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5. Pure and Applied Mathematics in the Philosophies of d'Alembert and Kant

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The practice of clarifying one's terms was evidently a sound one, but it was not a substitute for investigation of foundational problems. D'Alembert recognized that in mathematics one employed ideas that were not contained in sense perception. The straight lines and circles of geometry were idealizations that were realized only approximately in nature. He referred to the propositions of geometry as the "intellectual limits" of physical truths.(12) Incommensurable ratios were treated analogously. Although such ratios corresponded to no actual numbers, it was acceptable to regard them as such because "the difference between an incommensurable ratio and a number proper can be as small as one would wish".(13)

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early dynamics emerged implicitly in his presentation of a technical physical theory. His subsequent development of an explicit philosophy was conditioned by his interest in and commitment to empiricist epistemology. Historically the latter had been associated with the belief that geometry and the theory of extension formed the fundamental part of mathematics; all ideas originated in experience, and our most basic sensory impressions of continuous geometrical extension and temporal succession provided the primitive materials with which mathematics began.(14)

D'Alembert recognized the power of analysis and was quick to dismiss the British concern with synthetic geometry. His comments on algebra indicated some awareness of the intellectual aspects of the subject. A strong emphasis on the ultimate sensory,

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14. For accounts of the role of geometry in empiricist mathematical philosophies see Chikara Sasaki's "The Acceptance of the Theory of Proportion in the Sixteenth and Seventeenth Centuries", Historia Scientiarum No. 29 (1985), 83-116 (this study considers Isaac Barrow) and Craig G. Fraser's "Lagrange's Analytical Mathematics, Its Cartesian Origins and Reception in Comte's Positive Philosophy", forthcoming in 1990 in Studies in the History and Philosophy of Science (this study considers Auguste Comte).

geometrical character of mathematics nevertheless conditioned and ultimately limited his development of a coherent mathematical philosophy. He never attempted to analyze the numerical continuum or to explore the relations of algebra and logic; the turn to geometry led him away from the most promising directions of investigation.

More generally, the atmosphere of 18th-century empiricism and the close association of mathematics and physics during this period fostered an uncritical attitude towards the subject. As Grimsley has noted, d'Alembert insisted on "the elementary truth that the scientist must always accept the essential 'giveness' of the situation in which he finds himself."(15) This attitude manifested itself in contemporary mathematical analysis as a general and unconscious tendency to view the subject as something given from without, possessing its own autonomous and completed identity. (16)

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15. Op. cit. n.2, p.248.

16. This attitude is present in the approach during the period to questions of mathematical existence. Mathematicians were interested in the nature of particular analytic processes or the form that given solutions should take. Mathematical entities were

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While d'Alembert was a major scientist and a minor philosopher, Immanuel Kant was a major philosopher and a minor scientist, if indeed he could be called a scientist at all. Working slightly later than d'Alembert, he presented in 1781 his famous critical philosophy in the first edition of his Kritik der reinen Vernunft. This work was followed in 1783 by his Prologomena zu einer jeden künftigen Metaphysik die als Wissenschaft auftreten können, in 1786 by his Metaphysische Anfangsgründe der Naturwissenschaft and in 1787 by an enlarged and revised edition of the Kritik.

A vast philosophical literature exists that explores Kant's doctrines and their ramifications. His account of synthetic *a priori* judgements and his elaborate theory of the concepts and principles of the understanding are established parts of the philosophical cannon. The intent of the present paper

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regarded as things that were given, not as things whose existence needed to be established.

will be to indicate briefly some aspects of his mathematical philosophy, considering points of comparison and contrast with the views of d'Alembert discussed earlier.

Although Kant lived in the 18th century, the period known in the history of mathematics as the century of analysis, he made no attempt to address the major developments that had shaped contemporary mathematics. His discussions were limited to elementary geometry and arithmetic and could in this respect have been presented in antiquity. His goal in examining mathematics was to undertake a critical study of its foundations, of the question of how it was possible that we had certain mathematical knowledge, as preparation for a larger investigation of metaphysics. His account of mathematics and natural science were presented in the parts of the Kritik titled "Transcendental Aesthetic" and "Transcendental Analytic" respectively.(17) The purpose of these sections, he observed in the Prologomena, was to answer the questions

How is pure mathematics possible?

