arises in an interaction of hard bodies or surfaces. The interaction is described in terms of finite velocities and infinitesimal velocity increments where the lost motions are governed by d'Alembert's principle. A clear example of d'Alembert's treatment of continuous forces is the tenth problem of the *Treatise*. Here all of the features of d'Alembert's theory come into play—the concept of hard body, d'Alembert's principle, and the Leibnizian differential calculus. (p. 374) We are given an irregularly shaped object *KARQ* of mass *m* which is free to slide along a frictionless plane QR (Fig. 12.2). A body of mass *M* is situated on the curve *KQ* which forms the left edge of *KARQ*. A force acts on *M* in a direction perpendicular to *QR*. The two bodies possess given initial velocities (in the particular configuration adopted by d'Alembert it is assumed that the bodies initially possess a common motion to the right). The problem is to determine the motion of the system as *M* slides down *KQ*.

In d'Alembert's solution the trajectory traced by *M* is regarded as a polygon with an infinite number of sides in which the length of each side is infinitesimal. This geometric representation of *M*'s trajectory corresponds to the physical analysis in which *M*'s motion is understood to be the outcome of a succession of discrete dynamical events. The model of a continuous curve as an infinite-sided polygon is also used in the solution to represent the left edge of the body *m*.

Although d'Alembert's original principle involved a decomposition of velocities, he subsequently extended the principle in certain problems to what was in effect a decomposition of accelerations. He did so in some of the problems of his treatise involving pendulum motion, and adopted a similar formulation in his later researches in hydromechanics and theoretical astronomy. Assume that a body *m* is part of a constrained system of bodies and is acted upon by an external impressed force. At a given instant let $dv^{(I)}$ be the increment of velocity that would be imparted to the body by the impressed force if the body were free. Let v^{+} be the velocity of the body in the next instant, so that $dv = v^{+} - v$ is the actual increment of velocity experienced by the body. We have the decomposition of velocities $v + dv^{(I)} = v^{+} + w = (v + dv) + w$, or(12.11)

$$v + dv^{(I)} = (v + dv) + w,$$

where $w = dv^{(I)} - dv$ is the lost velocity of the body. By d'Alembert's principle equilibrium will subsist if each body of the system were subjected to the motion $dv^{(I)} - dv$. In any given problem we now invoke a suitable statical law and obtain a relation among $-mdv$ and $mdv^{(I)}$. In the problems d'Alembert considered this procedure gave rise to a system of differential equations that described the motion of the system.

(p. 375) We may re-express eq. (12.11) in the form(12.12)

$$dv^{(I)} = dv + w,$$

If eq. (12.12) is divided by *dt* and multiplied by *m* this becomes an equation involving forces:(12.13)

$$\mathbf{F}_a + \mathbf{F}_c = \mathbf{ma},$$

where $\mathbf{F_a}$ is the applied force acting on *m*, $\mathbf{F_c}$ is the constraint force on *m*, and $\mathbf{a}$ is the acceleration of *m*. (For convenience here and in what follows we use modern vector notation, although this notation was not used in the eighteenth century.) In this formulation d'Alembert's principle states that equilibrium would subsist if each of the bodies of the system were animated by the constraint force; that is, the constraint forces considered as a set of applied forces acting on the bodies result in a system in static equilibrium.

Although Lagrange was influenced by d'Alembert, his own development of mechanics occurred along lines that were significantly different from his older contemporary. Physical hypotheses about the ultimate nature of mechanical interactions were absent, and any adherence to a geometric-differential form of the calculus was rejected altogether. (Fraser (1983) and Panza (2003) explore the foundations of Lagrange's mechanics.) Lagrange's goal was to reduce mechanics to a branch of applied analysis in which the emphasis was primarily on the derivation and integration of differential equations to describe the motion of the system. Following his mathematical philosophy, he eschewed diagrams and geometric modes of representation in favour of a purely analytic approach involving operations, formulas and equations among variables. In his first extended memoir on the principles of mechanics Lagrange (1762) used what he called the principle of least action as the starting point for his analysis of the motion of a dynamical system. This principle was an integral variational law and its application was governed by the calculus of variations, a mathematical subject that Lagrange had pioneered in the very same volume. In subsequent investigations Lagrange abandoned the least action principle. Instead he combined d'Alembert's principle (in the form involving eq. (12.13)) and what is today called the principle of virtual work to arrive at the fundamental axiom of his presentation of mechanics. Thus in his mature theory of mechanics, he used methods and operations derived from the calculus of variations, but he did not develop the subject from an integral variational principle, as he had done in 1762.

Consider a set of external forces acting on a connected system of bodies, and suppose that the system is in static equilibrium under the action of these forces. Consider a small virtual velocity or displacement $\delta\mathbf{r}$ of a given body *m* of the system. Such a displacement is taken to be compatible with the constraints in the system. As before let $\mathbf{F}$ be the external or applied force acting on *m*. The principle of virtual velocities asserts that a general condition for equilibrium of the system is given by the equation(12.14)

$$\sum_m \mathbf{F} \cdot \delta\mathbf{r} = \mathbf{0}.$$

(p. 376) Consider now any constrained mechanical system, not assumed to be in equilibrium. For each body *m* of the system we have the decomposition $\mathbf{F_a} + \mathbf{F_c} = \mathbf{ma}$ (eq. (12.13)). By d'Alembert's principle the given system would be in equilibrium if each *m* were animated by $\mathbf{F_c}$, where this force is now understood as an external force acting on *m* within the connected system. By the principle of virtual velocities we have(12.15)

$$\sum_m \mathbf{F_c} \cdot \delta\mathbf{r} = \mathbf{0},$$

which we may, using eq. (12.13), express as(12.16)

$$\sum_m m\mathbf{a} \cdot \delta\mathbf{r} = \sum_m \mathbf{F}_a \cdot \delta\mathbf{r}.$$

Eq. (12.16) is a statement of the generalized principle of virtual work and is the starting point for Lagrange's theory of mechanics.

Beginning with eq. (12.16), Lagrange derived a system of differential equations to describe the motion of the system. Using the constraints, one reduces the description of the system to the specification of *n* 'generalized' variables $q_1, q_2, \ldots, q_m$. (If the system consists of *n* bodies moving freely then $m = 3n$ and the q_i would be the $3n$ Cartesian coordinates of the bodies.) Each of these variables is independent of the others, and each is a function of time. A system with suitably smooth constraints is then described by the m differential equations(12.17)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial Q}{\partial q_i} = 0,$$

Here $\dot{q}_i$ denotes the time derivative of q_i, the function T is what later would become known as the kinetic energy of the system and Q_i is what later would be called the potential, although these terms were not used by Lagrange. Eq. (12.17) became known in later dynamics as the Lagrangian equations of motion.

12.5 Statics of Elastic Bodies

The elastic behaviour of rods and beams emerged as a subject of interest in the early eighteenth century in two related problem of statics. In the problem of fracture one attempted to determine the maximum load that a beam of given material and dimensions can sustain without breaking. Typically it was assumed that the beam was cantilevered to a wall and that the rupture took place close to the wall. In the problem of elastic bending one was concerned with determining the shape assumed by a rod or lamina in equilibrium when subject to external forces. In the example of the elastica, where the elastic rod was treated mathematically as a line, the forces were assumed to act at the ends of the rod and to cause the rod to bend into a curve. The first problem had been considered by Galileo and had attracted the attention of (p. 377) Varignon and Antoine Parent, among others. Jakob Bernoulli initiated the study of the second problem, and his work became the basis for further researches by Euler. (For histories of elasticity in the eighteenth century see (Truesdell 1960) and (Szabo 1977). The subject of strength of materials is explored by Timoshenko (1953).)

