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The Emergence and Consolidation of Lagrange's Analysis 1770-1776

A Preliminary Sketch

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Introduction [1]

Lagrange's professional career was an exceptionally long one, spanning the years from 1754, when he was eighteen, to his death in 1813. The general outline of his biography is well known.[2] From his birth in 1736 until 1766 he lived in Turin, participating in the founding of the Turin Society in 1757 and then becoming one of its active members; from 1766 to 1787 he was mathematics director of the Berlin Academy of Sciences; from 1787 to his death he lived in France as a pensionnaire vétéran of the Paris Academy of Sciences.

The level of Lagrange's mathematical activity varied considerably during his professional life. Although his initial reputation was secured by several brilliant researches, his achievements during his time in Turin were in fact selective and relatively few in number. Thus his appointment to the Berlin Academy at age thirty was based as much on future promise as on prior record of achievement. As he became established in Berlin he settled into a pattern of steady and full mathematical productivity. During the years 1770 to 1787, when he was in his late thirties and forties, he contributed to a range of subjects in pure and applied mathematics. His reputation as a major scientific figure is based on his work during this period. In the decade that followed his move to Paris he was relatively inactive, a circumstance that was apparently connected to personal depression that he experienced at this time. In the late 1790s his research revived as he began to compose his major didactic treatises on the calculus. He completed significant research as a septuagenarian mathematician (his achievements are documentable) prior to his death in 1813.

Although Lagrange's analytical tendencies were apparent from the very beginning, his distinctive mathematical style really only became consolidated in the period 1770 to 1776, when he was in his late thirties and comfortably settled at the Berlin Academy. In these years the value of analysis becomes an explicit theme in his writings for the Academy on a range of subjects in pure and applied mathematics.[3] In a memoir of 1771 on Kepler's problem he distinguished three approaches to its solution, one involving numerical approximation, a second using geometrical or mechanical constructions, and a third that is algebraic, employing analytical expressions. The last he cited for its "continual and indispensable use in the theory of celestial bodies". In a paper the next year on the tautochrone, a problem first investigated geometrically by Huygens, he took as a starting point "analytical solutions" that had been advanced by Johann Bernoulli and Euler. In 1775 several memoirs appeared in which the value of analysis is promoted. In his paper on the attraction of a spheroid he attempted to show that the method of "algebraic analysis" provides a more direct and general solution than the "synthetic " or geometrical approach followed by Maclaurin. (This appears incidentally to be the first explicit appearance in his writing of the term "algebraic analysis".) In his study of the rotation of a solid he advanced an alternative to the mechanical treatment of d'Alembert and Euler, one that was "purely analytic", whose merit consisted "solely in the analysis" that it employed, and which contained "different rather remarkable artifices of calculation." In a memoir on triangular pyramids he noted that his "solutions are purely analytic and can even be understood without figures"; he observed that independent of their actual utility they "show with how much facility and success the algebraic method can be employed in questions that would seem to lie deepest within the province of Geometry properly considered, and to be the least susceptible to treatment by calculation."

The theme of analysis recurs in Lagrange's writings of the later 1770s and 1780s. In a 1777 study of cubic equations he described a method due to Harriot which avoided the geometrical constructions that had been used by mathematicians to investigate expressions for roots. In a memoir submitted to the Paris Academy in 1778 on the subject of planetary perturbations he offered a method for transforming the equations of motion that would "take the place of the synthetic methods proposed until now for simplifying the calculation of perturbations in regions beyond the orbit" and that "has at the same time the advantage of conserving uniformity in the march of the calculus". In 1780 he published a memoir on a theorem of Lambert's in particle dynamics. The result in question had been demonstrated synthetically, and Lagrange expressed concern that it might be regarded as one of "the small number [of theorems] in which geometric analysis seems to be superior to algebraic analysis". His purpose was to present a simple and direct analytical proof. In a study in 1781 of projection maps he offered a "research, equally interesting for the analytic artifices that it requires as well as for its utility in the perfection of geographical maps." In the preface to his famous Traité de la méchanique analitique, completed around 1783, he announced that in it "no figures would be found", that all would be "reduced to the uniform and general progress of analysis." In a memoir of 1788 he discussed successes and difficulties in treating analytically the various subjects of Newton's Principia mathematica, and offered a new analysis of the problem of the propagation of sound.

Directness, uniformity and generality are qualities that Lagrange associates with analysis; he sometimes also mentions simplicity. Analysis is cited not simply for the results to which it leads, but also for the methods that it offers. In the writings cited above he was affirming the value of analysis in situations where an alternative geometrical or mechanical treatment existed; it was the possibility of this alternative that led him to explicitly assert his own methodological preferences. One should also note the sheer preponderance of pure analysis in his work of the 1770s and 1780s in such topics as the theory of equations, diophantine arithmetic, number theory, probability and the calculus, subjects in which explicit questions of approach or methodology did not arise.

