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Mathematics

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Mathematics is the oldest science, stretching back to 2000 B.C. in Egypt and Babylon and reaching a very high state of development in ancient Greek civilization. A resurgence in mathematics began in the late Middle Ages and has continued from this period to the twentieth century. Throughout history, mathematics has played a fundamental role in the scientific investigation of the world. Beginning with a survey of geometry in Greek antiquity, the present article provides an overview of the development of mathematics from the sixteenth century to the present. The emphasis is on understanding the general character of research in each period rather than on the detailed presentation of particular results. The article provides a context for the more specialized developments described in the articles in this encyclopedia on arithmetic, calculus, algebra, statistics and probability, and mechanics. The article also covers several new subjects that have emerged in the last two centuries: set **theory**, logic, and **dynamical systems.**

* **Set theory.** Study of the structure of sets from the point of view of axioms imposed on them.

➢ **Dynamical systems.** See *Mechanics and Thermodynamics.*

Pre-1543 Roots

By the sixteenth century, a revival of mathematics in Europe was well under way, a revival that was connected to the more general flowering of culture and learning that was taking place at the time. The situation in mathematics was a result of the confluence of several particular factors: the renewed translation and study of Greek mathematical works; the influence of a sophisticated arithmetic–algebraic tradition with Islamic and Medieval roots; the demands of commerce, navigation, engineering, astronomy, and physics; the general value placed on mathematics in Greek philosophical writings; and the emergence of a research ethos that valued the production of new mathematical knowledge.

The great edifice of Greek mathematics was established by several figures in the period 300 to 200 B.C., most notably Euclid of Alexandria (fl. c. 295 B.C.), Archimedes of Syracuse (c. 287–212 B.C.), and Apollonius of Perga (fl. c. 200 B.C.). Euclid's *Elements* was a compilation of several diverse strands of earlier geometry and effectively established the Greek theoretical tradition. Archimedes, probably the greatest mathematician of antiquity, produced studies of a range of mathematical subjects. He used the method of exhaustion to verify the area of geometrical figures. This method was based on inscribing a polygon of known area in a curved figure and increasing the number of sides of the polygon so that the area of the figure was given to an arbitrary degree of accuracy. Archimedes also obtained a very good numerical estimate of the value of pi, provided a detailed study of the curve known as the **Archimedean spiral,** and initiated the quantitative study of **statics** in mechanics. The *Conics* of Apollonius (c. 250 B.C.), was a detailed investigation of a special class of curves obtained by intersecting a plane with a cone.

* **Archimedean spiral.** The curve spiraling into its origin, which in polar coordinates is given by the equation $r = \alpha\theta$.

* **Statics.** The part of mechanics that deals with bodies at rest and in equilibrium.

* **Polyhedron.** A solid bounded by faces that are polygons. A cube is a polyhedron with six faces.

Greek theoretical mathematics was defined by two characteristics. First, it was geometrical in both subject matter and foundation. The objects of study were plane figures, curves, and surfaces. Typical problems involved geometrical constructions, such as the construction of regular polygons in a circle using a straight edge and compass, or the construction of the five regular **polyhedra.** In addition, Greek mathematics was founded at the most basic level on a geometrical theory of quantity. The older traditions of the Egyptians and Babylonians were centered on problems in the arithmetic of whole numbers and fractions. A given magnitude was measured by assigning a number to it, rather than by representing it as a line segment or rectangle. It is also believed that an

early Pythagorean school of Greek mathematics was centered in arithmetic rather than geometry. The turn to geometry that is so fundamental in the surviving works is conventionally attributed to difficulties associated with the discovery sometime around 500 to 400 B.C. of the phenomenon of incommensurability. Simple arguments showed that it is impossible to express the diagonal and side of a right-angled isosceles triangle in **rational** terms, as the ratio of two whole numbers. The side and diagonal are said to be incommensurable magnitudes; in modern terms their ratio is the **irrational number** $\sqrt{2}$. Similarly, suppose a line of length a is divided at a point x so that $a{:}x=x{:}(a-x)$, that is, the line is divided so that the whole is to the larger part as the larger part is to the smaller part. The two resulting parts have no rational ratio. A line divided in this way is said to be divided in the golden ratio, and the value of this ratio is the irrational number $(\sqrt{5}+1)/2$. The existence of incommensurable magnitudes implied the impossibility of a purely arithmetic theory of quantity involving only whole-number proportions. It raised fundamental questions about an operation as basic as multiplication, since it was not at all clear how the traditional definition of multiplication as repeated addition would apply for two incommensurable magnitudes. The so-called "crisis of incommensurables" was resolved by representing magnitudes using line segments, and expressing the products of such magnitudes in terms of rectangles. All magnitudes possessed dimension, and only homogenous magnitudes could be added together—lines to lines, rectangles to rectangles, angles to angles, and so on. A systematic mathematical treatment of quantity was made possible by the further development of a theory of proportions, an achievement that is attributed to Eudoxus (c. 400–c. 347 B.C.) in the fourth century B.C. and was expounded by Euclid in the fifth book of his *Elements.*

* **Rational and irrational numbers.** Numbers in the form n/m, where n and m are integers. An irrational number is a number that can't be written in this form.

The focus on geometry in Greek mathematics was consistent more generally with the strongly spatial, sensual visual element that is found in classical architecture and sculpture. A quantity that is continuous may be divided endlessly; it contains no gaps and admits of arbitrary degrees of gradation. The Greek geometrical perspective led to a strict separation between the analysis of continuous quantity on the one hand and the study of properties of whole numbers on the other. Euclid defined *number* as a multitude of units; in this conception, 1 itself was regarded not as a number but as the origin or principle of number. Since 1 was not a number, there was no possibility of dividing the unit to obtain a numerical estimate of continuous quantity. The study of continuous quantity belonged to the domain of proportion theory, and the latter was an intrinsically geometrical subject.

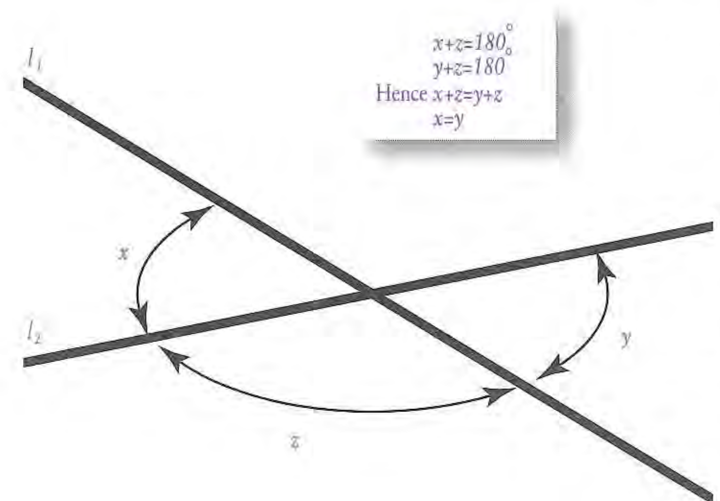

Figure 1. Two intersecting lines, l_1 *and* l_2, *showing that the opposite angles* x *and* y *are equal* ($x = y$).

The second fundamental characteristic of Greek mathematics was its use of deduction to establish results. The concept of deductive proof was a milestone in the history of Western thought and seems to have first appeared sometime in the period 550–400 B.C. It may have been connected to the discovery of incommensurability, but it seems to have arisen as part of a more general philosophical interest in dialectic, reasoned argument by means of question and answer. The nature of deductive proof may be illustrated by a simple example. Given two intersecting lines l_1 and l_2 (see Figure 1), show that the opposite angles x and y are equal, $x = y$. One way of showing this would be to make an empirical estimate of each of the angles, perhaps using the arms of a compass or a protractor, and to note that the resulting measures are the same. Using deduction, however, this result can be established once and for all, for all pairs of intersecting lines. Let z be the complement to x with respect to l_1. It is clear that $x + z = 180°$. But z is also the complement to y with respect to l_2, so $y + z = 180°$. It follows from these two equations that $x = y$. By means of simple deductive reasoning, the desired result has been established for any two lines anywhere.

An *axiom*, another word for *postulate*, is a statement that is accepted without proof. In the axiomatic method, the propositions of a mathematical system are proved deductively from axioms. Euclid's geometry provided the first example of the axiomatic method in the history of mathematics. Euclid and his ancient Greek successors raised deductive proof to an exalted position in their mathematical pantheon. Each treatise is strictly *synthetic*: It begins with definitions and postulates and proceeds to establish in a deductive order the various propositions of the subject. In such an approach, the way in which the proof was discovered or the particular construction devised is not made clear; all that counts is the finished deductive presentation of the result. The synthetic style was also followed in works that would today be regarded as part of physics. For example, Archimedes' treatise on the **equilibrium** of bodies began with a series of definitions and postulates followed by a strict proof from first principles of the law of the lever. He deduced this law for the case where the lever arms are commensurable in length; he then used a proof by contradiction to extend the result to the case where the lengths are incommensurable.

➢ **Equilibrium.** See *Mechanics and Thermodynamics*.

During the period A.D. 100–350 there was a revival of Greek mathematics associated with several figures in Alexandria. Ptolemy (c. 100–c. 170 B.C.) presented his **chord**-trigonometry in his famous work on geocentric astronomy, the *Almagest*. Diophantus (fl. 250) embarked on a detailed investigation of rational number theory in his *Arithmetic*. Perhaps the most important work of the period was Pappus's *Collection*. In it Pappus (fl. 300–350) described tools used by earlier mathematicians to solve the three great classical problems of Greek antiquity: duplication of the cube, trisection of the angle, and squaring of the circle. He also described the problem of the locus to three lines, a very important problem for the development of later mathematics [see *Geometry*]. The *Collection* was also important for initiating a discussion concerning mathematical method. In the seventh book, Pappus described the method of analysis. In its original meaning, analysis consisted of a "solution backwards": One began with the proposition to be proved or the construction to be effected and worked backward until one had arrived at something known. By reversing this process, one arrived at a direct proof or construction, which then formed the basis for the finished presentation in the favored synthetic style of Greek mathematical treatises.

* **Chord.** Given two points on a curve, the chord is the straight line joining them. In a circle, each chord subtends, or defines, a given angle at the center of the circle.

Following the decline and collapse of Greco-Roman civilization, the serious study of mathematics was taken up by scholars in the Islamic world in the period 750–1300. Although Islamic scholars are often credited primarily with having preserved the Greek mathematical heritage, they also enriched the subject with contributions of their own. Noteworthy was the development of a positional decimal arithmetic based on symbols of Hindu origin, including a symbol for zero, and a method of calculation or *algorism* for computing with these numbers. The term *algorism* derived from the name of al-Khwārizmī, a ninth-century Persian who was the first major Islamic mathematician. Al-Khwārizmī (fl. c. 800–47) also initiated the systematic study of solving equations. The name *algebra* derives from the Arabic word he used to denote the operation of balancing an equation by adding terms to each side in the course of obtaining a solution.

In the early Middle Ages the study of mathematics in the Latin West was rather rudimentary, relying on such things as the Roman philosopher Boethius's (c. 480–525) incomplete account of the first book of Euclid's *Elements*. Beginning in the eleventh century, Greek mathematical classics became available in the West, usually in Latin editions of Arabic translations of Greek originals. The revival of learning was stimulated by the

infusion of new ideas in arithmetic, algebra, and astronomy supplied by Islamic scientists themselves. The Crusades of the twelfth and thirteenth centuries increased contact between Western and Islamic culture, although there is disagreement among scholars concerning the precise importance of these contacts for the history of science.

Medieval thinkers were very interested in the didactic and philosophical aspects of mathematics. They were concerned with such questions as the nature of the continuum and whether the motions of the heavenly bodies are commensurable or incommensurable. Nevertheless, the work of two figures of the fourteenth century indicated that a resurgence of new ideas in mathematics was under way. Nicole Oresme (c. 1320–82), the bishop of Paris, developed a method of graphical representation of qualities, called the *latitude of forms*, which permitted the quantitative investigation of problems that traditionally had been treated in a purely verbal way. In England, Thomas Bradwardine (c. 1290–1349) considered the motion of a particle acted upon by a force, a problem in the qualitative Aristotelian tradition of physics, and analyzed the resulting motion using Euclid's theory of proportions [see *Overview: Physics*].

➢ **Trigonometry.** See *Geometry*.

* **Sine law.** Asserts that $a/\sin A = b/\sin B = c/\sin C$ in the triangle ABC, where a is the side opposite $\angle A$, b the side opposite $\angle B$, and c the side opposite $\angle C$.

* **Quadratic equation.** An algebraic equation in one unknown variable x of the form $ax^2 + bx + c = 0$. It is a quadratic equation because the highest power of x in the equation is 2. The term *quadratic* comes from the Latin word *quadratus*, meaning "square"; the area of a square of side x is x^2.