It is important to note that research on elasticity was carried out without the general theoretical perspective that is provided today by the concept of elastic stress. This concept, which underlies such basic modern formulas as the stress–strain relation and the flexure formula, only emerged explicitly in the 1820s in the writings of Claude Navier and Augustin Cauchy. Although one can discern in the earlier work some of the elements that enter into the modern concept of stress, the essential idea—that of cutting a body by an arbitrary plane and considering forces per unit area acting across this plane—was absent.

The divide that separates the modern theory and that of the eighteenth century is illustrated by the problem of elastic bending. Consider the derivation today of the formula for the bending moment of a beam. One begins by assuming that there is a neutral axis running through the beam that neither stretches nor contracts in bending. We apply elementary stress analysis and consider at an arbitrary point of the beam a cross-sectional plane cutting transversely the neutral axis. Elastic stresses distributed over the section are assumed to act across it. Calculation of their moment about the line that lies in the section, is perpendicular to the plane of bending, and passes through the neutral axis leads to the flexure formula, $M = SI/c$, where M is the bending moment, I is the moment of area of the section about the line, c is the distance of the outermost unit of area of the section from the line, and S is the stress at this outermost area.

In the problem of fracture eighteenth-century researchers obtained results that can be readily interpreted in terms of modern formulas and theory. Typically they assumed that the beam was joined transversely to a wall and that the rupture occurred at the joining with the wall. Here the physical situation directly concentrated attention on the plane of fracture—something of concrete significance and no mere analytical abstraction. The conception then current of the loaded beam as comprised of longitudinal fibres in tension is readily understood today in terms of stresses acting across this plane.

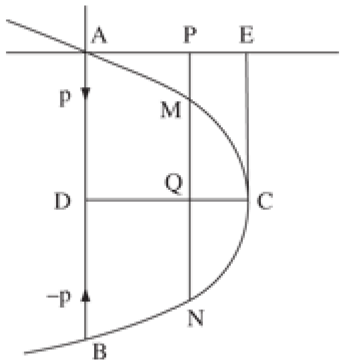

Fig. 12.3 . Euler's elastic curve.

By contrast, in the problem of elastic bending researchers were much slower to develop an analysis that connected the phenomenon in question to the internal structure of the beam. Here there was nothing in the physical situation that identified for immediate study any particular cross-sectional plane. In all of Jakob Bernoulli's seminal writings on the elastica the central idea of stress fails to receive clear identification and development.

Although a general theory did not emerge in the eighteenth century, there were many partial results and successful analyses of particular problems. We will consider one such result in some detail: the derivation by Euler of the buckling formula for a loaded column. (An account of Euler's results is given by Fraser (1990).) Euler obtained this result as a corollary to his analysis of the elastica. We are given an elastic lamina oriented vertically, in which the ends A and B are pinned and forces P and $-P$ (p. 378) act vertically at A and B to produce the stressed configuration depicted in Fig. 12.3. Consider a Cartesian coordinate system in which A is the origin, the positive y-axis is aligned vertically downward, and the positive x-axis extends horizontally to the right. Consider a point M on the lamina. The moment at M exerted by the external force P is equal to Px. This quantity is equated to the Ek^2/R^2, giving rise to the equation(12.18)

$$Px = \frac{Ek^2}{R^2}.$$

Here R is the radius of curvature of the lamina at M, E is a quantity that measures the stiffness of the lamina at M, and k is a constant related to the cross-sectional dimensions of the lamina at M. For x considered as a function of y the quantity R is given in terms of the first and second derivatives of x with respect to y at M. Eq. (12.18) is then a differential equation, and its solution will be an elliptic integral. Euler's primary goal was to undertake a detailed graphical analysis of the geometric forms assumed by solutions of this equation, thus obtaining a classification of the different shapes of the elastic curve.

If the tangent at A to the x-axis is close to 90°, then the resulting configuration of the elastica provides a model for a loaded column that is beginning to buckle. Euler showed that the size of the force P producing this configuration is(12.19)

$$P = \pi^2 \cdot \frac{Ek^2}{L^2},$$

where L is equal to AB, the length of the column. A remarkable implication of this result is that a finite force at least equal to eq. (12.19) is required to produce even the smallest bending of the column. In later mechanics eq. (12.19) would become known as Euler's buckling formula, and the value P in eq. (12.19) as the first Euler critical load. Euler believed there was something paradoxical about the result, since a load greater than a specified magnitude was necessary to produce any bending at all of the column.

Euler's project of giving a detailed enumeration of the different transcendental curves that satisfy the differential equation (12.18) failed to become a prominent concern of later research in the theory of elasticity, and does not seem to have been closely (p. 379) connected to the technological aspects of the problem. In engineering statics today one develops a theory for a given object—a beam, strut, or column—by laying down a coordinate system along its unstressed configuration and analyzing small distortions from this position. In Euler's approach the elastic curve provided a unified model for all of these objects and the coordinate system was oriented once and for all with respect to the direction of the external force. Because Euler did not have at his disposal the stress concept, he was not able to connect the quantity Ek^2 to the internal structure of the elastic lamina at M.

12.6 Dynamics of Elastic Bodies: Vibration Theory

In the first part of the eighteenth century researchers investigated a range of problems involving small vibrations: the propagation of pressure waves through continuous media; the vibrations of strings, rings, and rods; the oscillations of linked pendulums and hanging chains; the bobbing and rocking of a floating body. Typically, Newton's second law was applied successfully only to systems involving one degree of freedom, while various special methods were devised to extend the analysis to more general systems. It was not until the 1750s that the analytical equations which express the general principle of linear momentum had become established as the cornerstone of dynamical theory.

Progress in dynamics during the period is illustrated by investigation of the motion of an elastic string. (The following account is based on (Cannon and Dostrovsky 1981). See also (Fraser 1983).) Interest in this problem was motivated by music theory, as well as by the study of engineering structures involving elastic vibration. The detailed physical examination of the corpuscular structure of strings and wires was carried out by investigators such as Willem 'sGravesande (see (Bucciarelli 2003, 39–56) for details). We shall concentrate here on the more purely mathematical advances made in the description of the string's motion. Assume a string is stretched along the z-axis in the $z-y$ plane from $z=0$ to $z=l$ with tension P and mass density ρ. Consider an element ρdz of the string with coordinates z and y. The string is given a small displacement from its equilibrium position; the problem is to determine the shape it assumes in the ensuing small vibrations and to calculate the period and frequency of these vibrations. In a memoir published in 1732, Daniel Bernoulli first showed using elementary geometry that the force acting on the mass element ρdz equals $P(d^2y/dz^2)dz$ and acts in a direction perpendicular to the z-axis. Intuitively this result is plausible, since the restoring force will be proportional to the curvature (the greater the curvature, the greater the distortion and therefore the greater the restoring force), and for configurations of the string in which dy/dz is small, the curvature is given by d^2y/dz^2. Bernoulli imagined that each mass element of the string (p. 380) vibrates as would a simple pendulum of length L, where L is to be determined. If the displacement of an element ρdz suspended from such a pendulum equals y then the restoring force is given by $-(g/L)(\rho dz)y$, where g is the acceleration due to gravity. This force in turn must equal the actual force calculated above:(12.20)

$$P(d^2y/dz^2)dz = -(g/L)(\rho dz)y,$$

from which we obtain the equation(12.21)

$$d^2y/dz^2 = (-g/L)(\rho/P)y$$

which may now be integrated to give(12.22)

$$y = \pm c\sin(\sqrt{(g/L)(\rho/P)}z),$$

where we have expressed in functional notation what Bernoulli described in geometrical language. (Bernoulli stated that the solution to eq. (12.21) is the curve known as the 'companion to the trochoid' (cycloid)—a characterization of the sine function which survived well into the eighteenth century and is explained by Kline (1972, 351).) Using the end condition $y=0$ when $z=l$ we obtain a value for L:(12.23)

$$L = (g\rho/P)(l^2/\pi^2).$$

Bernoulli had discovered that the vibrating string assumes the shape of a sine curve. It remained to find the period and frequency of the vibrations. This, however, was now straightforward, since the device used in analyzing the string reduced the problem of finding these values to the analysis of the small vibrations of a simple pendulum of length