Three memoirs

Although Lagrange placed great emphasis on analytical mathematics broadly conceived it was in the various branches of the calculus that his distinctive conceptions most clearly emerged. A coherent vision of this subject informed three key memoirs that he wrote in the early 1770s: a treatise on the calculus of variations, finished in 1770 and published in 1773; a paper on formal calculation by means of infinite power series, published in 1774; and a memoir on partial differential equations, completed early in 1773 and published in 1774.

a) Calculus of variations (1773)

Lagrange's early variational writings -his letter to Euler of 1755 and his Turin memoir of 1762- had introduced his δ-algorithm within a framework of variational mathematics set forth in Euler's Methodus inveniendi (1744). He was here primarily concerned in showing how the different results of Euler's treatise could be recovered by his new method. By the 1770s however he had arrived at his own independent understanding of the subject. His memoir of 1773, "Sur la méthode des variations", is noteworthy for the boldness of its conception, for its explicitly abstract character and for the way in which the subject is systematically developed as a formal calculus.

The immediate stimulus for the research was provided by a paper of the French mathematician Borda which had appeared in the memoirs of the Paris Academy in 1770. Borda was correcting an error Lagrange himself had committed in his original Turin memoir of 1762. The problem in question was that of the brachistochrone in which the endpoints of the extremalizing arc are allowed to move along given definite curves. Lagrange had shown correctly that if the initial point is fixed and the endpoint is allowed to vary then the solution will be the cycloid passing through the initial point that cuts the end curve normally. His mistake occurred when he extended this result to the case where the initial point is also allowed to vary, and concluded wrongly that the solution will be the cycloid which cuts the initial curve normally. Borda showed that the condition which in fact must be satisfied is that the tangents to the initial and final curves at the points of intersection with the extremalizing arc be parallel.

Lagrange's mistake arose because he failed to notice that the integrand of the variational integral may contain constants that must themselves be varied in the variational process. Such is the case in the example of the brachistochrone with variable beginning point. In his memoir of 1773 he provided a rather abstract formulation of the variational problem that employed a general process of variation. All of the results of the subject, including Borda's example, were derivable from his theory.[4]

The idea of varying constants would prove to be a powerful one in Lagrange's subsequent work in the theory of differential equations. It is interesting that the idea should appear here -in a quite different mathematical context- in the calculus of variations. The common occurrence of the procedure suggests the existence of significant historical and mathematical links between his research in different branches of analysis.

b) On a new kind of calculus (1774)

The paper "Sur une nouvelle espèce de calcul" is usually cited as the original source for Lagrange's later programme of basing the calculus on Taylor's theorem. Although it is indeed true that he first introduced this idea here his primary intent in writing the memoir was not foundational. What seems to have most interested him was the way in which the study of formulas and analytical relations alone could generate substantial mathematical results.

The inspiration for the memoir originated in his very early discovery of an analogy between the mth power of the sum of two variables and the mth-order differential of their product:

$$(a+b)^m = a^m b^0 + m a^{m-1} b^1 + \frac{m(m-1)}{2} a^{m-2} b^2 + \frac{m(m-1)(m-2)}{2\cdot 3} a^{m-3} b^3 + \dots .$$

$$d^m(xy) = d^m x y + m d^{m-1} x dy + \frac{m(m-1)}{2} d^{m-2} x d^2 y + \frac{m(m-1)(m-2)}{2\cdot 3} d^{m-3} x d^3 y + \dots$$

In the introduction he writes:

> Although the principle of this analogy between the positive powers and the differentials, and the negative powers and the integrals, is not evident by itself, however, as the conclusion that one derives from it are not for that less exact, as one moreover may convince oneself <u>a posteriori</u>, I shall make use of it in this Memoir in order to discover different general Theorems concerning the differentiations and the integrations of functions of several variables, Theorems which are for the most part new, and which would be very difficult to arrive at by other paths.
>
> It is a particular type of calculus which appears to me to merit cultivation, and which can give rise to many useful and important discoveries in Analysis; the principal object of this Memoir it to provide several indications of this by showing the rules that one should follow in this calculus and the manner of applying it to different researches.