Following the fall of Constantinople to the Turks in 1453, there was a transfer of Greek manuscripts from Byzantium to the Latin West, and many of these works were translated directly from Greek into Latin. This program of translation was associated with, among others, Frederico Commandino (1509–75) and somewhat later, Claude Bachet (1581–1638). Particularly important were translations of Archimedes and Pappus, which had not been available in the Middle Ages and which opened up new vistas to Renaissance mathematicians. Another stream of research consisted of commentaries and improvements on Ptolemy's *Almagest*. The greatest mathematician of the fifteenth century, Johann Müller (1436–76) or Regiomontanus, produced treatises on **trigonometry** to assist the preparation of astronomical tables. His *On All Classes of Triangles*, composed in 1464, contained the **sine law** for plane and spherical triangles and established trigonometry as a science independent of astronomy. For the first time in the Latin West, algebra was used to solve a triangle, effected by Regiomontanus by reducing the given problem to a **quadratic equation**, which he then solved by established methods.

1543–1700

With the exception of number theory, the major mathematical advances of the early modern period were associated with applied subjects, motivated by and connected to work in commerce, engineering, astronomy, and navigation. This was particularly true of work in the Low Countries and Great Britain. In 1591, the Flemish mathematician Simon Stevin (1548–1620) published *La Disme*, a treatise that introduced calculation with decimal fractions in a systematic way. The book appeared in French and Dutch, vernacular languages that were accessible to a practical and commercial audience. At the conclusion of the book, Stevin emphasized the utility of decimal calculation for weights and measures, coinage, trigonometry, and other applied pursuits. Throughout this work he employed **Hindu-Arabic numerals**, which had become quite common in Europe by this time. A very significant theoretical feature of his method was the explicit rejection of the Euclidean distinction between discrete number and continuous magnitude. He announced boldly that "One is a number!", an assumption that enabled him to regard the unit itself as a number and thereby to divide it in order to obtain a numerical representation of continuous quantity.

* **Hindu-Arabic numerals.** The common symbols for numbers in use today (1, 2, 3, . . .); of Indian origin, they came into use in Islamic mathematics in the seventh and eighth centuries A.D.

Stevin's work on decimal arithmetic was parallelled by important work in numerical computation carried out by John Napier (1550–1617) in Scotland and Joost Bürgi (1552–1632) in Germany. Both men developed a system of logarithms that greatly reduced the labor involved in multiplying large numbers together. Napier's work provided the basis for further modifications introduced by Henry Briggs (1561–1630), and the Napier–Briggs logarithms became the preferred method for logarithmic computation. The basic principle is based on two simple ideas. First, the labor involved in adding two numbers is much smaller than that in multiplying them. Second, a geometric sequence ($1, a, a^2, a^3, \ldots$) may be correlated with an arithmetic sequence ($0, 1, 2, 3, \ldots$) by means of a table. The number $x = a^n$ here is matched to its exponent n, which is said to be the *logarithm* of x. The base of this system of logarithms is a. For any number x, one can look up its logarithm in the table. The multiplication of any two numbers $x = a^n$ and $y = a^m$ may then be reduced to the addition of n and m, according to the law of exponents $a^n a^m = a^{n+m}$. Given x and y, one looks up their logarithms n and m in the table, adds these two values together, and then, by returning to the table, identifies the number xy with this sum as its logarithm. In this way the product of two numbers is determined by doing an addition. Napier chose a base a that is very close to 1, so that the sequence of values in the geometric sequence is relatively dense. Later mathematicians chose other bases, and used **interpolation** to obtain numbers distributed within the geometric sequence.

* **Interpolation.** The process of finding the approximate values of a quantity between two known values of this quantity by a definite procedure: for example, to find values between a^n and a^{n+1} by approximation.

The method of approximation employed by Napier is quite interesting from a conceptual point of view. He introduced a kinematic model involving two particles, one moving uniformly from A to B and a second moving so that its velocity at each point is proportional to the remaining distance to B. The two particles generated arithmetic and geometric sequences. By skillfully exploiting the properties of nonuniform velocities, Napier was able to derive an inequality that enabled him to obtain precise bounds on the values of the numbers in his system.

VIÈTE, DESCARTES, AND METHOD IN MATHEMATICS

A profound shift in mentality during the period 1500–1700 was reflected in the increasing prominence of analysis within mathematics. In François Viète's (1540–1603) *Analytic Art* (1591), analysis became closely associated with algebra and the theory of equations, a subject established systematically four decades earlier in Girolamo Cardano's (1501–76) *The Great Art* (1545). In writing down a quadratic or cubic equation in one unknown variable, it is logically assumed at the outset that the thing designated by the unknown satisfies the condition expressed in the equation. To solve this equation is to follow a method that is "analytic" in the original sense of this word, to reason backward from an assumed solution or construction until one arrives at something that is known. The result is an expression for the unknown in terms of the coefficients of the equation, given in terms of arithmetic operations and the operation of taking roots.

The point is illustrated by a simple example. Suppose that it is necessary to find a line segment such that the square formed by this segment added together with a unit square is equal to a rectangle of area 5. To solve this problem, let x be the length of the desired segment, and formulate the condition of the problem as $x^2 + 1 = 5$. In positing this equation, it is assumed logically at the outset that the quantity x is a solution to the problem. This equation reduces to $x^2 = 4$, and the answer $x = 2$ is obtained by taking the square root of each side. The answer is reached by reasoning backwards from the presumed solution, following the analytic method.

The mode of solution just described should be contrasted with the procedure followed in the synthetic propositions of Book II of Euclid's *Elements* (e.g., II.11), where the answer was simply announced at the outset and then verified in a series of deductive steps. Influenced by Renaissance humanism, Viète was eager to link algebra and the theory of equations with classical mathematics; he found the suitable source not in Euclid but in the final major mathematical work of Greek antiquity, Pappus's *Collection*.

Propositions in Greek mathematics were typically divided into two types. In the first, it was required to establish some property of a given mathematical object: for example, to show that the angles of a triangle sum to two right angles, or to show that a certain class of numbers is perfect. (A number is *perfect* if it is equal to the sum of its proper divisors; for example, 6 and 28 are perfect because $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$.) In the second, it was necessary to find or construct something: for example, to find the point that divides a line in the golden ratio, or to determine a curve that satisfies a specified condition. It turns out that the second type of problem is encountered more frequently and is more fundamental to the whole of mathematics. The elevation of analysis within mathematics was based on the recognition that understanding and demonstration must reflect the way in which things are discovered or constructed. What was lacking in ancient mathematics and what is of the essence is the method by which the result is obtained. No one in the early modern period appreciated this fact more clearly than did the French mathematician and philosopher René Descartes (1596–1650). An explicit statement of Descartes's position is contained in his reply to the objections made by Antoine Arnauld (1612–94) to Descartes's *Meditations on First Philosophy* (1641). The *Meditations* was a philosophical essay in which Descartes set forth his famous method of radical doubt, his conception of the self, and several proofs for the existence of God. Arnauld criticized Descartes for failing to follow the classical method of synthesis. To this criticism Descartes replied:

> Analysis shows the true way by which a thing was methodically discovered and derived, as it were effect from cause, so that, if the reader care to follow it and give sufficient attention to everything, he understands the matter no less perfectly and makes it as much his own as if he had himself discovered it. . . Synthesis contrariwise employs an opposite procedure, one in which the search goes as it were from effect to cause. . . . It does indeed clearly demonstrate its conclusions, and it employs a long series of definitions, postulates, axioms, theorems and problems. . . . Yet this method is not so satisfactory as the other and does not equally well content the eager learner, because it does not show the way in which the matter taught was discovered.

It is important to note that Descartes regarded analysis not simply as a discovery procedure but also as an actual method of proof. His point of view is illustrated by the example $x^2 + 1 = 5$ considered earlier. Although the analytic procedure employed in obtaining the answer $x = 2$ would today be regarded as a solution rather than a demonstration per se, Descartes himself regarded it as a proof. By contrast, a synthetic solution in the traditional deductive Euclidean style would begin with the assertion that the given line segment is equal to 2 and verify this by observing that $2^2 + 1$ is equal to 5. Although our assent is certainly compelled, we are left with no indication of how the answer in question was obtained.

Descartes's crucial insight was that invention and discovery are the key to mathematical understanding, a fact that was not at all apparent in the formal deductive style of Greek works. An appreciation of Descartes's insight has been somewhat obscured today by the importance assigned to logical deduction in modern mathematical philosophy. However, astute thinkers in the seventeenth century recognized the full significance of the point in question. Descartes's conception of analysis as the true demonstrative method of mathematics was developed into an explicit program by Arnauld in his celebrated *Logic or the Art of Thinking* (1662), written with Pierre Nicole (1625–95). This book is sometimes known as the Port-Royal logic, so named because of the authors' association with the Jansenist abbey Port Royal of Paris. Whatever reservations Arnauld might earlier have harbored concerning the analytic method, he was by 1660 thoroughly convinced of its virtues. Although he admired the clear and simple principles used by geometers, he objected to geometry's reliance on synthesis for what he regarded as the failure of this method to spread enlightenment. He believed that genuine knowledge will occur only when we understand why something is true, when we have perceived the "order of discovery" revealed by analysis.

In mathematical practice, the new understanding was reflected in the emphasis placed on problems rather than theorems in Descartes's *La géométrie* of 1637. The guiding theme of this work was provided by Pappus's locus problem and its generalizations, and the traditional definition–postulate–proposition style of deductive treatises in the exact sciences was conspicuously absent. At the beginning of this work, Descartes proposed an important innovation, one that allowed him to simplify considerably the handling of problems. Instead of following Euclid's principle of homogeneity, in which each magnitude possesses a dimension, he introduced a geometry of line segments that allowed any magnitude whatsoever (rectangle, cube, etc.) to be represented by a line. In effect, Descartes's system of line segments embodied the notion of a general **real number** and provided an alternative geometric solution to the foundational problem resolved by Stevin with his concept of a decimal fraction. Given two line segments, Descartes showed by means of similar triangles that the product or area formed from these segments could itself be represented by a line segment. It should be noted that from a foundational viewpoint, his procedure was very incomplete: He provided no theory to account for basic propositions concerning ratios and similar triangles. The production of such a theory would have required him to go back and wrestle with the questions Euclid had finessed so brilliantly in the *Elements*. Descartes's procedure was a highly effective, pragmatic move that facilitated the solution of problems but rested at bottom on a rather uncritical conception of mathematical foundations.

* **Real number.** A number that is a *rational number*, one that can be expressed as a ratio of two integers, or an *irrational number*, one that cannot be expressed in that form.

The most important achievement of the seventeenth century was the invention of the differential and integral calculus. This subject had as its basic goal the determination of certain geometrical quantities associated with a curve: its tangent at a point, the path length along it, and the area bounded by the curve and an axis. Development of the calculus was based on the realization that powerful new methods of discovery lay at the foundation of this project. This point is clear when one considers tangent methods invented during the period. The Cambridge mathematician Isaac Barrow (1630–77) was a traditionalist, "between ancients and moderns," who in his *Geometrical Lectures* (1670) derived several elegant theorems concerning the tangents to various curves. In each case the result was stated and then proved via the skillful employment of properties of the curves in question in an impressive display of deductive inventiveness. Barrow's work represented the ultimate development of the classical Euclidean–Archimedean paradigm of mathematics. It stands in contrast to the slightly earlier work of the French legalist and mathematician Pierre de Fermat (1601–65), which stood in a direct line with the discoveries leading to the calculus. Fermat invented a technique involving a special quantity (it is customary to interpret this quantity as an **infinitesimal**, although Fermat did not do so explicitly) that reduced the problem of finding the tangent to a curve to the solution of a certain equation involving the special quantity. His approach provided a procedure based on direct understanding of the problem that could be applied by anyone to any problem without special knowledge or geometrical sophistication. The great power of this method was indicated by its successful and immediate application to a broad range of curves, both geometrical (represented in terms of polynomials) and mechanical (represented in terms of **transcendental expressions**).

* **Infinitesimal.** A magnitude that is smaller than any finite magnitude but is not zero. Infinitesimals were used by seventeenth-century researchers to determine tangents to curves and led to the invention of the infinitesimal calculus.

* **Transcendental expression.** An expression containing mathematical functions that are not expressible in terms of the elementary operations +, -, ×, ÷, and the taking of roots.

➢ **Differential equation.** See *Calculus*.

* **Geodesy.** Study concerning the shape, size, and weight of the earth.

* **Geodesic survey.** A survey determining the length of a degree of longitude at different places on Earth and so allowing for a determination of its shape.

An entirely new conception of method emerged during the early modern period that would underpin mathematics until well into the nineteenth century. The domain of mathematics itself was also enlarged in ways that were unforeseen or rejected by such otherwise advanced thinkers as Descartes. In particular, the calculus of Gottfried Wilhelm Leibniz (1646–1716) and Isaac Newton (1642–1727) made possible the study of a new class of curves and relationships. Mechanical or transcendental curves, generated by various kinds of motion, were describable by means of **differential equations** and could be analyzed using techniques of the new calculus. A vast domain of mathematical objects and methods was opened up for study, giving rise to a project of investigation that would occupy the attention of researchers for the next century.