L. The latter system, involving one degree of freedom, was known from Newton's second law to be governed by the equation(12.24)

$$\ddot{y} = -(g/L)y.$$

The solution of this equation, $y = h\sin(\sqrt{(g/L)}t + \delta)$, immediately yields the desired values for the period T and frequency υ:(12.25)

$$T = 2\pi\sqrt{(L/g)},\ \upsilon = \sqrt{(g/L)}/2\pi.$$

Using the value for L given by eq. (12.23) we obtain the final result for the vibrating string:(12.26)

$$T = 2l\sqrt{(\rho/P)},\ \upsilon = \sqrt{(P/\rho)}/2l.$$

Bernoulli had therefore derived by mathematical analysis the result known in musical theory of the seventeenth century as 'Mersenne's law', asserting the proportionality of the pitch or frequency to the quantity $\sqrt{(P/\rho)}/2l$.

To solve this problem Bernoulli assumed that the elements of the string undergo small vibrations as simple pendulums all of the same period. Such an assumption was used widely during the period to analyse oscillatory phenomena. Its adoption tended to be combined with certain restrictions on the motion. Thus Bernoulli assumed in his analysis that the elements of the string arrive simultaneously from one side at the equilibrium configuration along the z-axis.

(p. 381) In the 1740s d'Alembert developed an analysis of the vibrating string that proceeded along new and very different lines. His innovative work was made possible by two advances. First, the proportionality of force to the time rate of change of momentum had come to be recognized as a general principle that could be applied to a wide range of physical problems. Second, during the 1730s researchers gained experience with partial differential equations as they investigated such mathematical problems as the construction of orthogonal trajectories to families of curves (see (Engelsman 1984)). The vertical displacement y of the element ρdz of the string depends on both z and t. D'Alembert equated the restoring force $P(d^2y/dz^2)dz$ on ρdz to the time rate of change of its momentum, the latter being given as $\rho dz(d^2y/dt^2)$. For convenience we shall use the notation for partial derivatives that became standard in the nineteenth century. The partial differential equation d'Alembert had obtained may be written in the form:(12.27)

$$\frac{\partial^2 y}{\partial t^2} = k^2 \frac{\partial^2 y}{\partial z^2},$$

where $k^2 = P/\rho$. In later mathematical physics eq. (12.27) was called the wave equation. In a brilliant piece of analysis, d'Alembert integrated this equation, showing that its solution is given as(12.28)

$$y = \Phi(z + kt) + \Phi(z - kt),$$

where Φ is a function that is 'arbitrary', subject to the conditions that it be odd ($\Phi(z) = -\Phi(-z)$), zero at $z = 0$ and $z = l$, and periodic of period $2l$. The configuration of the string at time $t = 0$ is given by the function $y = \Phi(z)$. The class of physical solutions to the problem will therefore depend on the mathematical question of what an arbitrary function is—a question that became a matter of considerable discussion and controversy between d'Alembert and Euler.

In the eighteenth century a function was an object given by a single analytical expression—a formula constructed from variables and constants in finitely many steps using algebraic and transcendental operations and the composition of functions. Such functions were termed 'continuous', in opposition to 'discontinuous' functions—expressions defined piecewise over more than one interval of real numbers. (This definition of continuity is very different from the arithmetic conception of continuity in modern calculus.) D'Alembert insisted that only an equation $y = \Phi(z)$ in which $\Phi(z)$ was an analytical expression was acceptable as an initial solution to the wave equation; hence he would permit only functions that were given by single analytical expressions. Since the initial solution $y = \Phi(x)$ to the vibrating string had been shown to be periodic, one could allow only those functions that were, by virtue of their algebraic form, periodic. (Ravetz (1961A), Youschkevitch (1976), and Fraser (1989) explore the mathematical foundational issues in the debate between d'Alembert and Euler.)

Euler welcomed d'Alembert's derivation and integration of the wave equation as a major advance, but could not accept the restrictions that d'Alembert had imposed on the possible initial solutions. According to d'Alembert, an initial shape of the string (p. 382) that was given by an arc of the parabola $y = cz(l - z)$ would be unacceptable because the expression $cz(l - z)$ was non-periodic. Euler saw no reason why one could not translate the arc of the parabola along the horizontal axis, reflecting it about this axis on every second interval of length l, thus obtaining a new periodic curve. The curve as specified would be given by a periodic function that in analytical terms was defined piecewise over each interval of length l. In the solution $y = \Phi(z)$ the function symbol Φ would now refer to different algebraic expressions depending on the interval of real numbers to which z belonged.

On grounds of physical plausibility and mathematical generality, Euler advocated the acceptance of 'discontinuous' functions as initial solutions to the wave equation. D'Alembert maintained that such functions violated assumptions tacitly invoked in the derivation and integration of this equation. The calculus studied definite, given analytical expressions corresponding to natural modes of generation, and the inclusion of Euler's more general functions violated the basic principles of the subject.

The debate over the nature of arbitrary functions touched the very foundation of analysis and ranks as one of the most interesting episodes of eighteenth-century exact science. Nevertheless, the issues at stake remained somewhat isolated from the mainstream of contemporary mathematical practice. The general functions advocated by Euler raised foundational problems that could not be resolved given the current state of research. D'Alembert's opposition may have seemed obstinate, but it also displayed a clear sense for the spirit of the calculus. The distinguishing property of a 'discontinuous' function—that its algebraic form depended on the interval of real numbers to which the independent variable belonged—undermined the basis of the calculus as a subject explicable by formal analytical principles. To defend the introduction of these objects Euler appealed to the physical model, without any clear specification of the corresponding mathematical conditions. His notion of a more general function was never incorporated into the analytical theory presented in his famous mid-century textbooks, and indeed was at odds with its basic direction.

12.7 Mechanics of Fluids

At the beginning of the eighteenth century, hydrostatics, after the works of Simon Stevin, Evangelista Torricelli, and Blaise Pascal, was reasonably well understood. It was clear that a theory could be based on a few basic principles: the force exerted by a liquid is normal to the surface, it does not depend on the orientation of the surface element, and it is equal to the weight of the column of fluid above the surface. A general and concise formulation of these principles appears at the very beginning of Euler's *Naval Science* (1749). (For general overviews of fluid mechanics in the eighteenth century, see works by Truesdell (1954; 1955), Mikhailov (1983; 1994; 1999), Szabó (1977), Simón Calero (1996), Cross (2002), Darrigol (2005), Blay (2007), and Darrigol and Uriel Frisch (2008).)