Lagrange's analogical calculus is illustrated by his derivation of a result in the theory of finite differences. Assume u is a function of x. [5] The first finite difference Δu is the increase in the value of u when x is increased by the finite increment ζ. The second, third,... differences are defined iteratively to be $\Delta^2 u = \Delta(\Delta u)$, $\Delta^3 = \Delta(\Delta^2 u)$,...Using the Taylor expansion of u about x we obtain an expression for Δu in terms of ζ and the differential coefficients du/dx, d^2u/dx^2,...:

$$(1) \quad \Delta u = \frac{du}{dx}\zeta + \frac{1}{2\cdot 1}\frac{d^2u}{dx^2}\zeta^2 + \frac{1}{3\cdot 2\cdot 1}\frac{d^3u}{dx^3}\zeta^3 + \dots .$$

Lagrange wished to establish the converse of this result, an expression for the differential coefficient or derivative du/dx in terms of ζ and the finite differences Δu, $\Delta^2 u$,...:

$$(2) \quad \frac{du}{dx}\zeta = \Delta u - \frac{\Delta u^2}{2} + \frac{\Delta u^3}{3} - \dots .$$

It would be possible to derive (2) directly by calculating Δu, $\Delta^2 u$,... in terms of the derivatives of u and computing the expression on the right side. Lagrange preferred instead to begin the proof with the analogical equation

$$(3) \quad \Delta u = e^{\frac{du}{dx}\zeta} - 1.$$

He explained that the expression on the right side of (3) is to be understood as that which results when "having developed $[e^{(du/dx)\zeta}-1]$ according to powers of du, one applies the exponents of these powers to the characteristic d in order to indicate differences of the same order as the powers, that is, that one changes du^λ into $d^\lambda u$." He rewrote (3) in the form

$$(4) \quad e^{\frac{du}{dx}\zeta} = 1 + \Delta u,$$

and proceeded to take the logarithm of each side of (4):

$$(5) \quad \frac{du}{dx}\zeta = \log(1 + \Delta u).$$

The right side of (5) is to be understood as the expression that results when $(\Delta u)^\lambda$ is replaced by $\Delta^\lambda u$ in the expansion of $\log(1+\Delta u) = \Delta u - (\Delta u)^2/2 + (\Delta u)^3/3 + \ldots$. (5) therefore yields the desired theorem

$$(2) \quad \frac{du}{dx}\zeta = \Delta u - \frac{\Delta^2 u}{2} + \frac{\Delta^3 u}{3} - \ldots .$$

Using the principle of analogy Lagrange derived results considerably more complicated than (2). He noted that the derivation of each theorem although not "founded on rigorous and clear principles is for that however no less exact, as one may assure oneself a posteriori; but it would be perhaps very difficult to provide it with an analytical and direct demonstration."

Given the rather sleight-of-hand quality of Lagrange's derivation it is worthwhile to examine more closely why it works. (The following explanation is adapted from [Lacroix 1806, 530-531].) Note first that each $\Delta^n u$ (n a positive integer) is easily shown to be an expression of the form

$$\Delta^n u = \frac{d^n u}{dx^n}\zeta^n + a\frac{d^{(n+1)}u}{dx^{(n+1)}}\zeta^{n+1} + a'\frac{d^{(n+2)}u}{dx^{(n+2)}}\zeta^{n+2} + \ldots .$$

These relations may be used to eliminate successively $(d^2u/dx^2)\zeta^2$, $(d^3u/dx^3)\zeta^3, \ldots$ from the expression for Δu. In this way we obtain the equation

$$\frac{du}{dx}\zeta = \Delta u + b\Delta^2 u + c\Delta^3 u + \ldots,$$

where b,c,... are determined coefficients that are independent of u. To calculate the quantities we consider $u=e^x$ at $x=0$. Then $d^n u/dx^n = e^x$, $\Delta^n u = e^x(e^\zeta - 1)^n$ and

$$\left(\frac{d^n u}{dx^n}\right)\Big|_{x=0} = \left(\frac{du}{dx}\right)^n\Big|_{x=0} = 1$$
$$(\Delta^n u)\big|_{x=0} = (\Delta u)^n\big|_{x=0} = (e^\zeta - 1)^n.$$

Equation (4) here becomes

$$(4)^* \quad e^{\zeta}=1+(e^{\zeta}-1).$$

Equation (5) in turn becomes

$$(5)^* \quad \zeta=\ln(1+(e^{\zeta}-1)).$$

Expanding the right side of (5)* we obtain

$$\zeta=(e^{\zeta}-1)-\tfrac{1}{2}(e^{\zeta}-1)^2+\tfrac{1}{3}(e^{\zeta}-1)^3-\ldots$$
$$=(\Delta u)|_{x=0}-\tfrac{1}{2}(\Delta^2 u)|_{x=0}+\tfrac{1}{3}(\Delta^3 u)|_{x=0}-\ldots\ .$$

Hence the coefficients b,c,... equal -½, ⅓,... and equation (2) is established. Lagrange's derivation succeeds because it duplicates the formal procedure involved in this particular example.

c) First-order partial differential equations (1774)

One of Lagrange's major mathematical achievements during the 1770s and 80s was his development of a theory of first-order partial differential equations. His researches originated in a memoir of 1774, and were subsequently elaborated in papers of 1776 and 1787. In the 1774 treatise, "Sur l'intégration des équations à différences partielles du premier ordre", he took as his starting point earlier work of Euler and d'Alembert and introduced the ideas that he would subsequently elaborate and make the basis of his theory.

