ANNALS OF SCIENCE

BOOK REVIEW

Vector: a surprising story of space, time, and mathematical transformation, by Robyn Arianrhod, Chicago, University of Chicago, 2024, xxxvii + 376 pp., 13 halftones, 39 line drawings, $28.00 (hardcover), ISBN 9780226821108

The development of vector analysis as it is used in advanced calculus and physics has been documented in some detail by historians. In the nineteenth century there was the system of William Rowan Hamilton and Peter Tait's quaternions on the one hand, and the system of Hermann Grassmann and Willard Gibbs's vectors on the other, with the eventual consolidation and widespread adoption of vectors by mathematicians in Europe and America. The story here is one that does not involve differential geometry or ideas of invariance.

Developments in differential geometry and the theory of elasticity led in the last part of the century to the creation of what later would be called tensor analysis. Italian mathematicians Gregorio Ricci-Curbastro and Tullio Levi-Civita working in differential geometry developed a subject Ricci called the absolute differential calculus. The actual term tensor in something like the modern sense first appeared in 1898 in a book on crystallography by German mathematical physicist Waldemar Voigt. Voigt was investigating tension and deformation in elastic solids and made no use of differential geometry. The designation tensor analysis was adopted by Albert Einstein and Marcel Grossmann in 1915 in their work on gravity and general relativity. Tensor analysis is another name for the absolute differential calculus of Ricci and Levi-Civita.

General relativity was at the beginning an esoteric theory of limited interest to practicing physicists elaborated using unfamiliar mathematical tools characterized by their notational complexity. In the 1920s, authors such as George Birkhoff, Arthur Eddington and Hermann Weyl wrote expositions of the new tensor analysis and the general theory of relativity. Concerning these books historian Carl Boyer writes that 'their very popularity among the scientifically and philosophically oriented reading public did much to spread the notion of the incomprehensibility of mathematics and physics' (*History of Mathematics* 1989, p. 701).

In *Vector*, science writer Robyn Arianrhod charts the rise of vector analysis in the nineteenth century, examines developments in differential geometry centring on the work of Bernhard Riemann and Ricci and explores the rebranding of the mathematical theory as tensor analysis by Einstein and Grossmann. The theme of invariance (as it is understood in physics and differential geometry) is prominent in the narrative. There is something of an emphasis on British figures: Harriot, Wallis, Newton, Sommerville, Young, Faraday, Babbage, De Morgan, Peacock, Hamilton, Stokes, Cayley, Maxwell, Tait, Clifford, Heaviside, Chisholm and Dirac.

Vector is written in an engaging style with attention to the human side of the story and is intended for a broad audience. Biographical information is provided for the more prominent figures. The reader is assumed to have only limited mathematical training. One may not know that 'Σ' denotes summation, although one is expected to know how to multiply matrices. To follow the technical developments, it is necessary to consult the end notes where essential details are laid out. The endnotes also refer sometimes in considerable detail to historical work produced over the past decades. The book concludes with a timeline extending from 3000 B.C. to the present.

The book is mainly history but also includes wide-ranging discussions of selected current topics in the field. Michael J. Crowe's *A History of Vector Analysis (*1967) is a standard work devoted to nineteenth-century developments. Arianrhod goes over these events in chapters 4-8 of her book. Her account differs from Crowe's in providing less discussion of Grassmann's theory and more discussion of the physics. For example, there is an account of James Clerk Maxwell's work on electromagnetism with an explanation of Maxwell's equations. Tait was a friend of Maxwell's who in 1852 became Senior Wrangler (highest examination marks) at Cambridge University. We learn that Tait sent news of his success to his family in Edinburgh by the electrical telegraph, then a fairly recent invention. Another physicist highlighted in *Vector* is the Yale University professor Gibbs, who was influenced by Maxwell and published his *Elements of Vector Analysis* in the early 1880s.

Chapter 9 sets the stage for subsequent developments with an examination of Einstein's special theory of relativity and its further interpretation by Hermann Minkowski in terms of space-time geometry. Chapters 10–12 consist of accounts of Riemannian geometry, tensor analysis, and the general theory of relativity. In the entries in the index there are many more subentries for the word 'tensor' than there are for 'vector' or any other word. Significantly, there is no subentry for 'definition of'. In contrast to a scalar or a vector, the idea of a tensor is complicated in terms of concept, definition and notation. The subject of tensors can be daunting even to someone with a good mathematical background. Arianrhod devotes much effort to explaining the ideas and terminology of Riemannian differential geometry and tensor analysis in a way that conveys a meaningful sense for the subject to the general reader. Included are accounts of contravariant and covariant vectors and the conventions for summation used in tensor analysis.

In the first half of the book the author engages in the endnotes with the substantial historical literature that was consulted, although there are occasional omissions. For example, while Hamilton receives a good deal of attention, no mention is made of Thomas Hankins' authoritative 1980 biography. The chapters on differential geometry, tensors and relativity are written with somewhat closer attention to the secondary literature, which is discussed in some detail in the endnotes. Noteworthy is the recent historical work on the interactions between Einstein and David Hilbert. Also included are references to publications of Arianrhod herself on tensor physics. The historians and current scientists with a few exceptions are not listed in the index.

Chapter 11 titled 'Inventing tensors – and why they matter' includes an eclectic account of tensors in mathematical science and technology. There is discussion of current data science, natural language processing and large language models. Work in these areas is documented in the endnotes, where current interest in artificial intelligence is also highlighted. Tensors are viewed as something like generalized matrices in which the components possess multiple values or attributes corresponding to the system of objects under consideration.

In Chapter 12 Arianrhod chronicles the period from Einstein's initial collaboration in 1912 with Grossmann to his publication late in 1915 of the completed theory. Chapter 13 is titled 'What happened next'. Arianrhod examines Emmy Noether's work of 1918 on variational mathematics and discusses some aspects of her biography. She also tackles such difficult subjects as Levi-Civita's adoption in 1917 of general relativity and his use of tensors and the parallel transport of vectors to characterize curvature.

In the epilogue the author touches on cosmology and Einstein's theory of general relativity. In 1917 Einstein published a relativistic geometric model of the universe, a construction that was theoretical and seems to have been viewed even by Einstein himself as something of a curiosity. Nevertheless, the upheaval in cosmology following Vesto Slipher and Edwin Hubble's unexpected observational discoveries led to the invention (not envisaged by Einstein) of relativistic geometric models of an expanding universe framed in terms of the formalism of tensor analysis. Since the late 1950s general relativity has been confirmed empirically by radar time-delay sounding and other methods and has become the standard theory of gravity. Current measures of the expansion of the universe are derived and analysed using relativistic models. The latter are at the base of theorizing about the acceleration of expansion that was detected in the 1990s. As cosmology has over the past century grown steadily in interest and respectability, Einstein's theory of gravity and tensor analysis have become an essential part of the modern scientific world view.

Vector succeeds on several levels, as a stimulating survey of the development of the subject over the centuries, as a reference source for historical work that has been done, and as an accessible exposition of interesting but technical subject matter. In the history of physics and mathematics today the mathematical content is often viewed as something like a black box, with the primary focus being on the professional milieu or biographical and institutional details. It is commendable in a book for a wider audience that so much attention is devoted to looking at the mathematics itself and making it understandable.

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