Scientific Practice: Mersenne and the Institutional Roots of Modern Mathematics

Marin Mersenne (1588–1648) was educated at the Jesuit college of La Flèche and entered the Minim order in Paris. His monastic convent became the meeting place for leading French philosophers and scientists, including the mathematicians René Descartes, Gilles Personne de Roberval (1602–75), and Blaise Pascal (1623–62). Mersenne carried out a voluminous correspondence with scholars all over Europe, communicating knowledge of important discoveries to his network of acquaintances. The project of publishing his complete correspondence, begun in the 1930s, has resulted in 16 thick volumes. Mersenne is best known for his association with Descartes and for making Galileo's writings known to scientists of the northern countries. He was a one-man scientific clearinghouse who coordinated scientific research in Europe in the middle third of the seventeenth century.

In his writings, Mersenne advanced a measured defense of Christian faith, Aristotelian philosophy, and the new natural science. He defended mathematics against a group of Renaissance Italian thinkers who had challenged the certitude of geometry and arithmetic and denied the scientific character of these subjects. These people were proponents of traditional Aristotelian qualitative physics who opposed the mathematization of science then under way. For Mersenne, mathematical propositions were necessary and causal and satisfied the Aristotelian criteria for scientific knowledge. In his *Truth of the Sciences* (1625) he developed his philosophical views through an interlocutor named the Christian philosopher.

Figure 2. Portrait of Marin Mersenne. Courtesy of SPL/Photo Researchers, Inc.

The social group around Mersenne would become a prototype later in the seventeenth century for the formation of state-sponsored scientific academies. The Royal Society of London was established in 1660, and this was followed in 1666 by the creation of the Paris Academy of Sciences. The system of informal scientific communication adopted by Mersenne would be replaced by the official publications of these societies, as well as by independent journals founded for the timely dissemination of scientific knowledge. One of the most famous of these journals, the *Acta Eruditorum* of Leipzig, became the forum in which Gottfried W. Leibniz and his followers presented the differential and integral calculus to the world.

Further scientific societies were formed throughout the eighteenth century, the most important being the Berlin Academy (1700) and the St. Petersburg Academy (1724). The academies provided organization and support for science and permitted a system in which a certain degree of freedom in research was combined with rigorous evaluation by peers. They directed research through the setting of questions for scientific competitions and organized expeditions to make astronomical and **geodesic** observations and to complete **geodesic surveys**. Mathematics occupied a very significant place in the life of the academies. Leonhard Euler (1707–83) was a leading figure in both the St. Petersburg and Berlin academies, and Joseph Louis Lagrange (1736–1813) was prominent successively in the Turin Society, the Berlin Academy, and the Paris Academy of Sciences.

Eighteenth Century

Although there were several notable British mathematicians in the eighteenth century—Abraham De Moivre (1667–1754), James Stirling (1692–1770), Brook Taylor (1685–1731), and Colin Maclaurin (1698–1746) among them—the major lines of mathematical production occurred on the Continent, a trend that intensified as the century developed. Leadership was provided by a relatively small number of energetic figures: Jakob (1654–1705), Johann (1667–1748), and Daniel (1700–82) Bernoulli, Leonhard Euler, Alexis-Claude Clairaut (1715–65), Jean d'Alembert (1717–83), Johann Heinrich Lambert (1728–77), Joseph Louis Lagrange, Adrien-Marie Legendre (1752–1833), and Pierre-Simon de Laplace (1749–1827). Research was coordinated by national and regional scientific academies, of which the most important were the academies of Paris, Berlin, and St. Petersburg.

Although the great advances in mathematics in the eighteenth century occurred in analysis and mechanics, there was also a diffusion of quantitative methods and mentalities in a range of more practical subjects and pursuits. In navigation, experimental physics, engineering, botany, demography, government, and insurance there was an increasing emphasis on quantification and rational method. In the burgeoning industrial arts, instrument makers achieved new levels of precision measurement. In French engineering schools, sophisticated mathematics, including the calculus, was introduced into the teaching curriculum, a practice that would be widely followed in later education. The rational "quantifying spirit" of the Enlightenment would find a lasting and pervasive legacy in the adoption at the end of the century in France of the metric system, a development that took place under the direct supervision of prominent mathematical scientists of the time.

ANALYSIS AND MECHANICS

A distinctive feature of mathematical research in the eighteenth century was the close association that existed between analysis and mechanics, and more generally between mathematics and physics. The extensive cultivation of analysis was evident in the development of the calculus into more specialized branches of research: the theory of infinite series, ordinary and partial differential equations, the calculus of variations, and differential geometry. In addition, analysis was closely connected to several areas of physics, including the mechanics of particles and fluids, the theory of elasticity, celestial dynamics, and the study of machines. The leading mathematicians of the Enlightenment worked extensively in analysis and also made fundamental contributions to applied mathematics and physics. Mathematics itself was understood very broadly to include both pure and applied subjects. The modern disciplinary separation between mathematics and theoretical physics had not yet taken hold.

Jean d'Alembert was the scientific editor of a very important encyclopedia completed under the direction of Denis Diderot (1713–84). The first volume appeared in 1755. In the preliminary discourse to this work, d'Alembert included an organizational chart that outlined the various branches of knowledge. The traditional mathematical subjects of algebra and geometry were grouped together with astronomy, mechanics, and optics as "sciences of nature," and separated from logic, which was regarded as a "science of man." The mathematics section of the Paris Academy was constituted of geometry, astronomy, and mechanics. At the end of the century, Étienne Montucla's (1725–99) *History of Mathematics* grouped several applied subjects together under the traditional designation of "mixed mathematics," including mechanics, cartography, and optics.

ALGEBRAIC ANALYSIS: EULER AND LAGRANGE

By the early eighteenth century, symbolic methods were common in continental mathematics. In the infinitesimal calculus, especially, there were strong algebraic tendencies in the research of the Bernoullis, Pierre Varignon (1654–1722), Jakob Hermann (1678–1733), Count G. C. Fagnano (1682–1766), Jacopo Riccati (1676–1754), and others, tendencies that were combined, however, with pervasive geometric modes of representation. As the century progressed, a movement developed to separate analysis from geometry. This development, associated primarily with the names of Euler and Lagrange, derived from the conviction that analysis was an autonomous part of mathematics based on **algorithms**, concepts, and principles that were entirely nongeometrical in character.

In the 1740s and 1750s, Euler built the framework for an analytical formulation of the calculus: the **function** notation, a theory of trigonometric functions, a uniform procedure for introducing higher-order **differentials**, and a concept of **integration** as the algorithmic inverse of differentiation. Euler's efforts in this direction were taken up with renewed emphasis in the second half of the century by Lagrange, who proudly called attention to the absence of geometrical diagrams in his books on analysis. Lagrange developed his commitment to algebraic analysis into a thoroughgoing philosophy of mathematics that was expressed throughout his extensive writings in exact science.

A notable example of Euler's analytical approach is provided by his introduction in 1748 of the sine and cosine functions. Tables of chords had existed since Ptolemy's astronomy in antiquity, and Regiomontanus in the sixteenth century had established trigonometry as an independent branch of mathematics. With the advent of the differential calculus, trigonometric relations were expressed geometrically in terms of infinitesimal elements contained in a standard reference circle, or given implicitly by means of differential equations. Euler, by contrast, introduced the sine and cosine functions directly as formulas containing variables that were given independent of geometrical constructions involving triangles or circles. The familiar modern notation—sin x, cos x, tan x—originated with him. He also derived the standard power series for the trigonometric functions, using multiangle formulas and techniques that he had employed previously to obtain the exponential series.

* **Algorithm.** A set of rules for the solution of a problem in a finite series of steps.

* **Function.** Mathematical relationship that assigns each element of a set to one element of another set.

* **Differential.** An expression in calculus formed from the differential operator d; for example, the differential of $y = x^2$ is $dy = 2x\,dx$. Differentials are closely related to derivatives, which are obtained by dividing by a differential. In the preceding example, the derivative of y is $dy/dx = 2x$.

* **Integration.** In calculus, the process of finding the area under a curve, or the integral of a function.

ENLIGHTENMENT PHILOSOPHY OF MATHEMATICS: D'ALEMBERT AND KANT

An indication of general Enlightenment attitudes to mathematics is provided by the philosophical views of two leading thinkers of the century, the mathematician Jean d'Alembert and the East Prussian philosopher Immanuel Kant (1724–1804). D'Alembert's views may be inferred from his scientific writings, from his book *Elements of Philosophy* (1759), and from the articles he wrote for Diderot's *Encyclopédie*. Kant's conception of mathematics was set forth in his *Critique of Pure Reason* (1781), the greatest and most systematic treatise on philosophy to appear in the eighteenth century.

Following the encyclopedic ideal that prevailed in the French Enlightenment, d'Alembert was concerned primarily with producing a general synthesis of knowledge rather than undertaking a critical analysis of its foundations. He sought for each science a few principles that yielded the diverse phenomena within its domain. The success of this project depended on reducing these principles to the smallest possible number, in order to achieve for them the greatest extension and fruitfulness. Mathematics provided the model for this sort of investigation and served as the finest illustration of the "true systematic spirit" in science.

The two main schools of scientific philosophy in the eighteenth century consisted of empiricism and rationalism. Rationalism, which was rooted in the writings of Descartes and Leibniz, emphasized the power of thought alone to discover substantial truths about the world. Empiricism had been developed by British thinkers, John Locke (1632–1704) being the most prominent, and maintained that all of our ideas originate in sensory experience of the external world. Rationalism was associated with Continental writers, and empiricism with British writers. Like many French thinkers of his time, d'Alembert was deeply impressed by Locke's empiricist philosophy, and repeatedly emphasized the primacy of the world of experience and the sensory origin of ideas. There were, nonetheless, strong rational elements in his own conception of science. His concern with developing each science from a few clear and distinct principles, on the model of mathematics, situated him generally within the methodological tradition established by Descartes. In his *Treatise on Dynamics* of 1743 he introduced the concept of a body as an impenetrable extension moving in space. Such perfectly inelastic bodies interacted by contact according to laws that were derivable a priori, in a purely deductive way, with no reference to the concept of force or to experimental verification. Beginning with this conception, he developed a mathematical theory for obtaining the differential equations of motion of a wide range of dynamical systems. The clarity and simplicity of his method, its mathematical and nonexperimental character, indicated the influence of Descartes's rational philosophy on d'Alembert's thought.

D'Alembert's philosophy of mechanics was broadly consistent with the contemporary understanding of mathematics as a subject that included many physical areas of investigation. However, in his discussion of algebra, a somewhat more abstract perspective emerged. His conception of algebra was influenced by both empirical and rational considerations. Mathematics began with a concept of magnitude that was furnished by experience. At the most primitive level, magnitude consisted of a quantity that was capable of increase or diminution. In the article "Magnitude" in the *Encyclopedia*, d'Alembert distinguished between *concrete magnitude*, which referred to extension and time and was continuous, and *abstract magnitude*, which referred to whole numbers and was discrete. In his *Elements of Philosophy* he commented explicitly on the relationship of geometry and mechanics to algebra:

> The object [of geometry and mechanics] is material and sensible, and a perfect knowledge of this object belongs to those aspects of body and extension whose nature is unknown to us. But the principles of algebra involve purely intellectual notions, ideas that we form within ourselves by abstraction, by simplification and by generalization of first ideas...

D'Alembert noted that a flexible method was appropriate in algebra because "it is a purely intellectual and abstract science whose object does not exist outside of us." Algebra occupied a privileged place in the scheme of knowledge because it was known by our "natural lights" and functioned as the instrument of discovery in the study of magnitude.

In drawing attention to the special character of algebra within mathematics, d'Alembert was enlarging on certain attitudes that were found in the writings of the Malebranchistes, an important school of Cartesian thinkers in the generation that preceded his own in France. In his *Inquiry Concerning Truth* (1674–1712), Nicolas Malebranche (1638–1715) had emphasized the fundamental character of arithmetic and algebra. In contrast to geometry, which involved an appeal to our sensory faculties, these subjects required only the exercise of the intellect, of pure reason; for Malebranche they, not geometry, constituted the true foundation of mathematics.

D'Alembert died in 1783, just two years after Immanuel Kant had completed his *Critique*. Although Kant had serious interests in science, his work in this subject was largely descriptive and speculative; in contrast to d'Alembert, he was not an important mathematical researcher. He made no attempt in the *Critique* to address the major developments in analysis that had shaped contemporary mathematics, and confined his discussions to elementary geometry and arithmetic. His goal in considering mathematics was to undertake a critical analysis of its foundations, of the question of how it is possible that we have certain mathematical knowledge as preparation for a larger investigation of metaphysics. He wanted to answer two questions: How is pure mathematics possible? How is pure natural science possible?