In a survey of the history of partial differential equations of the first order Demidov [1982,26] distinguishes four stages in the development of the subject. The first stage, what he calls the formal-analytic period, involved the researches of d'Alembert and Euler between 1740 and 1770; the second stage was initiated by Lagrange and continued into the 1830s in the work of Monge, Pfaff, Cauchy and Jacobi; the third stage began with Jacobi and Hamilton and was characterized by close connections between the study of partial differential equations and analytic mechanics; the final stage commenced in the 1870s with Sophus Lie's synthesis of the theory.

Demidov uses the term "formal-analytic" to describe the researches of Euler and d'Alembert because these were developed purely analytically and without any underlying geometrical model. He classifies Lagrange as belonging to a subsequent phase of the subject because he was the first to produce a coherent theory, one moreover which was formulated in terms of a clear geometric interpretation.

Let us recall some of the important concepts of the classical theory as it was established by Lagrange. We begin with the general first-order partial differential equation $f(x,y,z,p,q)=0$, where $p=\partial z/\partial x$ and $q=\partial z/\partial y$. Integral solutions of this equation are of three kinds:

(i) The complete integral $F(x,y,z,a,b)=0$ contains two arbitrary constants a and b. It is the most general solution is the sense that all of the other solutions are derivable from it.

(ii) Beginning with $F(x,y,z,a,b)=0$ we let $b=\phi(a)$ where $\phi(a)$ is an arbitrary function, and consider the one parameter family of solutions $F(x,y,z,a,\phi(a))=0$. Taking the envelope of this family we obtain the general integral. By taking a specific function ϕ we obtain a particular case of the general integral.

(iii) By taking the envelope of the two-parameter family in (i) we obtain a singular solution.

Note that in the classical theory the general integral -involving an arbitrary function- is obtained from the complete integral -involving arbitrary constants- by means of a procedure that possesses a natural geometric interpretation, the taking of envelopes.

A striking characteristic of the classical theory is the way in which indications of its historical genesis are concealed from view. An examination of Lagrange's paper of 1774 reveals that the original idea for his investigation was entirely analytical and involved no geometrical interpretation whatsoever. His goal was to integrate the general equation $f(x,y,z,p,q)=0$ (1). His method was to use (1) to obtain q as a function of x,y,z and p and to substitute this expression for q into the total differential equation $dz-pdx-qdy=0$ (2). Regarding p itself as a function of x, y and z we obtain from the known condition for differentiability of (2) the first-order partial differential equation in p, $p(\partial q/\partial z)-q(\partial /\partial z)+(\partial q/\partial x)-(\partial p/\partial y)=0$ (3). Integration of this equation gives rise to a solution of the form $N(x,y,z,a)=0$ (4) containing the arbitrary constant a. Lagrange proceeded to obtain a further class of solutions by varying this constant. Thus he supposed that a is a function a(x,y,z) of x, y and z. By means of an argument that is explained in [Engelsman 1980, 13] he was able to show that a solution will result by eliminating a from the equations $N(x,y,z,a)-g(a)=0$ and $\partial N/\partial a-g'(a)=0$, where g is an arbitrary function. His argument was based on considerations regarding the integrability of certain expressions that is required for a solution to exist.[6]

In subsequent work Lagrange would significantly modify this derivation, discarding the idea of variation of constants and the argument about integrability and replacing them by an interpretation in terms of an integral involving two constants (the complete solution) and the taking of envelopes. It should be noted that the geometric aspect which is prominent in this later theory is entirely absent in the original 1774 memoir. While it is true that he transformed the subject from a collection of special techniques into a mathematical theory properly understood, it is also the case that the crucial idea originated in the earlier formal-analytic stage distinguished by Demidov.

