*tory* introduced by Ivor Grattan-Guinness. Episodes of tensions and discussions between mathematicians of meta-level issues of e.g., conceptions of mathematics, are not treated in detail in the book, but only alluded to at various places e.g., "the seventeenth-century tension between the traditional geometric style and the then-"modern" algebraic style" (p. 280), and Gordan's "retort" to Hilbert's existence proof of a finite basis in invariant theory: "this is not mathematics. This is theology." Interested readers who want to explore these issues will find directions to further study in the footnotes and the list of references.

From the Prelude (Chapter 1), and the technical level of the exposition, it can be inferred that the intended readers of the book are teachers of algebra at the school and college level, and college-level mathematics students. Katz and Parshall emphasize the importance of knowing the history of mathematics for the teaching and learning of mathematics. They claim that: "Using the history of algebra, teachers of the subject, [...], can increase students' overall understanding of the material." (p. 3). There is no doubt that teachers who want to use history to benefit students' learning of mathematics will find this book tremendously helpful because of the wealth of sources and cultures included. However, it would have suited the book if there had been some discussions or references to materials explicitly devoted to how this can be/has been done in teaching.

The book is very well written. It covers a lot of ground and deals with a huge amount of material and sources. It provides an extensive bibliography to original sources and research literature in the history of mathematics. This book is an excellent resource for teachers, students, researchers and anyone who has an interest in the history of algebra.

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**Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry**
By Glen Van Brummelen. Princeton University Press. 2013. 208 pp.

The book under review is a sequel to the author's 2009 *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*.[1] It is a primer on spherical trigonometry for the modern student, as well as a history of the subject addressed to a broad mathematical audience. Each chapter is accompanied by a detailed set of exercises to assist the student in engaging with the technical material.

Spherical trigonometry was founded by Menelaus of Alexandria around 100 AD, and the results that he presented were used by Ptolemy in the *Almagest*, his great work on mathematical geocentric astronomy. The Ptolemaic model for a heavenly body gives its ecliptic coordinates (longitude along the ecliptic and latitude above or below the ecliptic) as a function of time. Trigonometry comes into play but it is plane trigonometry. It is necessary to relate these coordinates to the planet's equatorial coordinates given as right ascension and declination. The two systems are measured along spheres that are inclined to each other, and the coordinates are related using Menelaus's theorem on spherical triangles. Van Brummelen devotes Chapter 3 to a study of this theorem and some of its uses in astronomy.

[1] Glen Van Brummelen, *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry* (Princeton University Press, 2009)

A significant advance took place around 1000 AD with the replacement of Menelaus's theorem by something called the rule of four quantities, a result that appeared in the writings of the Persian astronomer Abū Nasr Mansūr (960–1036). In his chapter on these developments Van Brummelen also includes an account of the use of spherical trigonometry to determine the *qibla*, or the direction to Mecca at a given point on the Earth. Van Brummelen notes that "the *qibla* – serves several purposes beside the daily prayers, including the direction in which Muslims should face when they are buried" (p. 67).

Although Menelaus and his Islamic successors did very important work, what we know today as spherical trigonometry was the invention of European mathematicians in the early modern period. A critical step was the recognition that the primary focus of study should be the triangle, as opposed to any polygonal figure, planar or otherwise. In chapter 5 Van Brummelen explores the basis of modern spherical trigonometry, an achievement of the Scottish mathematician John Napier (1550–1617), whose work was supplemented by his contemporary Henry Briggs. There is a chapter on areas of polygons on the surface of the sphere and how the associated result was used brilliantly by Adrien-Marie Legendre in 1794 to prove Leonhard Euler's polyhedral formula. A later figure of note was Jean Delambre, who in 1807 derived something known as Delambre's analogies (obtained independently by Carl Friedrich Gauss and Karl Mollweide). The final chapters are devoted to accounts of two more recent developments: Giuseppe Cesàro's work of 1905 on stereographic projection; and Marcq de Blond de Saint-Hilaire's invention in 1875 of the intercept method in celestial navigation for determining position at sea.

In Chapter 2 the concept of the polar triangle is attributed to Abū Nasr Mansūr. This is followed by three theorems of uncertain origin and historical provenance. Extended parts of the book are devoted to mathematical exposition. A discussion of Johannes Kepler's 1591 model of the solar system is not integrated historically with the other subjects in its chapter. In the section on Delambre's analogies Van Brummelen draws on Isaac Todhunter's derivation of these formulas. Van Brummelen states in the preface (p. xiii) that the book under review is not a scholarly work on the history of mathematics. It does contain a substantial amount of historical material, and the narrative follows a more or less chronological historical order.

The formula for the area of a triangle on the surface of a sphere is $A = R^2(a + b + c - \pi)$ $(*)$, where $A$ is the area, $R$ is the radius of the sphere, and $a$, $b$ and $c$ are the dihedral angles of the triangle in radian measure. In Chapter 7 this result is attributed to Albert Girard in 1626. It is not clear if the derivation Van Brummelen gives is Girard's but it is the one commonly found in modern textbooks. In the case of equilateral triangles the formula reduces to $A = R^2(3a - \pi)$. As $a$ tends to $\pi$ the triangle approaches a hemisphere of area $2\pi r^2$. Conversely, as $a$ tends to $\pi/3$ the triangle becomes more and more planar and its area tends to zero. Triangles that are similarly shaped are of the same size, so the familiar plane concept of similarity does not apply.