17. The Analytic was itself the first division of the Transcendental Logic, the second being the Dialectic.

How is pure natural science possible?

Kant asserted in the introduction to the _Kritik_ that human knowledge derived from sensibility and understanding; "through the former, objects are given to us; through the latter, they are thought."(18) The transcendental aesthetic was concerned with sensibility and the transcendental analytic with understanding. Kant therefore seemed to be suggesting, in the early parts of the _Kritik_, that mathematics was concerned directly with sense and made only secondary use of the concepts of the understanding. Since mathematical subjects did indeed arise in the _Analytic_, it would seem that his own thinking evolved in the course of writing the work.(19)

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18. A15/B29/KS61-62. (A refers to the first edition, B refers to the second edition, and KS refers to Kemp Smith's English translation of 1933; the latter is used in the present paper.)

19. To the extent to which concepts of the understanding are involved in mathematics an appreciation of Kant's philosophy of mathematics requires consideration of the _Analytic_. H. J. Paton (_Kant's Metaphysic of Experience_ Volume 1 (London: George Allen & Unwin Ltd.,1951), p.98) writes "In awareness of space and time as individual objects thought is always involved. Thought gives us the synthesis without which there is no unity in any object. The necessary synthesis is in the Aesthetic ignored...The provisional exclusion of the part played by thought must always be borne in mind." Kant

According to Kant mathematics concerned itself with the pure form of sensibility and as such was a priori, or independent of experience. The subject was based on two fundamental intuitions, an outer intuition of space and an inner intuition of time. From the former we obtained geometry and from the latter arithmetic. The propositions of mathematics, what Kant called judgements, were synthetic. The predicate of a mathematical proposition could not be obtained from the subject through analysis of its meaning but needed to be synthetically attached to the latter.

Mathematical activity involved construction. One obtained, for example, the concept of triangle by

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himself writes (A51-52/B75-76/KS93): "Thoughts without content are empty, intuitions without concepts are blind. It is, therefore, just as necessary to make our concepts sensible, that is, to add the object to them in intuition, as to make our intuitions intelligible, that is, to bring them under concepts. These two powers or capacities cannot exchange their functions. The understanding can intuit nothing, the senses can think nothing. Only through their union can knowledge arise. But that is no reason for confounding the contribution of either with that of the other; rather is it a strong reason for carefully separating and distinguishing the one from the other. We therefore distinguish the science of the rules of sensibility in general, that is, aesthetic, from the science of the rules of the understanding in general, that is logic."

constructing this concept, that is, by exhibiting it a priori in one's intuition of space. Numerical concepts were obtained by constructing the concept in one's intuitions of both space and time. Kant distinguished knowledge that involved construction from discursive knowledge, whose a priori synthetic character involved concepts of the understanding.

D'Alembert had grouped arithmetic and algebra together on the one hand, and geometry and mechanics on the other. The former were regarded as abstract and separate from experience; the latter as sensuous and material. Kant, by contrast, placed arithmetic and geometry together because both were based on the pure form of sensibility, on a priori intuitions of space and time. He sharply separated on philosophical grounds the science of motion from arithmetic and geometry:

> ...transcendental aesthetic cannot contain more than these two elements, space and time. This is evident from the fact that all other concepts belonging to the sensibility, even that of motion, in which both elements are united, presuppose something empirical. Motion presupposes the perception of something moveable. But in space, considered in itself, there is nothing movable; consequently the movable must be something found in space only

> <u>though experience</u>, and must therefore be an empirical datum. (20)

> ...in the [field of] appearance, in terms of which all objects are given us, there are two elements, the form of intuition (space and time), which can be known and determined completely <u>a priori</u>, and the matter (the physical element) or content - the latter signifying something which is met with in space and time and which therefore contains an existent corresponding to sensation. (21)

In these passages the modern distinction between mathematics and mathematical physics emerged clearly in a way that it had not in d'Alembert's writings.

