Introduction to "Über die Navigation in der Luft als Problem der Variationsrechnung," *Jahresbericht der Deutschen Mathematiker-Vereinigung* 39 (1930), 2nd section, 44–48 and "Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung," Zeitschrift für angewandte Mathematik und Mechanik 11 (1931), 114–124. In *Ernst Zermelo - Collected Works/Gesammelte Werke: Volume II/Band II - Calculus of Variations, Applied Mathematics, and Physics/Variationsrechnung.* Springer: Heidelberger Akademie der Wissenschaften, Heinz-Dieter Ebbinghaus et al. (Eds.), 2013, pp. 672-678.

Introductory note to 1930c and 1931a

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According to Heinz-Dieter Ebbinghaus (*2007*, 150) the navigation problem was suggested to Zermelo by the circumnavigation of the globe by the airship "Graf Zeppelin" in 1929. The problem concerns a blimp or plane that moves with a given velocity relative to the air, travelling between two points on the earth. Because of the action of wind, the motion of the airship over land is modified. Suppose that the strength and direction of the wind are given as a function of position and time. The problem is to find the trajectory followed by the airship and the corresponding steering angle such that the airship completes its journey in the least time. Following the Hindenburg disaster of 1937, transportation by dirigibles or zeppelins became less common. In later formulations of the problem the airship was often replaced by a boat and the wind by current, and the problem was one of navigation along water.

In his two papers Zermelo formulated the problem mathematically as follows. (Our account is restricted to the two-dimensional case, although it should be noted that Zermelo extended his analysis to three dimensions.) The ship moves with velocity k relative to the surrounding medium, the latter being air in the case of an airship, and water in the case of a boat. The ship must go from A to B. There is a wind (in the case of a airship) or a current (in the case of a boat) that affects the motion of the ship. We consider an $x-y$ coordinate system with A at the origin. Assume at time t that the ship moves relative to the medium at an angle $\varphi = \varphi(t)$ to the x-axis. This "steering angle" is the direction in which we power the ship, knowing that the actual direction followed by the ship will be modified as the result of the action of the wind or current. The magnitude of the wind or current at time t and position (x, y) is given in terms of its x and y components, respectively $u(x, y, t)$ and $v(x, y, t)$. If (x, y) are the coordinates of the ship then the equations that describe its motion are

$$\begin{aligned} \frac{dx}{dt} &= u(x,y,t) + k\cos\varphi\,, \\ \frac{dy}{dt} &= v(x,y,t) + k\sin\varphi\,. \end{aligned} \tag{1}$$

It is necessary to find the track and steering angle for which the ship travels from A to B in least time. The partial derivatives of u with respect to x and y are denoted u_x and u_y respectively, with similar expressions for the partials of v. Zermelo derived a formula that describes this optimal motion in terms

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of the steering angle φ:

$$\frac{d\varphi}{dt} = v_x \sin^2 \varphi + (u_x - v_y) \cos\varphi \sin\varphi - u_y \cos^2 \varphi \,. \tag{2}$$

Zermelo's derivation of (2) was complicated and carried out directly from first principles for the special case of this problem. To the knowledge of the present writer no subsequent investigator adopted Zermelo's solution. Immediately following its appearance Tullio Levi-Civita (*1931*) published an article in which he stated, "Zermelo's direct and elegant treatment is very interesting." However, Levi-Civita did not make use of this solution and showed that the result follows from standard results in the calculus of variations. A few years later Carathéodory (*1935*, 378) asserted that the navigation problem "was posed by Zermelo and completely solved by an extraordinarily ingenious method" (translation from the English edition (*1967*) of Carathéodory's book). Carathéodory also did not adopt this method and followed Levi-Civita in treating the problem by means of standard methods in the calculus of variations. Carathéodory's investigation was very complete and remains today the most detailed analysis of the navigation problem. Among other things, Carathéodory examined the question of sufficiency and extended and systematized Zermelo's discussion of this point. Other mathematicians of the period who worked on the navigation problem included Basilio Manià (*1937*), and Magnus R. Hestenes (*1937*). The problem also attracted the attention of physicists, and was the subject of papers by Richard von Mises (*1931*) and Philipp Frank (*1933*). The latter researchers did not use the calculus of variations at all, but made the basis of their investigation an analogy between the motion of the airship and the passage of a light ray through a medium of variable refractive index.