(p. 383) On the contrary, the dynamics of fluid motion was just taking its first steps. The only part of the subject that had been thoroughly studied was Torricelli's law for the efflux of water from a hole in the base of the vessel. Pierre Varignon, Johann Bernoulli, Jacopo Riccati, and Jakob Hermann had produced several proofs, which unfortunately mainly showed the need for general principles. A systematic attack on some of the fundamental questions of fluid mechanics, most of them completely new at the time, had been attempted by Newton in Book 2 of the *Principia*: the drag experienced by a body immersed in fluid, the outflow of water from orifices, the viscosity, the propagation of sound in air and of waves on the surface of a liquid. However, his solutions were difficult, not based on a fundamental theory, and sometimes flawed at critical points. The heritage he left to the next century was a list of problems. When, in the Preface to the *Principia*, he wrote 'If only we could derive the other phenomena of nature from mechanical principles by the same kind of [mathematical] reasoning!', he was perhaps thinking in large part of fluid mechanics. As we shall see, this situation was to change completely during the ensuing years, due to a handful of first-rate mathematicians.

Some features of works in fluid dynamics written in the first half of the eighteenth century are likely to confound a modern reader. First of all, principles, expressed in words, were far more important than equations. In addition, since the general principles of mechanics had not yet been discovered, special hypotheses had to be advanced almost for every problem. Results were usually formulated as proportions. Torricelli's law, to cite an important case in point, was expressed by saying that 'the velocities increase in the subduplicate ratio (the ratio of the square roots) of the heights'. Formulas were written in peculiar systems of units; thus, for example, a velocity was usually represented by $\sqrt{v}$, where *v* denoted the height from which a body had to fall to acquire that velocity. (A general history of systems of units in physics is given by Roche (1998).)

A problem in reading early works on fluid dynamics arises from the absence of the modern concept of internal pressure. In hydrostatics, pressure was the weight of a column of fluid. This exactness became blurred in the passage to dynamics. The word 'pressure' usually denoted the force exerted on the walls of the tube enclosing the liquid, and it was measured in units of length. The first hint of a more general idea occurs in Johann Bernoulli's *Hydraulics* (1742), where essentially it denotes the mutual force between parts of the fluid. A sufficiently clear description of the modern concept finally appears in Euler's memoirs of the 1750s. The importance of this line of development lies in the fact that the pressure in fluids is the first significant case of a contact force, that is, the main concept of modern continuum mechanics.

Despite these differences in methods and formalism, the level of critical thinking throughout the eighteenth century was high. Researchers of the period were fully aware of the importance of such physical phenomena as viscosity and capillarity, of the influence of temperature and of the need for an equation of state connecting temperature, pressure and density. The resistance experienced by a body moving through a fluid was usually assumed to be proportional to the square of the velocity (the 'common rule'), but it was clear that this was only a useful approximation. There (p. 384) were discussions about the composition of a fluid: was it a continuum or a collection of particles? Everybody agreed that matter was made up of small particles, but since it was practically impossible to derive the equations of motion from this hypothesis, researchers turned to the continuum model. While they took into account the results of experiments, they knew that a successful physical theory is simply a representation of reality. Even at this early stage, fluid dynamics had already generated important subfields: theory of tides (Aiton 1955), ship theory (Nowacki 2006 and Ferreiro 2007), and applications to medicine (Zimmermann 1996). The roles of experimenters and theoreticians were clearly divided: physicists did experiments, mathematicians created theories.

In considering these early results, we must always keep in mind that fluid dynamics was a difficult subject. As late as the 1740s, the motion of systems with two or three degrees of freedom was still a matter of discussion. The description of the dynamics of a continuum was a formidable mathematical problem.

The history of modern fluid dynamics properly begins with Daniel Bernoulli's *Hydrodynamics* (1738). (Problems in the period from Newton to Daniel Bernoulli are described by Maffioli (1994) and Mikhailov (1996), and an account of early experiments on fluids is given by Eckert (2006).) In method of research, Bernoulli is the eighteenth century's closest equivalent to a modern physicist. In his work we recognize many of the characteristic features that we now associate with classical theoretical physics: attention to experiments, modelling of phenomena, bold hypotheses, and powerful but unrigorous mathematics. Bernoulli began his work on fluids in the mid-1720s, motivated in part by problems he had encountered during his medical training. (Interestingly, he was probably led to the subject *because* of his medical training.) He composed the work itself at the beginning of the 1730s. In the *Hydrodynamics* we find most of the results on fluids obtained by physicists and mathematicians up to that point, and much more (for details see (Mikhailov 2002; 2005)).

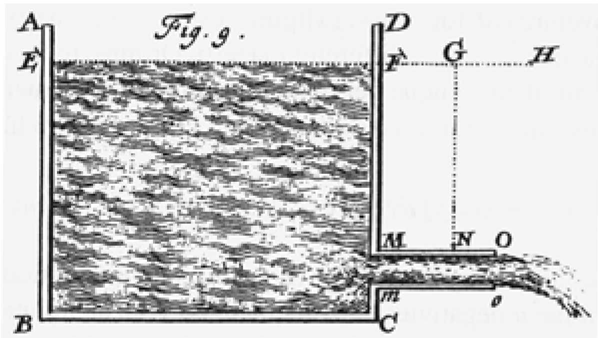

Fig. 12.4 . Bernoulli's principle.

The term 'hydrodynamics' was coined by Bernoulli to denote the union of hydrostatics (equilibrium under the action of forces) and hydraulics (description of motion). The dynamical principle employed was the conservation of mechanical energy in the Huygensian form of the equality between actual descent and potential ascent. This was a bold choice in a period when the principle was a subject of controversy; however, no other general principle for the motion of a system of bodies was then known. Bernoulli also adopted systematically the hypothesis of parallel sections introduced by earlier authors, according to which flow occurs by parallel slices. This was out of necessity, too: a more general flow would have required the use of partial differential equations, not yet known at that time. Bernoulli was fully aware of the limitations of these hypotheses. He believed, for example, that part of the 'living force' was lost because of the microscopic motion of the small particles making up the fluid.

Since today Bernoulli's name is associated with his famous theorem in fluid dynamics, this is probably the result that a modern reader is likely to search for first in the *Hydrodynamics*. However, the original form of the theorem looks quite different from what we are used to seeing. It is found in chapter XII, where Bernoulli set (p. 385) himself the problem of the simultaneous determination of pressure and velocity. Not having the full concept of inner pressure at his disposal, he resorted to an ingenious argument. In essence, Bernoulli considered a vessel, constantly kept full of water, discharging through a small horizontal tube (Fig. 12.4). First, he determined the increment of velocity in the horizontal tube by the conservation of energy. Then, to obtain the 'pressure' on a point in the wall of the pipe, he imagined at that point the tube suddenly broke off. The water would therefore flow out of the hole with a known velocity, and the height reached by the jet could be taken as a measure of the pressure.

Another famous result of the *Hydrodynamics* is the derivation of the Boyle–Mariotte law in the chapter on ‘elastic fluids’ (gases). Bernoulli’s gas consists of a virtually infinite number of small spheres ‘agitated in very rapid motion’. This model of matter was not unusual at the time, but Bernoulli was among the very few who could obtain definite mathematical results from it. On the basis of simple physical hypotheses, he demonstrated that in gases having the same ‘degrees of heat’ (temperature) the ‘elasticity’ is approximately proportional to the density and to the square of the mean velocity of the particles. For almost a century, this was the only result of note in the kinetic theory of gases.