A final point of interest concerns the relevance of Lagrange's classical theory to contemporary discussions of the function concept. Whereas Euler had called a solution complete if it contains an arbitrary function, Lagrange used this term for a solution containing two arbitrary constants. Such a terminological shift was natural for, as we have seen, an integral in Euler's sense is obtainable from Lagrange's complete solution. That the most general solution of a first-order partial differential equation is of the form $F(x,y,z,a,b)=0$ would have been a conclusion congenial to Lagrange. He viewed analysis at the very broadest level as the study of abstract relations between variables and constants. During the vibrating-string debate of the 1750s it had become evident that such a conception led to difficulties with respect to the question of the acceptance of arbitrary functions. The problem was that the notion of a fixed relation among variables resulted in a rather restricted interpretation of what could be counted as an arbitrary function. Because the complete solution of a first-order equation was shown to be of the form $F(x,y,z,a,b)=0$, the question of arbitrary functions -at least in the context of this particular theory- was displaced from consideration and removed as a fundamental point of foundational concern.

Conclusion

The years 1770 to 1776 have been identified as the significant period in the evolution of Lagrange's analytical philosophy and mathematical style. The theme of analysis would underlie much of his subsequent work, in particular his extensive development of the technique of variation of constants in the theory of perturbations. Indeed the question of his analytical style is directly relevant to his solution with Laplace of a central problem of 18th-century exact science, the stability of the solar system.

Given the importance of the decade of the 1770s in Lagrange's career it is of interest to consider his personal situation at this time. He lived in Berlin with his wife, Victoria Conti, a cousin of his from Turin whom he had married in 1767. His life was from all the evidence an extremely quiet one. In his twenty-one years in Berlin he never made a return visit to Turin nor apparently travelled anywhere.

Lagrange's correspondence with d'Alembert and Condorcet has survived in part as have letters that he wrote to his brothers and father in Italy.[7] Items that recur in his exchanges with his family include concern for their financial well-being, comments about his own health and expressions of anxiety -particularly at the time of Prussia's involvement in the late 1770s in the war of Bavarian succession- about political events. He expresses appreciation that Berlin has provided him with a quiet and comfortable place to work, and notes that he prefers the climate there to Turin's.

In a biographical essay entitled "Lagrange's Personality" George Sarton [1944] observes that Lagrange's "letters [to d'Alembert] written in Berlin might have been written just as well from Shangri-La." For insight into matters of mathematical substance the correspondence is a disappointment, carried out as it was following d'Alembert's main period of activity in science. Despite the tone of intimacy in which it is conducted the main link between the two men seems to have derived from their respective interest in the affairs of the Berlin and Paris academies. Responding to d'Alembert's tone of world weariness and false modesty, Lagrange frequently emphasizes the ambivalence he feels about his own achievements in mathematics.

There is nothing in Lagrange's correspondence concerning his mathematical philosophy or his belief in the superiority of analysis. The one mathematical subject in which he does express interest in letters of the early 1770s to both Condorcet and d'Alembert is his work on first-order partial differential equations. He seems concerned to receive their opinion of his work, and one can only assume that he was disappointed by their polite but reserved response.

Overall the impression that emerges in Lagrange's letters and in the reminiscences of his associates is of someone with only a limited engagement in anything outside of mathematics, whose work in this subject was itself of a solitary and individual character. Although a considerable quantity of his unpublished papers have survived they do not illuminate in any significant way his published oeuvre. The story of his mathematical development unfolds in the public record and one will look there for signs of innovation and direction. A broader understanding of his achievements will be found in a study of his formal education and early intellectual influences and in a survey of contemporary patterns of mathematical research. A detailed study of earlier work done on any given subject provides the best preparation for appreciating his own contributions.

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Notes

1. Lagrange's distinctive conception of analysis had definite mathematical and philosophical roots in the work of earlier researchers of the 18th century. This subject, which is beyond the scope of the present essay, will be explored in the author's forthcoming monograph on Lagrange.

2. See [Sarton 1944] for biographical information concerning Lagrange.

3. Detailed citation for the references which follow may be found in [Taton 1974, 10-20].

4. An account of the memoir is provided by Goldstine [1980, 129-138]. The author adheres very closely to Lagrange's original analysis; his account is descriptive rather than critical.

5. Lagrange actually assumes that u is a function of the several variables x,y,z,... For simplicity his procedure is here presented for the case where u is a function of the single variable x.

6. The emphasis of Engelsman's study is different from the present one. Whereas he is concerned with identifying the sources of Lagrange's concept of a complete solution, we wish to underscore the contrast between his later classical theory, with its essential geometrical features, and the purely analytical character of his original 1774 investigation.

7. The correspondence with d'Alembert and Condorcet was published in 1882 in volume 13 of Lagrange's Oeuvres. His correspondence with his family is contained in the archives of the École Polytechnique and has recently been published by Borgato and Pepe [1989].