The first of Zermelo's two papers is rather difficult to follow because it is really only an extended abstract and some details are omitted. Zermelo was motivated in part to continue work on the problem by an error that Levi-Civita found in the 1930 paper which he communicated to Zermelo in a letter. (See *Ebbinghaus 2007*, 151, where some other details of the genesis and reception of Zermelo's two papers may be found.) While the second paper provides a fuller account of the navigation problem, the analysis remains difficult to follow, requiring steps that are suited to the problem at hand but not otherwise part of a general theory. In his investigation of the problem Zermelo was returning to a branch of mathematics that he had worked on thirty years earlier. While he may have been somewhat out of touch with contemporary research in the subject, his formidable mathematical powers led him to interesting and useful results. Although his analysis was of a personal and even singular character, his papers are nonetheless worth studying and contemplating for their own sake and mathematical interest.

We now give a derivation of Zermelo's navigation formula using the idea that originated with Levi-Civita (*1931*). (Our derivation is based on the account given in *Funk 1962*, 282–284.) The following theorem was originally derived by Adolph Mayer (*1895*) and was presented by Oskar Bolza in his authoritative textbook *Vorlesungen über die Variationsrechnung* (*1909*, 572–574). We have a variational problem in which x is the independent variable and y_0, y_1 and y_2 are the dependent variables. The derivative of y_i with respect to x is denoted y_i'. The variables x and y_0, y_1, y_2 are connected by an equation of the form

$$g(x, y_0, y_1, y_2, y_0', y_1', y_2') = 0\,, \tag{3}$$

on the interval $[x_{1,}x_2]$. The variables y_1 and y_2 have prescribed values at the endpoints x_1 and x_2. The variable y_0 has a prescribed value at x_1. It is supposed that the value of y_0 at x_2 is a maximum or a minimum. A necessary condition for this to be the case is that there exists a multiplier function $\lambda = \lambda(x)$ defined on $[x_{1,}x_2]$ such that the following Euler equations are valid:

$$\begin{aligned} \frac{\partial(\lambda g)}{\partial y_1} - \frac{d}{dx}\Big(\frac{\partial(\lambda g)}{\partial y_1'}\Big) &= 0\,, \\ \frac{\partial(\lambda g)}{\partial y_2} - \frac{d}{dx}\Big(\frac{\partial(\lambda g)}{\partial y_2'}\Big) &= 0\,. \end{aligned} \tag{4}$$

The variational problem as formulated is known as a Mayer problem. Equations (4) are derived from a general multiplier rule for problems with constraints in the form of differential equations.

Consider now Zermelo's navigation problem. Because the time t itself is a variable that must be minimized we express t as a function of an independent variable τ, where τ lies on the interval $[\tau_1, \tau_2]$. The variables x and y are now to be regarded as functions of τ. From (1) we immediately obtain the following equation:

$$\Big(\frac{dx}{dt} - u\Big)^2 + \Big(\frac{dy}{dt} - v\Big)^2 - k^2 = 0\,. \tag{5}$$

Expressed in terms of the independent variable τ (5) becomes

$$\Big(\frac{x'}{t'} - u\Big)^2 + \Big(\frac{y'}{t'} - v\Big)^2 - k^2 = 0\,, \tag{6}$$

where the prime notation indicates differentiation with respect to τ. We now apply Mayer's result to this problem. In the application of this result the variables τ, t, x, y take the place of x, y_0, y_1, y_2 above. Here

$$g(\tau, t, x, y, t', x', y') = \Big(\frac{x'}{t'} - u\Big)^2 + \Big(\frac{y'}{t'} - v\Big)^2 - k^2 = 0\,. \tag{7}$$

Then there exists a multiplier function $\lambda = \lambda(\tau)$ such that

$$\begin{aligned} \frac{\partial(\lambda g)}{\partial x} - \frac{d}{d\tau}\Big(\frac{\partial(\lambda g)}{\partial x'}\Big) &= 0\,, \\ \frac{\partial(\lambda g)}{\partial y} - \frac{d}{d\tau}\Big(\frac{\partial(\lambda g)}{\partial y'}\Big) &= 0\,. \end{aligned} \tag{8}$$

These equations reduce to

$$\begin{aligned}\lambda\Big(\big(\frac{x'}{t'}-u\big)u_x+\big(\frac{y'}{t'}-v\big)v_x\Big)+\frac{d}{d\tau}\Big(\frac{\lambda}{t'}\big(\frac{x'}{t'}-u\big)\Big) &= 0\,,\\ \lambda\Big(\big(\frac{x'}{t'}-u\big)u_y+\big(\frac{y'}{t'}-v\big)v_y\Big)+\frac{d}{d\tau}\Big(\frac{\lambda}{t'}\big(\frac{y'}{t'}-v\big)\Big) &= 0\,.\end{aligned} \tag{9}$$