These results do not exhaust the content of the *Hydrodynamics*. Some of the topics look strikingly modern. Bernoulli studied hydraulic machines, defining the work done by a force in a given time and the efficiency of hydraulic engines. In particular, he considered the work done by compressed gases. Turning to applications, he examined the possibility of propelling carts and ships by means of jets: that is, the theory of rockets. These problems of variable-mass dynamics, considered tricky even today, were frighteningly difficult in the 1730s (see (Mikhailov 1976) and (Cerulus 2006)).

The appearance of the *Hydrodynamics* indirectly led to the publication of Johann Bernoulli’s *Hydraulics* of 1742. Its complete title is *Hydraulics, Now First Discovered and Proved Directly from Purely Mechanical Foundations, in the Year 1732*. However, despite the date 1732 he actually began writing it after the appearance of his son Daniel’s book. Johann had been deeply interested in fluid mechanics at least since 1715, but the finer points of the *Hydraulics* were suggested by Daniel’s book. (Indeed, as late as the middle of the 1720s he still believed that the pressure on the pipes was (p. 386) independent of the motion of the fluid.) Disgusted with his father’s behaviour, Daniel left the field to which he had contributed so much.

All things considered, it would be wrong to consider Johann a simple plagiarist, for there is enough novelty in his book to make it a masterpiece of fluid mechanics on its own. The *Hydraulics* is essentially a rethinking of Daniel’s theory of the relation between velocity and pressure. Though Johann had long advocated the principle of living forces, here he criticizes his son for using this ‘indirect’ method. In fact, the great achievement of the *Hydraulics* is the use of the ‘genuine’ method, that is, the proportionality of force to acceleration or what is known as the principle of momentum. Johann considered the resultant of the internal forces acting on an infinitesimal element of fluid, equated it with the acceleration and obtained a differential relation. This apparently simple idea pointed the way for future developments in general mechanics. Perhaps the most important result obtained by this method is Bernoulli’s equation for both steady and unsteady flows, in a form easily recognizable by modern readers.

A special place must be assigned to Alexis Clairaut’s *Theory of the Figure of the Earth, Derived from the Principles of Hydrostatics* (1743). (For detailed accounts of this work see (Greenberg 1995) and (Passeron 1995).) The first sections are dedicated to the establishment of a fundamental principle for the equilibrium of an incompressible fluid: ‘A fluid mass can be in equilibrium only if the forces acting at all places of a canal of arbitrary form mutually destroy each other' (translation from (Truesdell 1956, xx)). This is the final generalization of axioms due to Newton and Huygens. In modern terms, it states that $Pdx + Qdy + Rdz$, where P, Q, R are the components of the external force per unit volume, is an exact differential. From this principle, and from his work on differential forms, Clairaut was led to establish the relations(12.29)

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial x} = \frac{\partial R}{\partial z},\ \frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y}.$$

Clairaut's treatment is clear, simple, and general. The fluid is referred to a system of orthogonal Cartesian axes, thus foreshadowing the idea of a field of physical quantities, and the result is expressed in purely analytical terms by means of partial derivatives. It was then obvious that a satisfactory theory of fluid motion should include Clairaut's formulation as a special case.

With d'Alembert we reach an intermediate stage in the development of theoretical fluid dynamics, equally distant from the pioneering efforts of the Bernoullis and from the modern theory of Euler. On the one hand, he had a number of brilliant ideas, central for the development of the theory; on the other, he was unable to obtain much from them, and they can now be understood only within the general framework set up by Euler. While d'Alembert was striving for an abstract and general point of view, he was still at some distance from the modern analytical formulation. His philosophy of mathematics led him to obscure the concept of pressure in the derivation of major theorems. In addition, his technical style was intricate, sometimes downright confused, and full of detours. (In recent years, d'Alembert's contributions to fluid dynamics have been re-evaluated by Grimberg (1995; 2002).)

(p. 387) D'Alembert's first separate publication on fluid mechanics was the *Treatise on the Equilibrium and Motion of Fluids* (1744), in which he tried to obtain hydrodynamics from hydrostatics by his general principle of dynamics. In this he was partly successful, but the *Treatise* does not contain anything not to be found in the *Hydrodynamics*. His next publication was a memoir which won a prize at the Berlin Academy in 1746: *Reflections on the General Cause of Winds* (1747). Here he attributed the cause of the winds to the tidal force of the Moon and the Sun—an assertion for which he was ridiculed by Daniel Bernoulli in a letter to Euler ('When one has read it all, one knows as much about the winds as before'). However, the *Reflections* presents some notable results on the purely theoretical side, such as the use of the differential continuity condition and the application of the momentum principle to an element of fluid (the acceleration is equal to the force exerted by the rest of the fluid and by the tidal interaction.)

Up to this point, mathematical fluid dynamics had been a one-dimensional theory. The jump to three dimensions took place in a memoir in Latin that d'Alembert submitted in 1749 for a competition of the Berlin Academy on the subject of the resistance of fluids; but no prize was awarded, and d'Alembert reworked his results as a memoir in French, published in 1752: *Essay on a New Theory of the Resistance of Fluids*. (Euler had been a member of the jury, and this was the beginning of lifelong tension between the two scientists.) In essence, the *Essay* treats of an axially symmetric flow. Its importance consists in the fact that it is the first work where the motion of a continuum is described by means of a field of physical quantities. There are remarkable peaks: the condition of continuity is imposed by the constancy of the volume of a small parallelepiped of fluid in motion, Clairaut's condition is generalized to compressible fluids, and an early version of the dynamical equations for incompressible and axially symmetric flows makes its appearance. Astonishingly, d'Alembert introduced what would now be called a complex potential function for velocity. In pure mathematics, the *Essay* is known for the first appearance of the 'Cauchy–Riemann' equations of complex analysis.

The results obtained by the Bernoullis, Clairaut, and d'Alembert form the basis of the general formulation of fluid dynamics developed by Euler in the 1750s. (We can follow his interest in their results through his correspondence.) However, Euler's theory is much more than the sum of the previous efforts: it represents such a huge step forward that it is difficult today to appreciate the extent of the innovation. It was the classic—and, in a way, definitive—formulation of the elements of fluid dynamics.

The ingredients of Euler's success are easy to determine. First and foremost, he took the principle of momentum, asserting the proportionality of force and acceleration, as the fundamental dynamical principle. He adopted what is today termed the field viewpoint, made constant use of analytical methods, and formulated general results by means of partial differential equations. The theory of fluids lies at the centre of the analytical revolution.

Euler (1757A) was fully aware of what he had achieved, and his opinion deserves to be quoted in full:

> (p. 388) Although I envisage here such a great generality, both in respect to the nature of the fluid as well as the forces which act upon each of its particles, I fear not at all the reproaches often justly directed at those who have undertaken to bring to a greater generality the researches of others. I agree that too great a generality often obscures rather than enlightens, leading sometimes to calculations so entangled that it is extremely difficult to deduce their consequences in the simplest cases … But in the subject I propose to explain, the very reverse occurs: the generality which I undertake, rather than dazzling our lights, will the more reveal to us the true laws of Nature in all their brilliance, and we shall find therein even stronger reasons to admire her beauty and simplicity. It will be an important lesson to learn that principles which had been thought connected to some special cases have a much greater extent. Finally, these researches will require a calculation hardly more troublesome at all, and it will be easy to apply them to any particular case which one might propose. ('General principles of the state of equilibrium of fluids', translation from Truesdell (1956) lxxv)

Euler had been interested in fluid mechanics since the end of the 1720s, but he had decided not to publish anything so as not to compete with his friend Daniel Bernoulli. When he returned to the subject more than twenty years later, at first he limited himself to clarifying problems in hydraulics. (For a general history of hydraulics, see (Rouse and Ince 1957). On Euler's work in hydraulics, see (Eckert 2002, 2008).) However, even these 'minor' papers contain jewels. In the annotations (Euler 1745) which he added to his translation into German of Benjamin Robins' *New Principles of Gunnery* Euler introduced streamlines, stated that a body immersed in a steady stream suffers no resistance ('d'Alembert's paradox'), and presented a clear derivation of Newton's law of resistance. (D'Alembert's paradox derived its name from its appearance in d'Alembert's 'Continuation of researches on the motions of fluids' (1768). See (Grimberg, Pauls and Frisch 2008).) In a book published in 1749 Euler gave a neat proof of Bernoulli's theorem for the steady flow of an incompressible fluid in a circular tube, not much different from those found in modern textbooks. In 1751 he identified the path of each particle of a fluid in steady motion by means of a parameter (Euler 1767A)—an early form of what in modern theory is called a 'material description'.