Kant's thought may be viewed as a highly sophisticated synthesis of rational and empirical elements in contemporary philosophy. According to Kant, mathematics concerns itself with the pure form of sensibility and as such is a priori, or prior to experience. It is based on two fundamental intuitions, an outer intuition of space and an inner intuition of time. Geometry is derived from the former and arithmetic from the latter. All knowledge consists of judgments, that is, of statements asserting what is true formulated in subject–predicate form. *Analytic* judgments are those in which the predicate is contained in the subject and so are logically true by virtue of the meaning of the words contained in them. Consider, for example, the statement "A bachelor is an unmarried man." If we analyze the meaning of the subject bachelor, we arrive at the idea of an unmarried man; the predicate is contained in the subject and the statement is therefore analytically true. The propositions of mathematics, by contrast, are *synthetic*: The predicate of a mathematical proposition cannot be obtained from the subject through analysis of its meaning but needs to be attached to it synthetically. In a statement such as $12 = 7 + 5$, Kant reasoned that the predicate $7 + 5$ is not contained in a logical sense in the subject 12. Although the truth of mathematical judgments is known prior to experience, such statements are synthetic. Kant therefore distinguished clearly between statements that are logically true and statements that are mathematically true.

For Kant, mathematical understanding involved construction. One obtains, for example, the concept of triangle by constructing this concept, that is, by exhibiting it in one's intuition of space. Numerical concepts are obtained by constructing them in one's intuition of space and time. Kant distinguished knowledge that involves construction from discursive knowledge (e.g., "Every event has a cause"), whose synthetic character involves concepts of understanding.

D'Alembert had grouped arithmetic and algebra together, and geometry and mechanics together. The former were regarded as abstract and separate from experience; the latter as sensuous and material. Kant, by contrast, grouped arithmetic and geometry together because both were based on the pure form of sensibility, on a priori intuitions of space and time. He sharply separated, on philosophical grounds, the science of motion from arithmetic and geometry: "All other concepts belonging to the sensibility, even that of motion, in which both elements [space and time] are united, presuppose something empirical. But in space, considered by itself, there is nothing movable; consequently the movable must be something found in space only through experience, and must therefore be an empirical datum." Later, Kant wrote that physical matter signified "something which is met with in space and time and which therefore contains an existent corresponding to sensation." In these passages, the modern logical distinction between pure mathematics and mathematical physics emerged clearly for the first time.

Nineteenth Century

In his book *The Order of Things* (1966) the philosopher Michel Foucault (1926–84) suggested that in several branches of inquiry—biology, historical linguistics, and economics—it is possible to distinguish three simultaneous stages of development in the period from about 1700 to 1850: a descriptive phase, involving the holistic collection of all facts relating to the subject; a classical stage, in which well-defined categories and formal relationships are explored systematically; and finally, a stage characterized by more critical tendencies, in which researchers search for underlying principles and modes of explanation. Foucault described each of these stages in terms of an *episteme*, a kind of collective mentality or system of thought that characterized research during each period. Although he was concerned primarily with biological and social sciences, subjects that possessed what he called a "low epistemological profile," his ideas are relevant, at least in a general sense, to the case of mathematics. In particular, there is no question that a significant change in mathematical outlook occurred in the nineteenth century, a change that was marked by a self-conscious reinterpretation of the very nature of the subject. The emergence of a critical sensibility was evident in the traditional parts of mathematics, in analysis, geometry, and algebra, and was also apparent in the appearance of new and unprecedented areas of research.

In the late 1830s, William Rowan Hamilton (1805–65) began to investigate complex numbers; that is, numbers of the form $a + bi$, where a and b are real numbers and i is the imaginary number satisfying $i^2 = -1$. Such numbers had been known since the sixteenth century, when they appeared as the roots of quadratic and cubic equations. For example, the complex number $-2 + i$ is a root of the quadratic equation $x^2 + 4x + 5 = 0$. Complex numbers obey what is called the *commutative law for multiplication*. If you consider two such numbers $a + bi$ and $c + di$ and multiply them together, the result is independent of the order of multiplication. For example, take $1 + 2i$ and $3 - 4i$. We have $(1 + 2i)(3 - 4i) = 3 - (-8) + 6i - 4i = 11 + 2i$ and $(3 - 4i)(1 + 2i) = 3 - (-8) - 4i + 6i = 11 + 2i$, and the answer is the same. It was naturally assumed by everyone that every sort of number system that exists must obey the commutative law of multiplication.

Hamilton attempted to develop a generalization of complex numbers, and naturally tried to do so in a way that respected the commutative law. After a great deal of trouble and failed effort, he hit upon a revolutionary discovery: The number system he was looking for consisted of a perfectly satisfactory mathematical structure, but one in which the commutative law was not satisfied. Hamilton's *quaternions*, as he called his new numbers, turned out to be extremely useful in geometry and physics and opened up entire new vistas of research undreamed of by earlier mathematicians.

In geometry, Nikolai Lobachevsky (1792–1856) and János Bolyai (1802–60) recognized that there were valid alternatives to the time-honored geometry of Euclid. [It is customary to cite Carl Friedrich Gauss (1777–1855) here as well, although he did not publish his results and may not have appreciated their significance.] In the 100 years that preceded their work, researchers had investigated in detail the consequences that follow when Euclid's fifth postulate is rejected. Geometers from Girolamo Saccheri (1667–1733) to Franz Taurinus (1794–1874) had postulated for the sake of investigation that given a line l and a point P not on the line, it is possible to pass through P more than one parallel or nonintersecting line to l. These people believed in the absolute truth of Euclidean geometry and reasoned that the given postulate should lead to a contradiction, to an absurd and unacceptable geometrical system. Nevertheless, they derived many results and arrived at various constructions that seemed logically unobjectionable, although in the end they managed to convince themselves that they had arrived at a contradiction. Lobachevsky and Bolyai possessed the intellectual imagination and independence of mind to recognize that non-Euclidean geometry is a genuine conceptual alternative to traditional geometry.

The development of non-Euclidean geometry may be viewed as a partial illustration of the ideas about scientific change proposed by Thomas Kuhn (1922–96) in his *Structure of Scientific Revolutions* (1962). Kuhn observed that researchers in any given part of science belong to a community committed to a collective point of view or paradigm, the latter being inculcated in the education, training, textbooks, and social

organization of researchers in the field. In a typical or "normal" period, scientific activity consists in investigating and solving puzzles posed by the established paradigm. At certain times in the history of a subject, anomalies are detected, fundamental problems with the paradigm emerge, and a crisis results. In the ensuing revolution, the old paradigm is rejected, a new paradigm is established, and a new period of normal science begins. Kuhn's model of scientific change was suggested by his study of the history of astronomy, chemistry, and physics—empirical sciences—and it is clear that there are important **epistemological** differences that distinguish mathematics from these subjects. As many commentators have observed, there are no revolutions in mathematics. What has gone before possesses an enduring validity—the idea that Earth is the center of the celestial system is rejected as false, but Euclidean geometry remains with us forever. There are, nevertheless, important respects in which mathematics follows the Kuhnian model of change. Certainly, the commitment to Euclidean geometry constituted a paradigm that was shared universally by researchers before 1800. Also, the emergence of non-Euclidean geometry represented, if not a revolution, at least a radical conceptual change in mathematical understanding; the fundamental identity of mathematical and physical space, upheld by thinkers from Euclid to Kant, was rejected. Development of the new geometry was stimulated by discoveries that conflicted with Euclid but seemed to involve no logical inconsistencies. The new geometry was actively debated and even resisted by some sections of the scientific community; with the assimilation of the results of Lobachevsky and Bolyai came a new pluralism with respect to the nature of geometrical theories that was widely adopted by the mathematical community.

* **Epistemological.** Relating to theories concerning the nature and structure of knowledge.

GEORGE BOOLE AND MATHEMATICAL LOGIC

The new understanding of mathematics that emerged in the nineteenth century was manifested very clearly in the work of George Boole (1815–64). Bertrand Russell (1872–1970) wrote famously that "pure mathematics was discovered by Boole," a reference to his pioneering work in mathematical logic carried out in the 1840s and 1850s. In his application of new systems of calculation to the subjects of logic and probability, Boole exhibited an expanded conception of the nature of mathematics; both this conception and the technical system he developed have inspired generations of researchers up to the present day.

Boole, the son of a Lincolnshire cobbler, came from a provincial background, and like his contemporary Michael Faraday (1791–1867) benefited from the tradition of popular scientific education that had developed in early nineteenth-century England. He established an independent school in Lincoln, where in his spare time he carried out mathematical research in analysis, logic, and probability. These researches were of such significance that in 1849 he was awarded a position in mathematics at Queen's College, Cork, in Ireland, despite his lack of a university degree. Boole taught and pursued research at Cork up to his death in 1864.

English mathematics during the early nineteenth century was strongly shaped by a movement to embrace continental analysis—the analysis of Leibniz, the Bernoullis, Euler, Lagrange, and Laplace—and thereby to counteract what was seen as a significant decline in the level of research that had set in during the eighteenth century. This movement, centered at Trinity College, Cambridge University, included such figures as Robert Woodhouse (1773–1827), George Peacock (1791–1858), Charles Babbage (1792–1871), and John Herschel (1792–1871). In the years 1812–14, undergraduates of the Analytical Society actively promoted Leibnizian techniques in preference to the traditional **fluxional** and geometric approach of English mathematics. The views of the society were disseminated by its members upon graduation and became an established theme of contemporary mathematical work. Boole's early research developed within

➢ **Fluxion.** See *Calculus*.

this renewed tradition of analysis and consisted of a series of investigations in which he explored the use of formal operations in the solution of differential equations.

The early nineteenth century was also a period of renewed interest in logic among British philosophers and mathematicians. In his *Elements of Logic* (1826), Richard Whately (1787–1863), a professor at Oxford University, called attention to the serious value of logic and logical study in an attempt to redeem an ancient subject that had fallen into disrepute among empirically minded British thinkers. Whately defined logic as the formal science of reasoning and stressed its utility for all areas of rational investigation. Although the Scottish philosopher William Hamilton (1788–1856) (not to be confused with the Irish mathematician William Rowan Hamilton) was somewhat critical of Whately's doctrines, he accepted the value of logic and initiated the quantitative study of this subject. During the 1830s Hamilton became embroiled in a controversy with the mathematician Augustus De Morgan (1806–71) over questions of priority, an affair that also involved differences of opinion about the value of mathematics in liberal education. Boole's interest in logic was stimulated by De Morgan.

Boole developed his new mathematical understanding of logic in the books *Mathematical Analysis of Logic* (1847) and the *Laws of Thought* (1854). He explicitly established a radical new perspective with the following assertions: "On the principle of a true classification, we ought no longer to associate Logic and Metaphysics, but Logic and Mathematics" (1847); and "that Logic, as a science, is susceptible of very wide applications is admitted, but it is equally certain that its ultimate forms and processes are mathematical. . . . It is not of the essence of mathematics to be conversant with the ideas of number and quantity" (1854).

Boole's formal system of calculation consisted of a *class algebra* in which the traditional forms of the Aristotelian **syllogism**, as well as more complicated logical judgments, were expressed by means of algebraic equations. The variables $x, y, z, \ldots$ denote classes of any things, not simply numbers or quantities, and equations involving arithmetic operations indicate the relations between these things as formulated by propositions in ordinary language. The statement "All humans are mortal" may be thought of in terms of classes as asserting that the class of all humans is contained in the class of all mortals. Boole expressed this statement by the equation $x \bullet y = x$, where x denotes the class of humans and y denotes class of mortals. In terms of the familiar modern operations of set intersection and set union, the multiplication operation "$\bullet$" means intersection and the addition operation "+" means union. Hence $x \bullet y = x$ says simply that the intersection of the class of humans with the class of mortals is the class of humans itself. The symbols 0 and 1 indicate the empty set and the total universe of discourse, respectively. The complement of a class x is equal to $1 - x$, denoting the class of all objects in the universe that are not x.

* **Syllogism.** A valid deductive argument containing two premises and a conclusion. For example: All humans are mortal; Socrates is human; therefore, Socrates is mortal.

A fundamental law of Boole's algebra is formulated by the universal equation $x \bullet (1 - x) = 0$, expressing the fact that the intersection of any class x and its complement $1 - x$ contains no elements. This relation may be rewritten as $x \bullet 1 = x \bullet x$ or $x = x^2$ and expresses the logical fact that "Every x is an x." It was here that his system departed from the ordinary algebraic rules satisfied by numerical quantities; in the algebra of real numbers the equation $x^2 = x$ is not generally valid. Boole's system constituted a new kind of algebra, one that described logical statements but differed from the traditional numerical algebra of Cardano, Viète, and Descartes. Implicit in these different types of algebra was the radically new notion of an uninterpreted formal system of calculation, one that satisfied certain specified rules but whose symbols and operations could be interpreted differently in different contexts.

Boole's algebraic logic was an important pioneering work, but it possessed certain limitations, the failure to express quantification being the most significant of these. Quantification refers to statements of the form "For all x it is the case that. . ." and "There exists x such that. . ." There were also technical difficulties in some of Boole's procedures and derivations. His system would be modified and extended in the nineteenth century by Stanley Jevons (1835–82), John Venn (1834–1923), Charles S. Peirce (1839–1914), and Ernst Schröder (1841–1902). Noteworthy here was Venn's use of geometrical diagrams to depict relations among propositions expressed by Boole's algebraic equations. Such diagrams would become important in modern mathematical education in representing logical and set-theoretic relations in a visual way. Another far-reaching feature of Boole's systems consisted of utility in the design of automatic logic and calculation machines. In 1869, Jevons constructed on Boolean principles a logic machine the size of a piano, which he exhibited at the Royal Society in London. Boolean algebra is basic to the design and operation of electronic computing devices and has played a prominent role in the development of modern computer science.