From (1) we have $\big(\frac{x'}{t'}-u\big)=\big(\frac{dx}{dt}-u\big)=k\cos\varphi$ and $\big(\frac{y'}{t'}-v\big)=\big(\frac{dy}{dt}-v\big)=k\sin\varphi$. Introduce the new multiplier function $\mu=\frac{\lambda}{t'}$. Note that for any variable z we have $\frac{dz}{d\tau}=t'\frac{dz}{dt}$. Then equations (9) simplify to

$$\begin{aligned}\mu\cos\varphi u_x+\mu\sin\varphi v_x+\frac{d}{dt}(\mu\cos\varphi) &= 0\,,\\ \mu\cos\varphi u_y+\mu\sin\varphi v_y+\frac{d}{dt}(\mu\sin\varphi) &= 0\,.\end{aligned} \tag{10}$$

Elimination of μ and $\frac{d\mu}{dt}$ from these two equations leads to Zermelo's navigation formula (2).

It is useful to consider some examples. We will examine two given by Zermelo (*1930*), although we present them in more detail than Zermelo did himself. For the sake of discussion, we will illustrate his formula for the case of a ship or power boat crossing a river. In the first example we are given a river that is straight with constant width a lying parallel to the x-axis, the latter being oriented in a west-east direction. The beginning point A on the south side of the river is taken to be the origin of the coordinate system; the y axis runs in a positive direction northwards, the x axis runs in a positive direction eastwards. The destination is the point B on the opposite side of the river with coordinates (b, a). The speed of the boat relative to the water (its speed over land in still water) is k. The current is constant in time and place, flowing from east to west. Its value at each point (x, y) is $-c$, where $k > c \geq 0$. The goal is to find the so-called "steering direction" $\varphi = \varphi(t)$. This is the direction in which we power the ship, knowing that the actual direction followed by the ship will be modified as the result of the action of the current. Equations (1) here become

$$\begin{aligned}&\text{(a)}\quad \frac{dx}{dt} = -\,c+k\cos\varphi\,,\\ &\text{(b)}\quad \frac{dy}{dt} = k\sin\varphi\,.\end{aligned} \tag{11}$$

The navigation formula (2) becomes $\frac{d\varphi}{dt}=0$, or $\varphi=$ const. It is apparent from (11) that $\frac{dy}{dx}=$ const., and the path followed by the ship in least time is the straight line joining A and B. If we let $a = b$ then $\frac{dy}{dx} = 1$. Dividing (11a) by (11b) we have $\cos\varphi - \sin\varphi = \frac{c}{k}$. Solving for the steering angle φ we obtain

$$\varphi = \arccos\left(\frac{\frac{c}{k} + \sqrt{2 - \frac{c^2}{k^2}}}{2}\right). \tag{12}$$

For example, if $k = 10$ knots and $c = 3$ knots then $\varphi = \arccos(.841) = 32.8°$. Hence we would place the ship on a steering bearing of 57.2° (90° – 32.8°) with respect to true north for it to follow the track $y = x$ of least time from A to B.

Zermelo (*1930*, 48) refers to the second example as "the simplest non-trivial example of our theory." In this example the current in the river increases as a linear function of y, being 0 at A and reaching its maximum value at B. Assume again that the river flows from east to west. Equations (1) here are

$$\begin{aligned} &\text{(a)} \quad \frac{dx}{dt} = -y + \cos\varphi\,, \\ &\text{(b)} \quad \frac{dy}{dt} = \sin\varphi\,, \end{aligned} \tag{13}$$

where the units have been adjusted so that $k = 1$. The navigation formula gives $\frac{d\varphi}{dt} = \cos^2\varphi$. We integrate this to obtain $\tan\varphi = t + \tan\varphi_0$, which gives the steering angle as a function of time. To find the optimal track we begin by noting that from (13b) we have $\frac{dy}{d\varphi} \cdot \frac{d\varphi}{dt} = \sin\varphi$, or $\frac{dy}{d\varphi}\cos^2\varphi = \sin\varphi$. Hence $y = \frac{1}{\cos\varphi} + \text{const.}$, or

$$y = \frac{1}{\cos\varphi} - \frac{1}{\cos\varphi_0}\,. \tag{14}$$

We now divide(13a) by (13b) and use (14) to obtain

$$\frac{dx}{dy} = \frac{1 - y(y+f)}{\sqrt{(y+f)^2 - 1}}\,, \tag{15}$$

where $f = \sec\varphi_0$. (15) describes the tracks followed by the ship in least time from A to points B on the opposite bank. Fig. 1 provides graphs of solutions to (15) giving the optimal curves for values of f from 1.0 to 1.45 in increments of 0.05.