The real beginning of the modern theory came with the memoir '*Principles of the motion* of fluids', presented in 1752 but published much later (1761), in which Euler systematically represented a fluid as a continuum referred to a fixed set of orthogonal Cartesian axes and governed by partial differential equations. The first part was about pure kinematics: by a reasoning similar to that employed by d'Alembert, Euler obtained the continuity equation for incompressible fluids. In the dynamical part, Euler applied 'Newton's second law of motion' to an element of fluid and obtained the general dynamical equations for ideal incompressible fluids in the special case where the only external force is gravity.

Never content with simply obtaining new results, Euler recast, completed, and organized the theory in a sequence of three papers in French, which became the main source for the study of theoretical fluid mechanics in the second half of the eighteenth century. The first paper (Euler 1757A) presented all that was then known on fluid statics, giving the general equations of hydrostatics,(12.30)

$$\nabla p = \rho \mathbf{F},$$

(p. 389) where p is the pressure ρ the density and $\mathbf{F}$ the extraneous force per unit mass. In the application of his principles to the atmosphere, Euler implicitly corrected d'Alembert's faulty hypothesis by attributing the cause of the winds to 'different degrees of heat'. The second paper (1757b) was mainly important for giving, in the case of compressible fluids, the equation of continuity(12.31)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = \mathbf{0},$$

and the general dynamical equations(12.32)

$$\rho \mathbf{F} - \nabla p = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v \bullet \nabla \mathbf{v} \right),$$

where **v** is the velocity. From these rock-solid foundations, it was possible to calculate the first exact solutions: rotation about the origin and rectilinear motion in absence of external forces. The third paper (Euler 1757C) contained the Bernoulli equation for the streamlines.

The importance of these works for the general foundations of physics is difficult to overestimate. In the second paper, for example, we find clearly stated the distinction between external and internal forces acting on a mechanical system. On a more advanced level, in the *Guide to Natural Science*—probably written not long after 1755 but not published until 1862—Euler applied his general equations of fluid dynamics to an hypothetical ether filling all space and responsible for every kind of physical interaction. (On this work, see (Radelet-De Grave and Speiser 2004)). This was the first serious attempt at a unitary theory of physics based on ideas that correspond to the modern field concept. Eulerian fluid dynamics would remain the model for such theories for almost two centuries.

The young Lagrange was indirectly responsible for the next advance. In August 1759 he sent to Euler a memoir in which the one-dimensional propagation of sound was represented by a weightless string loaded by mass-points equally spaced. This provided the stimulus for Euler to set to work on wave propagation through a compressible fluid. He sent some of his results to Lagrange in a letter of January 1760. To describe the disturbance propagating through the particles of air, Euler needed a new kind of mathematical description; in fact, the letter contains the first appearance of the material coordinates (that is, the initial coordinates of an element of fluid) for the motion of a continuum. The complete research was published, after a delay of a few years, in a sequence of three papers published in 1766 (Euler 1766A, 1766B, 1766C). They are remarkable for the first appearance of the three-dimensional wave equation and of the material equation of continuity.

Euler had succeeded in describing waves of all kinds by means of the wave equation. He recognized that sound is a succession of pressure pulses, and that its mathematical description—involving the integration of a wave equation—would require ‘discontinuous ’functions, or functions defined piecewise using analytic expressions. As we (p. 390) saw in the problem of the vibrating string, such functions were somewhat controversial, and Euler was hesitant to employ them in his theory. However, the new discrete model advanced by Lagrange in 1759 provided a natural motivation to adopt such functions, and therefore to accept the wave equation as a valid mathematical description of a range of phenomena. Lagrange’s model gave Euler the confidence to accept discontinuous functions and to embrace the general applicability of the wave equation in mathematical mechanics.

At the beginning of his *Mechanics* (1736), Euler had stated that he planned to write general expositions of the advanced parts of mechanics. With respect to fluid mechanics, he fulfilled this promise with the publication of a comprehensive treatise presented to the St Petersburg Academy in 1766. Since he could not find a printer, he divided the work into four lengthy memoirs which were published in the proceedings of the Academy (Euler 1769–1772). Here in his usual style he presented all the aspects of the subject in a masterfully clear way. The third memoir is a theoretical treatment of hydraulics from the new point of view, and the fourth memoir is the first systematic presentation of mathematical acoustics. The second memoir contains the dynamical equations in material coordinates, which had also appeared in a memoir on the propagation of sound published by Euler in 1766.

Research on the mechanics of fluids was also important for providing impetus to the study of partial differential equations. Thus, for example, in the second French paper on fluids Euler was led to consider 'a very curious analytic question' which is nothing else than the problem of solving the general first-order linear partial differential equation with constant coefficients. Also, in Euler's third memoir on sound there is the earliest example of solution of a partial differential equation by separation of variables. Finally, the now classic linear transformation for reducing the one-dimensional wave equation to the simple form(12.33)

$$\frac{\partial^2 z}{\partial u \partial v} = 0$$

first appears in a letter of Euler to Lagrange of November 1762 (Euler 1892).

The next investigator to cultivate the subject at a fundamental level was Lagrange. For almost twenty years he was the only mathematician who could understand Euler's theory to the point of being able to use it as a starting point for further research. His first contributions appeared in a lengthy memoir on dynamics published in 1762. Lagrange derived Euler's dynamical equations for incompressible fluids from his method of using conjointly d'Alembert's equilibrium of the lost forces and the principle of virtual work, with the condition of incompressibility as a constraint. However, his most important contributions to fluid mechanics appeared in a memoir published in 1783. The first part is a concise presentation of Euler's theory. Building on this general background, Lagrange obtained some classic theorems. Preeminent among them are the velocity-potential theorem (under general conditions, a particle once in irrotational motion is always in irrotational motion), the necessary and sufficient condition for a surface to be material (formalizing intuitive ideas of continuity), and the impulse theorem (the flow produced by impulsive hydrostatic pressure is (p. 391) lamellar). (For details, see (Truesdell and Toupin 1960).) These were followed by the introduction of perturbation methods into fluid dynamics.

This memoir of Lagrange closes the century. The new theory became generally known through Jacques Cousin's *Introduction to the Study of Physical Astronomy* (1787), Lagrange's *Analytical Mechanics* (1788), Laplace's *Celestial Mechanics* (vol. 1, 1799) and Siméon-Denis Poisson's *Treatise on Mechanics* (1798). Waves on the surface of liquids and viscosity were left to the next generation.