PHYSICAL SCIENCE STANDS APART FROM MATHEMATICS

With the emergence in the nineteenth century of an internal and logically self-contained conception of mathematics, there also developed a corresponding notion of theoretical physics, in which a highly mathematical approach was combined with a clear disciplinary separation of physics from mathematics. The history of physics in the period 1790–1840 has been termed a "second scientific revolution," a period characterized by the use of sophisticated mathematical models in several areas of physical investigation traditionally treated qualitatively: the theory of heat, electricity, and magnetism. People who were primarily mathematicians, such as Augustin-Louis Cauchy (1789–1857), Carl F. Gauss, and Georg Bernhard Riemann (1826–66) certainly made important contributions to physical science, but the central participants were increasingly researchers who trained and practiced as physicists. The consolidation of mathematical physics as the century progressed was accompanied by the emergence of theoretical physics as a new disciplinary category within science. The situation is aptly summarized by Russell McCormmach and Christa Jungnickel in their social history of nineteenth-century physics: "The position of intermediary between mathematics and physics, as Riemann was seen to hold, was increasingly taken over by a new kind of specialist, the theoretical physicist. The theoretical physicist might consult or even collaborate with a mathematician, but he always worked as a physicist rather than a mathematician. As a physicist, he was knowledgeable in mathematics, and although he did not do original research in mathematics, he was capable of adapting new mathematics to physical uses and, in the process, of offering the mathematician new mathematical opportunities."

This new disciplinary alignment was apparent in the growing distinction between mathematical physics, a subject practiced by mathematicians, and theoretical physics, a subject of concern to physicists. Whereas in the eighteenth century the leading mathematicians were also the leading physicists, 100 years later the two communities had coalesced along different lines. A survey of some of the major researchers in the second half of the nineteenth century indicates the divide clearly: Among mathematicians there were such figures as Karl Weierstrass (1815–97), Leopold Kronecker (1823–81), Charles Hermite (1822–1901), Arthur Cayley (1821–95), Eugenio Beltrami (1835–1899), and Sophus Lie (1842–99); among physicists, Wilhelm Weber (1804–91), James Clerk Maxwell (1831–79), Josiah Gibbs (1839–1903), Ernst Mach (1838–1916), Ludwig Boltzmann (1844–1906), and Pierre Duhem (1861–1916). The German experimentalist Wilhelm Wien (1864–1928) publicly discussed the disciplinary dichotomy between physics and applied mathematics in 1915 in a thoughtful article entitled "Goals and Methods of Theoretical Physics."

CANTOR AND SET THEORY

Georg Cantor (1845–1918) began his career at the University of Berlin, where he completed a doctorate in number theory in 1867. He soon turned to problems in analysis, a subject that fell in Berlin within the province of the great mathematician Karl Weierstrass. In the second half of the century Weierstrass delivered a series of lectures that reshaped the contours of mathematical analysis. He recognized that subtle but profound questions of existence were involved in many mathematical techniques and reasonings employed uncritically by researchers. Among the fields to which he contributed was the theory of Fourier series. This subject concerned the way in which a given mathematical function may be expressed in terms of an infinite sum of the basic trigonometric functions sin x and cos x. Invented in the early nineteenth century by the French mathematician Joseph Fourier (1768–1830), the theory had many applications in physics and engineering and soon developed into a major branch of mathematical research. Cantor's first work in mathematics involved the study of collections of real numbers that arose in problems in Fourier series.

After an initial period of research in Berlin, Cantor procured a position in 1869 as lecturer and later professor at the University of Halle. He was deeply engaged in the study of real numbers and provided a new and fundamental definition of such numbers. A turning point in his career occurred in late 1873, when he began to think seriously about the notion of infinity and what it meant to say that two infinite sets are of the same size. The central tool in all of his work was the concept of a one-to-one correspondence (introduced explicitly by Cantor in 1878). Cantor realized that the size or *cardinality* of an infinite set may be specified in terms of this concept: Two sets have the same cardinality if their members can be put in a one-to-one correspondence. Although the set of all positive integers {1, 2, 3, 4, . . . } may seem to be twice as large as the set of even integers {2, 4, 6, . . . }, according to Cantor's definition, the two sets have the same cardinality. This is because the respective members of the two sets can be matched up in a one-to-one fashion: 1↔2, 2↔4, 3↔6, 4↔8, and so on. Similarly, the set of all integers {...-3, -2, -1, 0, 1, 2, 3, . . . } has the same cardinality as the set of positive integers {1, 2, 3, 4, . . . } because the two sets may be matched up one to one: 1↔0, 2↔1, 3↔-1, 4↔2, 5↔-2, and so on.

In late 1873, Cantor realized that there are different orders of infinity, and that infinite sets may be ranked hierarchically with respect to cardinality. He did this by showing that a one-to-one correspondence between the integers and the real numbers does not exist. The integers are said to be a *countable* set, since they can be enumerated or counted in a definite list. The real numbers, as Cantor showed, are *uncountable*: They cannot be enumerated or counted in a list. Since the integers are embedded in the reals, this implied that the size or cardinality of the real numbers is greater than that of the integers. Cantor used the first letter of the Hebrew alphabet, ℵ (*aleph*), to designate an infinite cardinal number. (Cantor used Hebrew letters to designate infinite cardinals, because the Latin and Greek alphabets were already used for more familiar mathematical quantities.) The cardinal number of the set of integers is the smallest infinite cardinal number, which Cantor designated as $\aleph_0$ (pronounced "aleph-zero"). The real numbers form a **continuum**, and their cardinal number was designated simply as c, where c stands for the continuum. Cantor had shown that $\aleph_0 < c$.

Cantor's initial discovery was followed by further revelations. He turned his attention to the *rational numbers*, that is, numbers of the form n/m, where n and m are integers. The rational numbers are said to be *dense*, in the sense that given any two rational numbers, no matter how close together, a new rational number can be formed in between them by taking the average of the two. It would appear that the rational numbers are very numerous, much more so than the discrete sequence of positive integers {1, 2, 3, . . . }. Nevertheless, Cantor was able to show that the rational numbers are countable by giving a simple process that allowed one to enumerate these numbers in a list. Hence the rational numbers have the same cardinality as the integers: Their cardinal number is $\aleph_0$.

* **Continuum.** The entire real line, or the open interval of numbers between zero and one. On a line with an origin and no endpoints, every point corresponds to a real number, and to every real number there corresponds a point on the line.

Cantor next examined the set of *algebraic numbers*, that is, numbers that are the roots of polynomial equations with integral coefficients. The numbers 2, -3, 3/4 and $\sqrt{3}$ are algebraic because they are the roots of the equations $x - 2 = 0$, $x + 3 = 0$, $4x - 3 = 0$, and $x^2 - 3 = 0$, respectively. It is evident that algebraic numbers are rather numerous, since in addition to the rational numbers they include all of the roots of such quantities. Real numbers that are not algebraic are called *transcendental*. Examples of such numbers are π (pi), the ratio of the circumference of a circle, and e, the base of the system of natural logarithms. It turns out that it is very difficult to show that a given number is transcendental. [It was not until 1873 that Charles Hermite (1822–1901) showed that e was transcendental; the transcendence of π was established by Ferdinand Lindemann (1852–1939) in 1882.] Cantor proved in 1874 that the algebraic numbers are countable. They can be enumerated in a list, and so have the same cardinality $\aleph_0$ as the positive integers. Since the real numbers are uncountable, and since the real numbers are the union of the algebraic and transcendental numbers, this result implied that the transcendental numbers are uncountable and possess the same cardinality as the real numbers themselves. Cantor's conclusion formed the basis for a quite unorthodox existence proof: He had established by set-theoretic reasoning not just that transcendental numbers exist but that there is a great profusion of such numbers—without ever actually exhibiting a single example!

In 1877, Cantor showed that the unit square can be put in a one-to-one correspondence with the unit interval, so that the two sets have the same cardinality. This result seemed to conflict with one's intuitions about dimensionality, since a two-dimensional continuum had been shown to have the same cardinality as a one-dimensional continuum. Cantor himself said, "I see it, but I don't believe it!" In the next two decades he singlehandedly developed a theory of transfinite cardinal arithmetic. He carried out this research despite problems with his health—he was afflicted with manic-depression and was incapacitated by a series of nervous breakdowns, the first of which occurred in 1884 when he was 39 years old. As he grew older, his mental health deteriorated and he spent the last years of his life in a Halle psychiatric institute. Cantor's equilibrium may also have been disturbed by professional tensions stemming from the original and controversial character of his work. Cantor was particularly concerned about resistance to his ideas expressed by the distinguished Berlin algebraist Leopold Kronecker, who opposed the publication of some of his results. Despite the presence of influential supporters and access to leading journals, Cantor was obsessed by criticism of his research, and resented the fact that he was never offered a position at a major university.

Cantor also experienced considerable stress as a result of concrete difficulties in his set-theoretic investigations. As his career progressed, he became very concerned with several fundamental problems, the most famous of which is called the *continuum hypothesis*. If the sequence of infinite cardinal numbers is listed, beginning with the smallest and increasing in size, $\aleph_0, \aleph_1, \aleph_2, \ldots$, the question naturally arises: Where does the cardinal number c of the continuum of real numbers appear in this list? Cantor had shown that c is larger than $\aleph_0$. He supposed very naturally that c is the second largest infinite cardinal, $c = \aleph_1$, a conjecture that became known as the continuum hypothesis. To deny this hypothesis would be to suppose that there is a subset of the real numbers that is larger in cardinality than the set of integers but smaller than the cardinality of the set of real numbers itself. No amount of research and study revealed the existence of such a set: Every subset examined had either the cardinality of the integers or the cardinality of all of the real numbers. However, try as he might, Cantor was unable to rigorously prove the continuum hypothesis, and the strain of the effort weakened his mental health.

The next stage in the history of set theory would be characterized by the axiomatic development of the foundations of the subject. Cantor was developing a new field and had proceeded in an exploratory manner, without worrying a great deal about careful definitions and explicit postulates. By 1900, it was becoming apparent that an informal approach was no longer acceptable; the continuum hypothesis and other problems had raised considerable anxiety about the very foundations of the subject. The solution was to follow the model of Euclidean geometry and to develop set theory rigorously in terms of a small number of definitions and axioms. In 1908, Ernst Zermelo (1871–1953) introduced a new axiom that would become known as the *axiom of choice*, and with it was able to resolve some fundamental problems. Zermelo proposed a list of axioms for set theory, a list that was revised and augmented by Abraham Fraenkel (1891–1965) in 1922. The standard axioms for set theory today are called the *Zermelo-Fraenkel* (Z-F) *axioms*. By the early part of the twentieth century, set theory was accepted by the mathematical community as an important achievement and would come to be recognized by many researchers as the preferred way of formulating mathematics in a completely general way.

FREGE AND LOGICISM

Kant had considered a class of statements that he termed *analytic*, statements that are true logically or by virtue of the meaning of the terms contained in them. As discussed earlier, the statement "A bachelor is an unmarried man" is analytically true, since the idea of an unmarried man is contained logically in the idea of a bachelor. For Kant, mathematical judgments were not of this sort. In a statement such as $12 = 7 + 5$, he reasoned that the predicate $7 + 5$ is not contained in a logical sense in the subject 12. Although the truth of mathematical judgments is known prior to experience, such statements are *synthetic*. As noted earlier, Kant distinguished clearly between statements that are logically true and statements that are mathematically true.

The great Jena philosopher and mathematician Gottlob Frege (1848–1925) rejected this fundamental tenet of Kantian philosophy and argued that the propositions of arithmetic are logical truths. He developed his mathematical logic or "conceptual writing" in 1879 as part of a program to show that all of arithmetic consists of analytical judgments. Whereas Boole had advocated the application of mathematics to logic, Frege was advancing the much more radical thesis that all of mathematics may be reduced to logic. This philosophical position became known as *logicism*.