Kant also differed from d'Alembert in his conception of dynamics. In his <u>Metaphysische Anfangsgründe der Naturwissenschaft</u> he envisaged a dynamics that was derived from Newtonian particle mechanics and was based on the concept of force. D'Alembert's whole mission in the <u>Traité de Dynamique</u> had been to explain mechanical interaction without using force, through an abstract analysis of the properties of impenetrable extension moving in space. These differences indicated a profound divergence in their conception of physical theory and the role of

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20. A41/B58/KS81-82.

21. A723/B751/KS583.

mathematics within it, a subject that is beyond the scope of the present paper to explore.

Kant's account of mathematics shared this general characteristic with d'Alembert's, that it was an attempt to understand the subject within a philosophical programme that was influenced by empiricism. Kant was able to provide a sophisticated account of elementary geometry, of the processes that are involved in understanding a Euclidean theorem, that integrated the sensory and intellectual aspects of this activity. His discussion of arithmetic and algebra, although highly sketchy, furnished some suggestions that have been developed in modern intuitionist philosophy.(22)

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22. On number Kant writes: "...the pure <u>schema</u> of magnitude (<u>quantitatis</u>), as a concept of the understanding, is <u>number</u>, a representation which comprises the successive addition of homogeneous units. Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition." [A142/B182/KS183-184] On algebra he writes: "...in algebra by means of a symbolic construction, just as in geometry by means of an ostensive construction (the geometrical construction of the objects themselves), we succeed in arriving at results which discursive knowledge could never have reached by means of mere concepts."[A717/B745/KS579] Also: "Mathematics alone, therefore, contains demonstrations, since it derives its knowledge not from concepts but from the construction of them, that

Although he achieved substantial results, Kant never presented a systematic philosophy that illuminated contemporary advanced mathematics. The subsequent development of his doctrines in the 19th century, like that of other empiricist-influenced mathematical philosophies, occurred not in mathematics but in the philosophy of physics, more particularly in positivist-empiricist theories of physical explanation.

References

Cassirer, E. 1951. The Philosophy of the Enlightenment. Beacon Press, Boston.

D'Alembert, J. 1805. Oeuvres Philosophiques, Historiques et Littéraires. Paris.

Fraser, C. G. 1985. D'Alembert's principle: The original formulation and application in Jean

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is, from intuition, which can be given a priori in accordance with the concepts. Even the method of algebra with its equations, from which the correct answer together with its proof, is deduced by deduction, is not indeed geometrical in nature, but is still constructive in a way characteristic of the science. The concepts attached to the symbols, especially concerning the relations of magnitudes, are presented in intuition; and this method, in addition to its heuristic advantages, secures all inferences against error by setting each one before our eyes."[A734/B762/KS590]

d'Alembert's Traité de Dynamique (1743). Centaurus 28, 31-61, 145-159.

Fraser, C. G. 1990. Lagrange's analytical mathematics, its Cartesian origins and reception in Comte's positive philosophy. Forthcoming in Studies in the History and Philosophy of Science.

Grimsley, R. G. 1963. Jean d'Alembert (1717-1783). Clarendon Press, Oxford.

Hankins, T. L. 1970. Jean d'Alembert Science and the Enlightenment. Clarendon Press, Oxford.

Kant, I. 1781. Kritik der reinen Vernunft. Second revised and enlarged edition appeared in 1787. English translation by Kemp Smith published in 1929.

McRae, R. 1961. The Problem of the Unity of the Sciences: Bacon to Kant. University of Toronto Press, Toronto.

Paton, H. J. Kant's Metaphysic of Experience Volume 1. George Allen & Unwin Ltd., London.

Sasaki, C. 1985. The acceptance of the theory of proportion in the sixteenth and seventeenth centuries. Historia Scientiarum No. 29, 83-116.