Non-Euclidean geometry is a modern subject where spherical trigonometry comes into play. At the conclusion of the book Van Brummelen identifies this as a topic for further study. In hyperbolic geometry the sum of the angles of a triangle is always less than 180° and its area is $A = k(\pi - a - b - c)$, where $k$ is a positive constant and $a$, $b$ and $c$ are the angles of the triangle. If we go back to spherical triangles and let $R = ki$, where $i$ is the square root of minus one, then we have from $(*)$, $A = k(a + b + c - \pi)$, which is formally identical to the expression for the area of a triangle in hyperbolic geometry. Even before non-Euclidean geometry was invented, researchers such as Johann Lambert noticed curious similarities between spherical trigonometry and the properties of triangles under the "hypothesis of the acute angle." Nikolai Lobachevsky's new geometry was sometimes called "imaginary geometry" because it seemed to correspond to spherical geometry on a sphere of imaginary radius.

The subjects presented in the second half of the book are probably better suited to reviving interest in spherical trigonometry. The early material on rising times, astronomical coordinate transformations and Menelaus's theorem will likely be a hard sell to today's students. Overall Van Brummelen's book is written in an engaging and clear manner, with pithy observations about the mathematical results and their history. The text is accompanied by many illustrations and a set of 11 color plates. As I worked through the book

I sat at my desk with a set of plastic balls, drawing triangles and circles on them with a fine felt-tip pen. The book can be read with profit by anyone interested in moving beyond the plane and should become a preferred text for future courses on spherical trigonometry.

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**Mathematische Schriften, Reihe 7, sechster Band, 1673–1676, Arithmetische Kreisquadratur**
By Gottfried Wilhelm Leibniz. Berlin (Akademie-Verlag). 2012. Siegmund Probst and Uwe Mayer, Eds.

Bearing in mind Leibniz's saying, according to which: "those who know me on the basis of my publications, don't know me", we can confidently state that volume six of Leibniz's mathematical works (Gottfried Wilhelm Leibniz (2012), *Mathematische Schriften, herausgegeben von der Leibniz-Forschungsstelle Hannover der Akademie der Wissenschaften zu Göttingen beim Leibniz-Archiv der Gottfried Wilhelm Leibniz Bibliothek (Hannover), Reihe 7, sechster Band, 1673–1676, Arithmetische Kreisquadratur*, Siegmund Probst and Uwe Mayer, eds.), hereinafter VII6, fills in another important *lacuna* in our knowledge of Leibniz's mathematics, by providing a wealth of previously unpublished manuscripts.

The criteria chosen for collecting the texts which constitute volume VII6 are both chronological and thematic. As the title suggests, we find here Leibniz's tracts, notes and extracts related to the arithmetical quadrature of the central conic sections (namely, the expression for the area of a circular, hyperbolic or elliptical sector via an infinite converging series of rational numbers). All texts relate to research conducted between 1673 and 1676, during Leibniz's sojourn in Paris.[1]

Especially noteworthy is the last tract (n. 51), that offers a new critical edition of the treatise *De Quadratura Arithmetica Circuli, Ellipseos et Hyperbolae* (*De Quadratura Arithmetica*, hereinafter). It stands as an ideal completion of the textual studies devoted to this work, started almost a century ago with the partial edition by L. Scholtz in 1934 (Scholtz, 1934), and pursued with the magisterial first complete critical edition prepared by E. Knobloch, in 1993 (Leibniz, 1993). The text prepared by Knobloch has been translated into German (a translation made by Otto Hamborg is available online: (Leibniz, 2007)), into French (Leibniz, 2004) and into Spanish (in (Leibniz, 2015)).

As the editors tell us in the instructive preface to the present volume, the treatise *De Quadratura Arithmetica* never saw publication during Leibniz's lifetime, but had nevertheless a complex editorial history. In fact, we can count at least six versions of this treatise composed between 1673, when Leibniz discovered the convergent series for the area of the circle ($\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$, cf. VII6, n. 1) and 1676, when the last and most extensive draft of the *De Quadratura Arithmetica* was completed (n. 51). With the publication of other notable sketches of the same treatise (these are n. 1, 4, 8, written between Autumn 1673 and October 1674, and n. 14, 20, 28 composed, in Spring and September 1676, respectively), this volume completes the

[1] Most of the texts are redacted in Latin, with notable exceptions, in particular: n. 5 and 7, which contain notes related to the French draft of the Arithmetical Quadrature sent by Leibniz to Huygens in Autumn 1674 (AIII1, n. $39_2$); two texts (n. 17, 26) are in Dutch, and contain excerpts on the approximate measurement of arcs and angles made by the mathematician Georg Mohr for Leibniz; one text is in German (n. 38), and reports annotations that the mathematician and philosopher K.E. von Tschirnhaus made for Leibniz too.