12.8 Celestial Mechanics

In the eighteenth and nineteenth centuries the study of the motion of planetary bodies moving under the mutual attraction of gravity was regarded as a branch of physical astronomy. The theoretical study of the gravitational interaction of bodies following definite force laws distinguished this subject from positional or descriptive astronomy. The subject became know as celestial mechanics, after the title of Simon Laplace's major work of 1798. In broad outline, celestial mechanics in the eighteenth century was marked by three major developments. First, the semi-geometrical theory of Newton's *Principia* was recast in analytical terms using the methods and notation of the Leibnizian calculus. Second, several problems involving the motion of systems of more than two bodies were analysed. The greatest effort was devoted to the Earth–Moon–Sun system and the Sun–Jupiter–Saturn system, and the motion of comets also attracted considerable attention. In the case of the Moon, investigation focused on the motion of its apogee, and on its secular or long-term acceleration. Of particular concern for planetary motion was the problem of stability, of showing that the three-body systems of interest were not subject to any irregularities that over the long term would result in a progressive derangement of the system. Third, phenomena such as libration and precession were analysed using the dynamical theory of rigid bodies that had emerged by the middle of the century.

The modern theory of celestial mechanics is based on the formal methods of Hamilton–Jacobi theory—a subject developed by William Rowan Hamilton and Carl Gustav Jacobi in the first part of the nineteenth century, and subsequently applied to celestial mechanics by Félix Tisserand, Charles Delaunay, and Henri Poincaré (see (Nakane and Fraser 2003) and (Fraser and Nakane 2003).) During the eighteenth century, researchers employed differential equations expressing the proportionality of force and acceleration. A major goal was to show that the inverse-square law of gravitation formulated in Book 3 of the *Principia* was sufficient to account for the various planetary phenomena. The general triumph of Newtonianism often identified by historians with eighteenth-century physical science in fact refers rather specifically to the verification in physical astronomy of the inverse-square law. Innovations in research were based on skilful reduction of the differential equations of motion, and on the development of perturbation methods involving the method of variation of arbitrary (p. 392) constants. Some important advances in the formulation of fundamental principles of mechanics occurred in physical astronomy, particularly in the study of the motion of a rigid body.

Commenting on the historical outlook of modern astronomers, Stanley Jaki (2000, 329–330) observes that 'the very tools with which they work (satellite-directed radar techniques) are too different from the methods of their forebears to prompt them to look back to the eighteenth and nineteenth centuries.' Celestial mechanics of this earlier period is a somewhat curious subject to a reader today. In a broad sense everyone acknowledges the subject's success in confirming the inverse-square law of gravitation. However, the basic results in anything resembling their original historical form are not included in modern expositions of celestial mechanics. Furthermore, the focus on special problems (now treated numerically using computers) and the rather formidable technicalities involved in their solution give rise to a subject resistant to easy contemplation and understanding. (For historical accounts of eighteenth-century celestial mechanics, see (Grant 1852), (Waff 1976), (Forbes 1980), (Linton 2004, Chapter 9), and (Taton and Wilson 1995).)

What is perhaps more meaningful to a modern reader than the details of particular problems is the progress made in the formation of fundamental principles and methods of solution. We consider in some detail two problems that were particularly fruitful from this point of view: the motion of the Moon's apogee, and the changes in average motion of Jupiter and Saturn.

12.8.1 Earth–Moon–Sun System

A basic characteristic of the Moon—one that it shares with the Sun—is that its movement along the ecliptic varies in velocity from day to day. However, unlike the case of the Sun, the point of minimum velocity does not remain fixed but advances by an amount of about 3° each tropical or sidereal month. The anomalistic month—the time taken by the Moon to move from apogee to apogee—is longer than the sidereal month—the time taken by the Moon to travel 360° relative to some fixed point on the celestial sphere. (Two additional lunar periods of note are the synodic month, the time between two full Moons, and the Draconitic month, the time between the Moon's circuit from node to node.) The values of the anomalistic and tropical months were know to the Seleucid Babylonians, and the relations between the different lunar periods were tabulated by Ptolemy in the *Almagest* and *Handy Tables*, and built into his geometric models of the Moon. (See (Pedersen 1974, chapter 6) for details.)

In the *Principia* Newton investigated the motion of the Moon about the Earth, and tried to evaluate the gravitational effect on the Moon exerted by the Sun. In Book 1 he had shown that if a particle moves in an ellipse under the action of a central force located at one focus, then the force varies as the inverse square of the distance of the particle from the force centre. He also showed that for nearly circular orbits, a force law of the form $F \propto \frac{1}{r^{2+\epsilon}}$ (where F is the force and r is the distance from the body to the force centre) will result in a rotation of the line of apsides, no matter how small ϵ is. That is, even the slightest departure of the force law from an inverse-square (p. 393) form leads to a steady progression of the line of apsides. The effect of the Sun's gravity on the Moon could be interpreted in terms of a model involving a slight departure of the force acting on the Moon from a strict inverse-square law given as a function of the distance between the Moon and the Earth. A major difficulty with this theory—one that Newton was unable to resolve—was that the apsidal motion predicted by the model was only one half of the observed value of about 3° per month.

During the 1740s the problem of the motion of the Moon occupied the energies of three leading savants: Clairaut and d'Alembert in Paris, and Euler in Berlin. Newton's investigation was recast in terms of systems of differential equations to describe the Moon's motion. The three men each made important progress in the development of a lunar theory. However, like Newton, they arrived at a value for the apsidal motion that was one half of its observed value. (Clairaut, incidentally, was rather critical of Newton's *Principia*, observing (1749, 329) that in the difficult parts Newton 'employs far too few words to explain his principles, whereas he appears to surrender himself complaisantly to details and facts of calculation that readers will have no qualms accepting'.) By the 1740s the problem of the lunar apogee had become a major challenge to the validity of a physical astronomy founded on the inverse-square law of universal gravitation. It should be noted that the concern here was not a subtle variation in the orbital motion, but a phenomenon evident from basic observation and documented as far back as Hipparchus and Ptolemy in antiquity. (Later historical research would show that it was also well known even earlier, by the Seleucid Babylonians.) If the current gravitational theory was unable to account for gross observations, it was unlikely to succeed at predicting the other more subtle variations experienced by the Moon in its motion.

During the 1740s Clairaut speculated that it might be necessary to abandon the inverse-square law. Euler arrived at a similar conclusion, believing that ultimately gravity resulted from some sort of vortical effect of a ponderous ether, and that it was *a priori* unlikely that an exact inverse-square law would be valid. (In other instances where the theory encountered difficulties, Euler was inclined to attribute the discrepancies to the resistance encountered by planetary bodies as they moved through an interplanetary medium.) D'Alembert was more hesitant to abandon the accepted law of gravitation, though he believed that it was possible that an additional force such as magnetism was influencing the motion of the celestial bodies.