Frege presented his views with a remarkable vividness and clarity, no more so than in his 1884 classic *Foundations of Arithmetic*, a work that even today provides an excellent introduction to the philosophy of mathematics. A central thesis of this book concerned the close connection between mathematics and language. Traditionally, mathematics was linked with empirical or quantitative subjects, and logic was grouped with linguistic arts such as rhetoric and grammar. Mathematics was the study of quantitative phenomena: shapes, numbers, musical harmonies, planetary orbits, and so on. It was associated with communication only in a metaphorical or figurative sense, as, for example, in Galileo's remark that the "book of nature is written in the language of mathematics." Frege intended on philosophical grounds to establish a direct and fundamental connection between arithmetic and language. He rejected empirical accounts of number, but adhered strongly to the belief that our knowledge of numbers is objective. What is objective is "what can be conceived and judged, what is expressible in words. What is purely intuitable is not communicable." The basis of his argument is something that has become known as the *context principle*: "never to ask for the meaning of a word in isolation, but only in the context of a proposition." When we consider a word in isolation, it calls up some sort of idea in us; and we accept this idea as the meaning of the word. Frege observed: "But we ought always to keep before our eyes a complete proposition. Only in a proposition have the words really a meaning. It may be that mental images float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough if the proposition as a whole has a sense; it is this that confers on its parts also their content." Ideas and mental images provide no basis for an understanding of number; the meaning of a number emerges only in the context of a proposition. At the end of the *Foundations* he wrote: "Only by adhering to this [the context principle] can we, as I believe, avoid a physical view of number without slipping into a psychological one." Frege's interpretation of arithmetic in terms of language was grounded in his belief that the objects of arithmetic are independent of the physical world around us, but are objective nonetheless.

Central to Frege's technical notion of number was the concept of one-to-one correspondence. Frege recognized that this notion is a fundamental one and even precedes the basic act of counting. A very young child who is unable to count is still able to produce the same number of knives as forks in a table setting, because he or she knows that each place must have a matching knife and fork. The concept of a one-to-one correspondence was central to Cantor's work on infinite sets, and it also appeared in contemporary writings of Richard Dedekind (1831–1916) on real numbers. According to Frege, two sets have the same number if their elements can be put in a one-to-one correspondence. Particular numbers are defined in terms of *inclusive classes*, so that the number three, for example, is defined as the class of all sets containing three elements. The logicist conception of number satisfied the formal criteria required in a rigorous theory of definition and would be regarded as one of the very positive achievements of Frege's mathematical philosophy.

Frege also introduced many of the central technical notions of modern mathematical logic, first in his *Conceptual Writing* (1879) and later in his *Fundamental Laws of Arithmetic* (1893–1903). Perhaps his greatest achievement was to develop logic in a way that allowed one to express quantification, that is to say, statements of the form "For all x it is the case that . . . ," or "There exists x such that . . . ". The locution "For all x . . . " is known as the *universal quantifier* and the locution "There exists x . . . " is known as the *existential quantifier*. The statement "All swans are white" may be written using the universal quantifier in the logical form "For all x, if $S(x)$, then $W(x)$," where $S(x)$ means "x is a swan" and $W(x)$ means "x is white." Frege's system was much more powerful than Boole's and turned out to be just what was needed to express mathematical statements in a formal way. It is the basis of the standard logic used today in the foundations of mathematics.

Twentieth Century

In 1900 the distinguished German mathematician David Hilbert (1862–1943), then 38 years old, was asked to deliver the plenary address at the International Congress of Mathematicians held in Paris. Hilbert chose as the subject of his address 23 unsolved problems that he offered up to the researchers of the new century. The problems provided a cross section of advanced mathematics as it was conceived at the time by one of its most astute and eclectic practitioners, and set the agenda for a broad range of research in the years to come. The problems were divided into roughly four groups: foundations, number theory and algebra, geometry, and analysis.

Hilbert's interest in the foundations of mathematics had developed in the 1890s, culminating in the publication in 1899 of his famous *Foundations of Geometry*. Following the lead of Moritz Pasch (1843–1930), Hilbert provided a critical study of classical Euclidean geometry according to modern standards of rigor. In the opening problems of the Paris address he considered two outstanding foundational questions of contemporary mathematics. The first concerned the status of Cantor's continuum hypothesis; the second, the problem of establishing the consistency of arithmetic. Much later, in the 1920s, Hilbert would return to the subject of the foundations of mathematics, proclaiming the need to establish the consistency of mathematics using the concept of a formal system in logic. Hilbert emphasized the importance of obtaining decision methods, general procedures that allow one to decide in a finite number of steps whether a given formula is true. This program of research, known as *Hilbert's program*, was an active focus for logical research during the 1920s.

It is beyond the scope of the present article to convey the character and diversity of Hilbert's problems, covering as they did many of the most technical questions of contemporary mathematics. A prominent concern in much of Hilbert's work was the question of establishing the existence of mathematical objects. His first major result, in 1888, consisted of a novel and highly original proof of the existence of a finite basis for a given system of algebraic invariants. Such invariants consist of expressions that remain unchanged in value under the transformation of coordinates and were studied extensively by mathematicians during the nineteenth century. A major goal of the theory was to find a given set or basis of invariants in terms of which all others could be expressed. Hilbert's proof involved new ideas from algebraic number theory and completely redirected research in the subject. In the twentieth problem he raised the question of Peter Lejeune Dirichlet's (1805–59) problem, of establishing the existence of a solution to a given partial differential equation with a specified value on the boundary. Examples of such a problem arose often in physics. The function giving static electrical charge must have a certain distribution on the surface of a conductor and satisfy the potential equation—a partial differential equation—in the interior of the conductor. During the nineteenth century, mathematicians relied on arguments from the theory of optimization to justify the existence of such a solution; they reasoned that the solution must exist because it maximized or minimized a given integral quantity. Karl Weierstrass in the 1870s had introduced examples in the **calculus of variations** showing that such reasoning was in general flawed. Work on the Dirichlet problem would be a highly productive field of analysis after 1900, with important results by Sergei Bernstein (1880–1968) and Hilbert himself. The problem of the existence of mathematical objects, not just in analysis but also in other branches of mathematics, remains a deep and unresolved issue in modern mathematical philosophy.

➢ **Calculus of variations.** See *Calculus*.

POINCARÉ, EINSTEIN, AND MATHEMATICAL PHYSICS

Although pure mathematics and theoretical physics diverged in the second half of the nineteenth century, important high-level links remained between the two subjects. Examples are provided by the work of the French mathematician Henri Poincaré (1854–1912) and the German physicist Albert Einstein (1879–1955). Poincaré was the leading mathematician of France and perhaps all of Europe at the end of the nineteenth century. Although the formative period of his work took place before 1900, he broke in decisive ways with past research, introducing new ideas in analysis and topology that would stimulate mathematics for the next 100 years. Einstein's formulation of the general theory of relativity involved the development and application of tensor analysis, a branch of differential geometry with origins in the work of Gauss, Riemann, and Beltrami.

Poincaré's early work in mathematics concerned the theory of differential equations. By 1880 this subject had become a very extensive field of research, containing several distinct parts and possessing numerous applications in physics, engineering, and astronomy. Poincaré's most notable contributions occurred in celestial mechanics, more particularly the famous **three-body problem.** In 1889, he was awarded a prize for an essay on this problem submitted to a scientific competition to celebrate the sixtieth birthday of King Oscar of Sweden. He proceeded to write two extended works, *Treatise of Celestial Mechanics* (1892–97) and *Lessons on Celestial Mechanics Given at the Sorbonne* (1905–10).

➢ **Three-body problem.** See *Astronomy and Cosmology: Eighteenth Century.*

The three-body problem arises in astronomy and concerns the interaction of three bodies under the force of gravity. Examples of such systems are the Sun–Earth–Moon system and the Sun–Jupiter–Saturn system. Using Newton's law of gravitation, it is possible to describe the motion of a three-body system in terms of a set of differential equations. These equations must be solved to determine accurately the position of the system as a function of time. It turns out that it is very difficult, if not impossible, to do so. The challenge then becomes obtaining information about the system by analyzing the differential equations without actually solving them. One way to do this is to obtain approximate solutions and to develop methods to improve the approximation, a difficult task that occupied the labors of many eighteenth- and nineteenth-century mathematical astronomers.

Poincaré initiated an entirely new approach to the differential equations of the three-body problem, developing what are known as *qualitative methods of analysis*. The basic problem was the nonlinearity of the equations. If two particular solutions are known, it is not the case that the sum of these solutions is also a solution. Poincaré discovered that the analysis of the equations was assisted by considering periodic orbits, a subject explored with some success by the American astronomer George William Hill (1838–1914) in 1877. A periodic orbit is a solution that repeats itself after a certain specified interval of time. The local situation around a given periodic orbit will consist of a sequence of periodic orbits of successively higher period. Poincaré succeeded in obtaining important qualitative information about the three-body system by analyzing the properties of periodic orbits.

Around 1900, Poincaré began to write on a range of questions in the philosophy of science and mathematics. He developed a conventionalist interpretation of physical theory, according to which physical constants and units of measurement must be interpreted in the context of the theory as a whole. In this vein he reasoned that the laws of motion of bodies are not absolute but relative. This idea was given a cogent formulation in 1905 by a contemporary of Poincaré's, a young worker in a Zurich patent office named Albert Einstein. Trained as a physicist, Einstein's famous early discoveries in relativity theory and quantum physics concerned central problems of mechanics and electromagnetism. At this stage of his career he was not deeply versed in mathematics and indeed, stood somewhat self-consciously apart from advanced mathematics. In the years following 1905 he attempted to extend his principle of relativity to a theory of gravitation, and began to develop the idea of connecting the geometry of space with the physical action of gravitational forces. To do this it was necessary to draw on more sophisticated mathematical tools than he possessed, and he turned to his mathematician colleague Marcel Grossmann (1878–1936) for assistance. In their subsequent collaboration, Einstein and Grossmann made use of a new subject, which they called *tensor analysis*, a part of differential geometry that had been developed in the preceding decades under the name *absolute differential calculus* by a group of Italian researchers. The concept of the tensor itself was drawn from physics. In writings on crystallography during the period 1898–1910, Woldemar Voigt (1850–1919) had used the term *tensor* to denote the six numbers required to describe mathematically the elastic tensions in the interior of a deformed solid. The term *tensor* had, in fact, been used earlier, by Josiah Gibbs in a book on vector analysis, and even earlier by William Rowan Hamilton in his research on quaternions.

A completed version of the general theory of relativity was published by Einstein in 1916. In general relativity, the motion of a system of particles moving under the action of gravity follows a **geodesic trajectory** in space and time governed by a variational law. This motion is described by a set of equations written in tensorial form, known as the *field equations*. The influence of gravity is expressed in terms of quantities appearing in these equations in such a way that gravitational force is conceived as arising from the intrinsic geometry of space and time. Although mathematically complicated, the theory had definite experimental consequences, concerning primarily such astronomical phenomena as the motion of the **perihelion** of Mercury and the bending of starlight in the Sun's gravitational field. In 1917, Einstein also applied the theory to the construction of models for the whole universe, thus launching a subject known as *relativistic cosmology*. Initially a rather arcane area of study, this subject has, with the amazing discoveries in observational cosmology that began with Edwin Hubble (1889–1953) in 1929 and have continued up to the present day, become central to research on the large-scale structure of the universe.

* **Geodesic trajectory.** A path between two points in a mathematical space that is the shortest distance between these points. If the space is the surface of Earth and distance is measured along the surface, the geodesic trajectory connecting two points is the great circle path (intersection of a plane through the center and the surface of the sphere) connecting them.

* **Perihelion.** The point in a planet's orbit about the Sun where it is closest to the Sun.

RUSSELL, GÖDEL, TURING, AND MATHEMATICAL LOGIC

Although Frege was undoubtedly the most important logician of the nineteenth century, his work was not widely known, and the particular notational system he used was never adopted by the scientific community. The thesis that mathematics may be reduced to logic was promoted in a more influential way by the English mathematician and philosopher Bertrand Russell. Russell began thinking seriously about logic around 1900, inspired in part by the writings of the Italian researcher Giuseppe Peano (1858–1932). Upon reflecting on such fundamental notions as class membership and inclusion, Russell realized that there were difficulties in unrestricted use of the concept of class. Suppose that classes are divided into two sorts. The first consists of classes that do not contain themselves as members, such as the class of cups on a shelf or the class of numbers less than 100. The second consists of classes that do contain themselves as members, such as the class of concepts, or the class of all classes that may be defined in 50 words or less. Clearly, every class must be of one sort or the other. Consider now the class of all classes of the first sort. Russell reasoned that the class defined in this way was inconsistent, since the assumption that it was either of the first or second class led directly to a contradiction.

Russell's paradox seemed more disturbing than some of the inconsistencies that were being found in set theory at the time because it involved the most fundamental notions of class and set inclusion. Russell communicated what he subsequently referred to as his "contradiction" to Frege in a letter dated June 16, 1902. Frege was extremely concerned by Russell's discovery and embarked on a reexamination of the very foundations of his logical system.

Russell's discovery also stimulated his own work in mathematical logic. He began an ambitious collaboration with Alfred North Whitehead (1861–1947) that would culminate in 1910–13 with the publication of their three-volume *Principia Mathematica*. This work was an attempt to formalize all of mathematics within a logical system. Russell and Whitehead employed logical notation that since has become standard. Their systematic development of formal logic constituted a benchmark that provided a focus for further work in the field.

In 1931, the young Austrian logician Kurt Gödel (1906–78) proved a fundamental theorem that more than any other shook the foundations of mathematical logic. Although his result was a very technical one, its basic character may be summarized as follows. By assigning numbers to the symbols and formulas of Russell and Whitehead's system through a coding procedure, Gödel showed that statements about the system could be expressed by formulas in the system itself; that is, the system had the ability to refer to itself. He was inspired by the paradox of the liar, which originated in Greek antiquity. A woman says that she is lying. Is what she says true or false? Gödel exhibited a formula that is true in the usual interpretation but is not provable within the formal system. The formula in question asserts informally that "the formula with my coding number is not provable." This formula is undecidable, in the sense that neither it nor its negation is provable. As a direct corollary to this result, Gödel showed that the consistency of a formal system powerful enough to express arithmetic cannot be established within the formal system itself.

Gödel's undecidability theorem implied that some of the central goals of formalized mathematics are unachievable. Like Heisenberg's uncertainty principle in physics, it seemed to indicate that there are irreducible limits on what we can know. It has come to be seen as a characteristic result of intellectual modernity, indicating that a final and complete understanding of the physical and mathematical world is unattainable. In 1950, *Fortune* magazine would call Gödel's 1931 result the most important mathematical discovery of the century. Its implications for philosophy and artificial intelligence were explored in 1979 in a semi-popular vein in Douglas R. Hofstadter's (b. 1945) Pulitzer Prize–winning *Gödel, Escher, Bach*, a book that examined the phenomenon of self-reference in a range of cultural contexts.

The undecidability theorem essentially announced the end of the Hilbert program, of the goal of proving the consistency of mathematics in a finite way by formal means. Subsequent negative results during the 1930s indicated further limitations on formal procedures. The most famous of these was a result obtained independently in 1936 by Emile Post (1897–1954), Alonzo Church (1903–95), and Alan Turing (1912–54). These authors showed that no general algorithm exists to decide whether a given proposed inference is valid. Turing's solution was particularly interesting because it involved concepts that would be important in the later development of computer science. Turing became interested in the decision problem while he was an undergraduate at Cambridge University in England. He devised the notion of an idealized machine known as a *Turing machine* that can be instructed to perform a given calculation in a finite number of steps. The machine operates on a tape or one-dimensional strip of paper divided into squares; at each step the machine scans a given square and moves the system to a new configuration. The final result is a sequence of squares containing the answer to the calculation. Turing posited the existence of a universal machine, in which the instructions for a given algorithm along with the particular data are entered together into the machine. Turing showed that such a universal machine is in principle incapable of generating all effectively computable numbers; he did so using an argument similar to the one Cantor had employed to establish the uncountability of the real numbers.

Turing's researches were inspired by Gödel, who continued to work on mathematical foundations in the 1930s. During this period, Gödel alternated between academic positions in Vienna and the newly created Institute for Advanced Study at Princeton, New Jersey. He was becoming deeply engaged in the study of set theory and Cantor's celebrated continuum problem, the first of Hilbert's 1900 list of problems for the new century. In 1937, he established the fundamental result that the continuum hypothesis is consistent with the usual Zermelo–Fraenkel axioms of set theory. This is known as a *relative consistency* result: If the axioms of set theory are consistent, the system obtained by adding the continuum hypothesis to them is also consistent.

In 1940, Gödel moved permanently to Princeton, where he became a close friend of Albert Einstein's (Einstein served as a witness when Gödel applied for U.S. citizenship in 1948). Gödel contributed relatively little to mathematics after his move to the United States; he suffered from hypochondria and depression and died in 1977 from malnutrition caused by a "personality disturbance." In 1963, the American mathematician Paul Cohen (b. 1934) complemented Gödel's 1937 result by showing that the negation of the continuum hypothesis is also consistent with the axioms of set theory. The results of Gödel and Cohen established the independence of the continuum hypothesis: Either it or its negation can be added to the usual Zermelo–Fraenkel axioms to obtain consistent but conflicting systems of set theory. To accomplish his result, Cohen introduced a new technique called *forcing*, a method that proved very fruitful in establishing the independence of other fundamental conjectures in set theory and general topology.

MODERN MATHEMATICS, ABSTRACT ALGEBRA, AND THE BOURBAKI INFLUENCE

Around 1900, mathematicians began to refer self-consciously to their work as part of a "modern" program of investigation. The notion of modernism in mathematics is associated with such things as the axiomatic method, the use of set-theoretic ideas, and an emphasis on the study of general mathematical structures. These characteristics were expressed very clearly in a tradition of research in abstract algebra, beginning with Hilbert and continuing with Emmy Noether (1882–1935), B. L. Van der Waerden (1903–96), Emil Artin (1898–1962), Garrett Birkhoff (1911–96), and Saunders MacLane (b. 1909). Van der Waerden wrote *Moderne Algebra* (1930), and Birkhoff and MacLane authored *A Survey of Modern Algebra* (1941), the latter a highly successful university textbook that explained higher algebra to a generation of Anglo-American students. The emphasis was on the study of mathematical structures—groups, fields, rings—rather than on particular problems or special results. In the preface to their book, Birkhoff and MacLane wrote: "Modern algebra also enables one to reinterpret the results of classical algebra, giving them far greater unity and generality." In 1942, MacLane, in collaboration with Samuel Eilenberg (1913–98), carried this idea one step further with the development of the *theory of categories*. The idea of this subject is to construct an abstract general model that focuses on those relations or mappings—known as *morphisms*—between structures that preserve certain basic properties. The theory is then applicable whether the structures are sets, algebraic groups, topological spaces, and so on. For example, the structures may be groups, and the morphisms are mappings (called *homomorphisms*) that preserve multiplication: If a maps to a^* and b maps to b^*, then ab maps to a^*b^*.

Modern mathematics is also strongly identified with a group of French mathematicians who authored a series of volumes under the pseudonym "Nicolas Bourbaki." Prominent in the original group were Henri Cartan (b. 1904), Claude Chevalley (1909–84), André Weil (1906–98), and Jean Dieudonné (1906–92); later members included Alexander Grothendieck (b. 1928), Pierre Samuel (b. 1921), and Jean-Pierre Serre (b. 1926). The Bourbaki project began in the 1930s in a plan to write a new textbook on analysis, in the French tradition of producing systematic treatises in this subject, and developed into a more ambitious project to formulate all of mathematics in a grand compendium. Bourbaki emphasized the unifying concept of structure, the importance of axiomatics, and the value of set theory as a language for mathematics. The authority of its distinguished members has been used to support its claims about the nature of mathematics. The titles of the volumes produced by the group provide an overview of advanced mathematics in the middle of the twentieth century: algebra, integration, topological vector spaces, Lie groups and algebras, general topology, commutative algebra, set theory, spectral theory, and differential and analytical varieties. Important characteristics of the Bourbaki vision include a lack of emphasis on mathematical applications and a neglect of formal logic; the latter seems to have been the consequence of contingent factors rather than an expression of philosophical conviction—logicians happened to be absent from the Bourbaki coterie of mathematicians in the period of its formation.

Although Bourbaki was the product of a centralized and elite system of research, its view of the nature of mathematics influenced a popular educational movement known as the *New Mathematics*. The New Math originated in the United States in the 1950s and was also important in the British Commonwealth and parts of Europe. It was the latest in a series of reform movements in U.S. mathematical education. Its initial goal was to bring the more elementary parts of the mathematics curriculum into line with the character and concerns of advanced research. It began in the early 1950s in Illinois as a program to improve the preparation of high school students for university study. Its philosophical adherents tended to be people who were involved in abstract areas of work, such as abstract algebra, set theory, and topology. Although it began at the college and high school levels and was intended for academically superior students, it soon became widespread and eventually filtered down to the primary grades. The New Math was given a strong impetus by the increased financial support allocated to science education following launching of the Soviet Union's *Sputnik* in 1957, when concerns emerged about the state of science teaching in the United States.

Mathematics and Society: The New Math

At the more elementary levels, the New Math, introduced following the research of the Bourbaki project, was guided by the belief that a child would learn mathematics more effectively if he or she understood why the algorithms and procedures worked, rather than being subjected to rote instruction. Set theory, which had not existed before 1870 and was regarded as a rather abstruse branch of mathematics when it was invented, was seen as an appropriate source of ideas for introducing the young beginner to basic notions of quantity and magnitude. Emphasis was placed on understanding the axiomatic laws of algebra, such as the commutative and associate laws for multiplication. Conceptual distinctions, such as the distinction between number and numeral, were seen as very important: The student should know that the number four is a concept, and that 4 or IV are only particular numerals to denote this concept.

The New Mathematics became a popular sensation; stories about it appeared on the cover of *Time* and it was the subject of satirical songs. Problems with implementation developed: There were difficulties with teacher training and parental incomprehension, and criticism of it grew among certain mathematicians and educators. In 1973, Morris Kline (1908–92), an applied mathematician at New York University, wrote *Why Johnny Can't Add: The Failure of the New Math*, a work that summarized some 15 years of criticism. Kline, who was not at all deterred by his lack of first-hand experience with precollege science education, firmly believed that the New Math was too abstract and that it did not take into account the extensive historical links between mathematics and physics.

The criticism was, from certain perspectives, warranted. The new practice, indeed, tended as a practical matter to forget the extent to which mechanical facility in applying an algorithm or performing a calculation precedes actual recognition of the underlying mathematical principle. Perhaps its main weakness was to overlook the concrete and empirical character of mathematics. There is no question, however, that it was an idealistic and fascinating educational experiment, and certain of its ideas have become permanently rooted in mathematical pedagogy.

The New Math reflected a research ethos that was very influential in the 1950s but is less so today.

In the last decades of the century there was a resurgence of interest in applied mathematics, not just in the traditional applications of partial differential equations to physics emphasized by Kline, but also in such subjects as probability, statistics, optimization, operations research, computer science, linear and nonlinear programming, theory of circuits, graph theory, finite mathematics, and so on.

FRACTAL GEOMETRY AND CHAOS THEORY

At the very time that the Bourbaki influence was at its peak, the seeds of an influential new movement in mathematics were being sewn by researchers in several areas of science. The new subject, designated very broadly under the title nonlinear analysis or dynamical systems, possessed roots in nineteenth- and early twentieth-century mathematics and was stimulated in the 1960s and 1970s by the sudden widespread availability of powerful calculators and computers. Research was infused with a spirit of experimentation, not the traditional laboratory experiments of physics and chemistry but a new approach in which functional algorithms and procedures were tested numerically or expressed graphically using computers.

The term *fractal* was coined by the French-American mathematician Benoit Mandelbrot (b. 1924) in 1975 to describe a geometrical object that possesses self-similar properties under repeated magnification. The term *fractal* referred both to the broken shape of the fractal image and to the fractional dimension of its topological description as a point set. Fractals had been produced in all but name a century before by Georg Cantor, Helge von Koch (1870–1924), and Waclaw Sierpinski (1882–1969). Perhaps the best known fractal is von Koch's snowflake curve, which is constructed as follows (see Figure 2): We begin with an equilateral triangle, and erect smaller equilateral triangles on the middle third of each side. In the resulting 12-sided figure we erect equilateral triangles on each side, obtaining a 48-sided figure. We repeat this process ad infinitum, arriving in the limit at a closed curve with the following unusual property noted by von Koch: The curve encloses a finite area but has a perimeter that is infinite in length.

The immediate origins of fractal geometry were located in the research of two French mathematicians in the years after World War I working on the theory of functions in the complex domain. Pierre Fatou (1878–1929) and Gaston Julia (1893–1978) considered a function $w = f(z)$ that maps the complex number z to another complex number w. Each complex number z is of the form $x + iy$, where x and y are real numbers and $i^2 = -1$. This number can be specified as a point in the *complex plane*, following a procedure that was first invented in the early nineteenth century. The distance along the horizontal axis gives the real coordinate x and the distance along the vertical axis gives the imaginary coordinate y. The number z is then located at the point with coordinates x and y. The mapping $w = f(z)$ takes each point z in the complex plane and maps it to another point w in this plane. For a given z (called a *seed*), the sequence (called the *orbit*) of values $f(z)$, $f(f(z))$, $f(f(f(z)))$, . . . is generated by iterating the function f. It turns out that for some values of z, the orbit is bounded—the associated sequence of values all lie within a fixed radius of the origin in the complex plane. For other values of z the orbit increases indefinitely. The complex plane is divided in this way into parts, and the boundary between these parts is known as a *Julia set*. For many simple examples of f, such as the quadratic map $f(z) = z^2 + c$, the Julia set possesses interesting fractal-like properties.