In his researches of the late 1740s Clairaut had reduced the analytical description of the Moon's motion to the equation of a rotating ellipse (Linton 2004, 301):(12.34)

$$\frac{k}{r} = 1 - e\cos q\theta,$$

where r is the distance of the Moon from the Earth, e is the eccentricity of the Moon's orbit, and θ is the mean anomaly of the Moon. q is a constant slightly less than one, which implies that the Moon must travel slightly more than 360° to move from apogee to apogee or perigee to perigee. Clairaut's derivation of eq. (12.34) involved a skilful integration that attracted Euler's admiration (for details see (Linton 2004, 301), as it was an approximate one obtained by neglecting certain terms, and resulted in a value for apsidal motion that was one half its true value. Clairaut believed that the (p. 394) terms neglected in the derivation of this equation did not contribute substantially to this motion. Nevertheless, in 1748 he went back and examined a modified version of the equation which included the missing terms. After a great deal of labour involving successive approximations he made the exciting discovery that the neglected part did in fact contribute the missing part of the apsidal motion

Clairaut's result was rightly heralded as a breakthrough, and led Euler and d'Alembert immediately to revise their lunar theories. All three men produced treatises on the Moon in the early 1750s. Euler's book employed a Cartesian rectangular coordinate system and introduced new methods for integrating the differential equations of motion. More generally, Euler devised a calculus of trigonometric functions that he employed extensively throughout his celestial researches and made a common part of the subject. D'Alembert concentrated on the algebraic development of the theory, producing a literal theory in which the values of the terms were not replaced by approximate numerical values. (The lunar theorist Brown (1896, 239) observes that 'while [Clairaut] worked out his results numerically, d'Alembert considered a literal development and carried out his computations with more completeness.') D'Alembert derived the following equation (see (Linton 2004, 303)) connecting the sidereal month T_s and the anomalistic month T_a:(12.35)

$$\frac{T_a}{T_s} - 1 = \frac{3}{4}m^2 + \frac{225}{32}m^3 + \ldots$$

Here m is the ratio of the mean motion of the Sun to the mean motion of the Moon (a number slightly less than $\frac{1}{12}$). If the cube term in m is neglected, the older value for the motion of the apogee is obtained. Because the coefficient of the cube term is large, the quantity $\frac{225}{32}m^3$ is substantial and must be retained, and by including it, the observed value for the motion of the apogee is derived. (For a detailed history of the lunar apogee problem see (Waff 1976 and 1995).)

12.8.2 Sun–Jupiter–Saturn System

It had become well known by the middle of the century that the values of the average motions in longitude of Jupiter and Saturn were changing over time. The motion of Jupiter was increasing slowly, while the motion of Saturn was decreasing. This observational phenomenon was called the 'great inequality of Jupiter and Saturn'. Explaining it using Newton's theory of gravity was a major focus of research in celestial mechanics in the second half of the century. Euler laid some of the groundwork for the theory, but the most important progress was made by Lagrange and Laplace in the 1770s and 1780s. Lagrange fashioned the method of variation of parameters into a fundamental analytical tool in the theory of perturbations. Laplace succeeded in applying this theory to the motions of Jupiter and Saturn, and deduced that the inequality was periodic with a period of about 850 years. The three-body system consisting of these two planets and the Sun was the dominant gravitational grouping in the solar system and the one subject to the greatest variation. Laplace's result meant in effect that the (p. 395) stability of the solar system had been derived from the inverse-square law of universal gravity—a result that was the crowning achievement of celestial mechanics in the eighteenth century. (For an account of this history see (Morando 1995).)

Linton (2004, Chapter 9) distinguishes two approaches to the treatment of perturbations. In the first, called the method of absolute perturbations, the orbital parameters are taken as constant, and an approximate solution is obtained for the differential equations involving these parameters. This solution is then substituted back into the differential equations, and a second approximation follows. Proceeding in this way, one arrives at a successively more accurate description of the motion of the perturbed body. (Brown (1896, 47) refers to this method as the 'method of successive approximation'.) This was the method that Clairaut followed in his investigation of the Moon's apogee, in which the second approximation was found to be the key to removing the discrepancy between theory and observation.

The disadvantage of the method of absolute perturbations is that the successive approximation procedure can give rise to terms that increase indefinitely with time. In the eighteenth century such terms were referred to as 'arcs of circles', because they appeared as angles that could be regarded as the arcs of a circle of radius 1. It was necessary to avoid these quantities because they were non-periodic and increased indefinitely with time. A familiar refrain in treatises on celestial mechanics of the period was the desirability of obtaining expressions containing no arcs of circles.

A second approach to perturbations is based on the method of variational of arbitrary parameters, introduced by Euler and developed more fully by Lagrange. The basic idea is the familiar one encountered today in a first course in differential equations when one solves a non-homogenous differential equation using variation of parameters. One takes the general solution of the homogenous equation and then regards the constants of integration appearing in this solution as functions of the independent variable. If the expression considered in this way is substituted back into the non-homogenous equation, one is able to solve for the parameters and obtain a solution of the non-homogenous equation. In the astronomical context, one begins by assuming the orbit of the given body is Keplerian, corresponding to unperturbed motion, so that it is specified with respect to six orbital parameters. These parameters are then regarded as functions of time, and the Keplerian solution is substituted back into the differential equations of perturbed motion. The goal is to integrate these equations in order to obtain a solution of the original problem.

Although the method of variation of orbital parameters was introduced by Euler, it was Lagrange who developed the method into a highly effective tool. An exposition of it was presented in his *Analytical Mechanics* (1788) and in the subsequent editions of this work. The method was at the base of the researches carried out by Lagrange and Laplace during the 1780s in their investigation of the great inequality of Jupiter and Saturn. In the case of the Moon the distance from the Earth to the Moon was negligible in comparison with the distance from the Earth to the Sun, and this fact simplified some of the approximations. By contrast, the ratio of the distances of Jupiter and Saturn to the Sun was not negligible, leading to greater complications in the analysis of their mutual motions. Working from some results of Lagrange, Laplace **(p. 396)** discovered the following general equation connecting the masses *m* of the planets of the solar system and the lengths *a* of their semi-major axes (Morando 1995, 139):(12.36)

$$\frac{m}{a} + \frac{m'}{a'} + \frac{m''}{a''} + \ldots = const.$$

Because the masses m and m' of Jupiter and Saturn are very large compared to the other planetary masses, the above relation is approximately $\frac{m}{a} + \frac{m'}{a'} = const.$ Using Kepler's third law Laplace was led to the conclusion that the variation in the mean motion *n* of

Jupiter is related to the variation in the mean motion n' of Saturn by the equation $\delta n' = -2.33\delta n$. Moreover, this relation accorded with what was known from observation, and Laplace inferred that the changes in the mean motions of Jupiter and Saturn were a result of their gravitational interaction—a conclusion that both he and Lagrange had rejected explicitly at an earlier stage of their investigations. With this insight, Laplace launched an all-out attack on the problem of the great inequality. He had shown previously that the commensurability or near-commensurability of the motions of two bodies in the three-body problem can lead to long-term periodic terms for their mean motions. The ratio of the mean motion of Jupiter to the mean motion of Saturn was very close to 5:2, and Laplace was able to show after very considerable effort that this leads to a long-term periodic expression for the mean motions with a period of about 850 years.

Although Laplace deserved full credit for resolving the mystery of the great inequality of Jupiter and Saturn, it is worth noting how many of the key ideas and methods used by him were drawn from Lagrange's writings. Today it is the results and methods pioneered by Lagrange that tend to attract the attention of mathematicians and physicists, while Laplace's achievement occupies an honoured place in the mausoleum of past science. A very active area of modern research concerns general mathematical solutions of the three-body problem, treated in a qualitative way using methods of analysis and topology. A noteworthy theoretical result in this direction is a problem analysed mathematically by Lagrange (1777) in a lengthy memoir that won the Paris Academy prize for 1772. Lagrange considered a system in which the three distances between the bodies are equal, and found solutions in which one of the bodies occupies what are today called the 'Lagrange points'. Given two massive bodies, the Lagrange points are located in the orbital plane of these bodies, at the vertices of the two equilateral triangles in this plane taken with respect to the line joining the bodies. Although Lagrange's result was primarily of mathematical interest, in the early twentieth century it was discovered that asteroids are present at the Lagrange points of Jupiter's orbit about the Sun (see (Linton 2004, 326) for details).

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