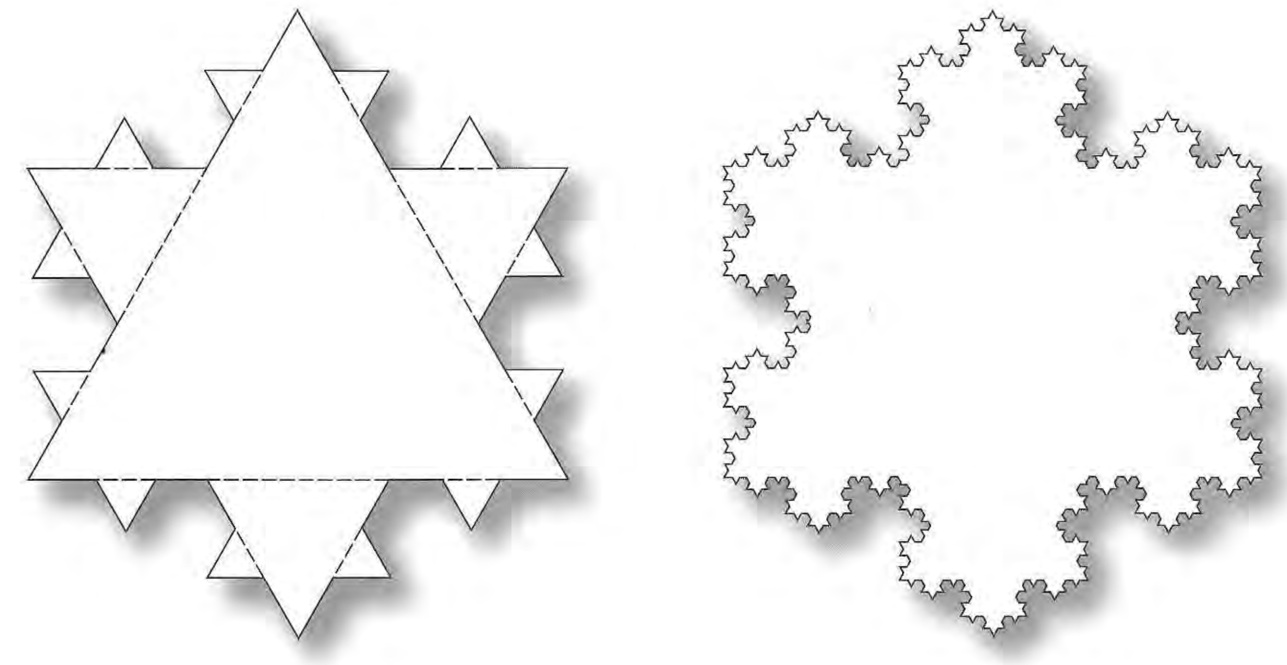

Figure 3. Diagram of the construction of Koch's snowflake curve.

Mandelbrot was educated at France and was the nephew of Szolem Mandelbrojt (1899–1983), a member of the Bourbaki circle in Paris. Although he was strongly attracted to mathematics and a product of elite schools, he felt no affinity with the prevailing mood of French mathematics. In the 1950s, Mandelbrot moved to the United States, where he took up a position in the physics research department of IBM in New York State. In the decades that followed he worked on problems as diverse as electronic noise, Brownian motion, and cartography, experiences that led him to the study of self-similar mathematical sets and curves. The product of these labors was his 1977 monograph on fractal curves and their properties. Among the groups that first recognized the value of fractals as a tool of applied research were geophysicists and geologists. The properties of surfaces in contact with each other may be analyzed by means of fractal geometry, a field of study pioneered by researchers at the Lamont–Doherty Geophysical Observatory in the late 1970s.

In 1979, Mandelbrot began to investigate the classical Fatou–Julia theory and made an important discovery concerning an object that in his honor would be named the *Mandelbrot set*. This discovery stimulated a resurgence of interest in complex dynamics. The class of generalized fractals to which the Mandelbrot set belongs has been the subject of considerable research in the last decades of the century, a project that has been assisted by very powerful computers.

Chaos theory has a somewhat different history from fractal geometry, developing as a result of attempts by physicists and meteorologists to interpret the nonlinear differential equations appearing in their work. The major event in its early development occurred in 1963 with the publication by American meteorologist Edward Lorenz (b. 1917) of a paper in which he summarized several years of work on weather prediction. Lorenz's results were extended in the 1970s by physicist Mitchell Feigenbaum (b. 1944) as well as by several other researchers in the United States and Europe. Working in virtual isolation at Los Alamos National Laboratory in 1974, Feigenbaum arrived at some important discoveries concerning the periodic behavior of numerical functions under iteration, an accomplishment that was aided by a handheld electronic calculator. The actual term *chaos* was introduced by T. Y. Li and J. A. Yorke in 1975 in a paper in the *American Mathematical Monthly* titled "Period 3 Implies Chaos."

The key idea of chaos is the sensitive dependence on initial conditions that frequently occurs in nonlinear systems. As early as 1892, Henri Poincaré had written: "It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible. . . ." This idea was at the base of Lorenz's work. The situation is illustrated by the following problem. Beginning with the real-valued function $f(x)$ and a value $x = x_0$, calculate $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, and examine the behavior of the sequence of values thus generated. The problem provides a model for feedback, in which the result of a given process is entered back into the process at each successive stage. The example of the parabola $y = kx(1 - x)$ illustrates some of the main ideas. [Note that because $f(x)$ contains the quadratic term $-kx^2$, the relationship between y and x is nonlinear.] For certain values of k, the iterations of $f(x)$ for each x on a given interval will tend to cluster around a given point, called an *attractor*. For other values of k, the behavior of $f(x)$ under iteration will be chaotic: Points x that are periodic with respect to the iteration will have very long periods and appear random; furthermore, points very close to a given value of x will give rise to an entirely different orbit, even after a very small number of iterations. In the second case, $y = kx(1 - x)$ is said to be a *chaotic dynamical system*.

The process of iteration can also be applied to two-dimensional systems in the x–y plane. In certain systems, the values iterate to a point set known as a *strange attractor*—"strange" because the set possesses unusual fractal-like properties. The behavior of the system on the attractor itself is quite chaotic. In complex dynamics the Julia set for a given map will be chaotic, in the sense that the orbits on this set are unstable—a slight variation in the seed produces a very different orbit. It is also possible to produce quite regular fractal patterns by means of random and apparently chaotic processes. Michael Barnsley (b. 1946) in the 1980s described a simple process called the *chaos game* in which fractal patterns in the plane are generated by a procedure that involves choosing points randomly by the throw of a dice. There are fundamental connections between the study of chaotic systems and fractal geometry, and the two subjects are usually considered together in books on dynamical systems.

If the Bourbaki influence represented mathematical modernism at its height, the advent of applied nonlinear analysis indicated that mathematics had moved into a thoroughly postmodern phase. During the 1970s and 1980s, dynamical systems demanded increasing attention from researchers and began to attract the interest of the general public. In 1987, *New York Times* journalist James Gleick proclaimed in his book *Chaos* that a new science had been created, and chronicled this event in a dramatic narrative emphasizing the determination and revolutionary insight of the new practitioners. Beautiful pictures of fractals are now found everywhere, and simple algorithms to produce graphical images of iterated functions on personal computers are readily available. The academic fields of dynamical systems and nonlinear analysis today occupy a significant place in mathematics and physics departments, and there is an ongoing program to apply the theory to subjects as varied as economics, botany, and geology. This trend has been strengthened by the movement to the West of scientists from the former Soviet bloc, where there was an active and independent tradition of research in dynamical systems.

The success of the new science has, in turn, attracted critics. In 1996, John Horgan (b. 1953), a writer with *Scientific American*, published *The End of Science*, in which he emphasized the pop-culture status of chaos theory and the related field of complexity, suggesting that these subjects were examples of what he called *ironic science*,

Scientific Biography: John von Neumann and Alan Turing

Two of the leading pioneers of the modern computer, John von Neumann (1903–57) and Alan Turing (1912–54), were mathematicians who began their careers in formal logic and set theory, von Neumann in Germany and Turing in England. A central idea of the modern computer, the notion of a stored program, was implicit in Alan Turing's idealized conception of a Turing machine, in which the instructions for performing a computation were included with the data in the information input to the machine. Turing's researches took an applied turn with the outbreak of World War II, when he became involved with the British decoding group at Bletchley Park in Buckinghamshire. The Bletchley workers built machines or *bombes* (so-called because they made a loud ticking noise) that were designed to simulate a German Enigma coding device. Experience with these machines provided valuable experience in enabling electronic devices to perform logical and numerical functions.

During the 1920s, von Neumann had carried out research on logic and set theory in Berlin and Zurich. In 1933, he joined Princeton's Institute for Advanced Study, where his command of a range of research subjects made him at age 31 a leader of U.S. mathematics. In 1944, he joined a group involved in electronic computation at the Moore School of Engineering of the University of Pennsylvania. Under the direction of engineers John W. Mauchly (1907–80) and J. Presper Eckert (1919–95), the Moore School developed the most sophisticated electronic calculator to date, the Electronic Numerical Integrator and Calculator (ENIAC). This machine was a digital device of 15,000 vacuum tubes designed specifically to do calculations needed to prepare artillery range tables.

Von Neumann arrived at the Moore School as planning was getting under way for a successor to ENIAC known as the Electronic Discrete Variable Automatic Computer (EDVAC). He prepared an influential document outlining a logical design for this machine titled "First draft of a report on the EDVAC." His approach was summarized by Eckert and Mauchly at the time: "In his report, the physical structures and devices . . . are replaced by idealized elements to avoid raising engineering problems which might distract attention from the logical considerations under discussion." The idea was to divide the code entered into the machine into two parts: The first consisted of a set of instructions, while the second consisted of the values required in a computation involving a particular problem. Both parts were stored in the computer's memory. An arithmetic unit performed the actual computations, while a control unit coordinated application of the instructions (i.e., the program) to the given data. Numbers were stored internally in binary form, a numerical mode ideally suited to the on–off capability of vacuum tubes.

Following the end of the war, von Neumann established a computing project at the Institute for Advanced Study. The IAS computer came on line in 1952 and was used to solve problems in matrix inversion, astrophysics, number theory, fluid dynamics, and other scientific subjects. A British computer project was directed by Alan Turing, who, following the end of the war, took a job with the National Physical Laboratory on the design of the Automatic Computing Engine (ACE). Although familiar with American work, Turing steered this project in a somewhat different direction, emphasizing a more complicated logical architecture. His ACE report of 1945 was one of the signal documents in the early years of computer science. The Pilot Ace became operational in 1950, by which time Turing had joined a project at Manchester University to produce the Manchester Automatic Digital Machine (MADAM). Turing died in 1954 at age 41.

science in which established theory is reinterpreted and in which no new fundamental discoveries are made. (Horgan took this notion of irony from literary critic Northrop Frye, who used it to refer to the possibility of multiple interpretations.) Although high-level recognition has been granted to researchers in affiliated topics in topology and differential geometry, no one working in applied nonlinear analysis has been awarded the Fields Medal for mathematics or the Nobel prize in Physics.

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Physics

David Topper

If we think of physics as originating in the minds of individuals, it is customary to begin with the early Greek philosophers. Perhaps the first of these philosohers was Thales of Miletus (c. 625–c. 547 B.C.) who spoke of all things as being composed of water. Later thinkers used a fourfold division: earth, water, air, and fire. In one sense they might be considered the first physicists, since their quest was analogous to the present-day search for the smallest building blocks of matter. But in another sense they were more like chemists, since the concept of a chemical element is a closer analog to these substances; indeed, the idea of earth, water, air, and fire as the elements of nature continued into the eighteenth century.

Pre-1543 Roots

Two early groups who had a major impact on the development of physics were the Pythagoreans and the atomists. Pythagoras (c. 560–c. 480 B.C.) and his followers set the stage for mathematizing the world by asserting that the essence of all things is number (specifically, geometry). Although their view of mathematics bordered on the spiritual (in somewhat the way that we speak of lucky numbers), a look at any of today's physics textbooks, pages filled with many mathematical formulas, will make clear the extent of their insight. In contrast, Leucippus of Miletus (fl. fifth century B.C.) and Democritus of Abdera (fl. late fifth century B.C.) conceived of a universe composed only of lifeless matter moving randomly in a void. This matter could be broken down into many invisibly small indivisible (atomic) units. According to the atomists, neither mind nor spirit existed in nature: This was a materialist view of the world.

ARISTOTLE'S QUALITATIVE PHYSICS

The science of physics as taught from the late ancient world for centuries—well into the seventeenth century—was based primarily on the writings of Aristotle (384–322 B.C.). Although specific aspects of his ideas and thoughts relating to physics were often challenged, the overall picture was not displaced until the works of Galileo, Newton, and others during the Scientific Revolution of the sixteenth and seventeenth centuries. By then, Aristotle's physics was seen as a belief system grounded on ancient texts, taught by scholars in universities, but having no basis in experience or experiment. Yet that was not how it began.

Aristotle's physics was mainly a science of motion; this science was based on common sense. If you push an object lying near you, it moves as long as you push it; no push, no motion. So a push (or a force) produces motion (or speed). The harder you push (i.e., the greater the force), the faster the object moves. Stated as a proportion: Speed (S) is proportional to force (F); in contemporary notation, $S \propto F$. Aristotle also realized that air was a factor in motion, retarding or resisting it; thus the same force would not produce the same motion in, say, water. So the greater the resistance of the medium (R), the slower the speed; thus S is inversely proportional to R (or $S \propto 1/R$). Combining the two proportions, we obtain $S \propto F/R$, which is Aristotle's law of motion for such horizontal motion. The history of the truth or falsehood of these laws forms a central focus in the physics of motion from ancient times through the Scientific Revolution.