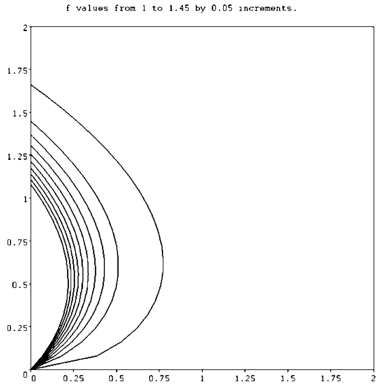

Fig. 1

Zermelo's navigation problem appears in several later textbooks on the calculus of variations. *Funk 1962* and *Klötzler 1970* are representatives from the German literature. Funk's treatment is very clear but relatively brief. A notable characteristic of Klötzler's account is its abstruseness, a result of the author's decision to develop the calculus of variations as part of functional analysis. English-language authors seem united in presenting the problem as one in nautical navigation. Some authors of note here are Fox (*1950*), Pars (*1962*), Young (*1969*), and Sagan (*1969*). With the exception of Funk, Zermelo's navigation formula is derived in none of these books. The only thing the authors seem to owe to Zermelo's original analysis is the formulation of the problem as one in the calculus of variations. In the English literature the navigation problem is presented for a ship moving through water against a current. There is no recognition that the problem was originally formulated for an aircraft or dirigible. Young states the problem but provides no solution. The treatment of Fox is very clear and derives from *Carathéodory 1935*. Fox (p.152) deduces the interesting fact that the ship can be steered along its optimal track by blind reckoning, that is, by a knowledge only of the time independent of any information about its external position. Pars's exposition is essentially a recapitulation of Fox's account. Of these later treatments, Sagan's is the simplest in its approach. This author solves for the time t and then uses the standard Euler equations to obtain a description of the motion of the boat.

It should be noted that the canonical problems of the calculus of variations—the isoperimetric problem, the hanging chain, the brachistochrone—go back centuries and appear at an early stage in the history of the subject. The navigation problem is somewhat unusual in providing a simple and signature example of very recent vintage, arising from technological developments of the twentieth century.

In the theory of control Zermelo's navigation problem appears as a standard problem of interest. Indeed, in modern mathematical science this subject would probably be the most likely place one would encounter the problem. For a sample of this literature see *Hsu and Meyer 1968, Bryson and Ho 1975*, and *Lewis 1986*. Writers refer to the Zermelo problem as "classic"; it is invariably presented for the case of a power boat moving against a current. In a recent book on optimal control and aerospace applications Ben-Asher (*2010*, 101) writes: "This problem was proposed by the German mathematician Ernest [*sic*] Zermelo in 1913 [*sic*]. Although formulated for a boat, it can also describe an aircraft flying in wind, assuming a fast response to heading changes. Therefore, it will be used extensively in this textbook." Historical misinformation is also present in the *Wikipedia* article on Zermelo, where we find the following: "Proposed in 1931, the Zermelo's navigation problem is a classic optimal control problem. The problem deals with a boat navigating on a body of water, originating from a point O to a destination point D. The boat is capable of a certain

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1930c

(Auszug. Eine ausführliche Darstellung erfolgt demnächst in der „Zeitschrift für angewandte Mathematik".)

Das hier behandelte Problem ist das folgende. *In einer unbegrenzten Ebene, in welcher die Windverteilung durch ein Vektorfeld* u, v *als Funktion von Ort und Zeit gegeben ist, bewegt sich ein Luftschiff oder Flugzeug mit der konstanten Eigengeschwindigkeit* k *relativ zur umgebenden Luftmasse. Wie muß das Fahrzeug gesteuert werden, um in kürzester Zeit von einem Punkte* P_0 *zu einem anderen* P_1 *zu gelangen?* Wird mit φ der Winkel bezeichnet, den die „Steuerrichtung" des Fahrzeuges, d. h. der Vektor seiner Eigenbewegung mit der x-Achse bildet, so ergeben sich unmittelbar durch Vektoraddition für die Geschwindigkeitskomponenten der Gesamtbewegung die „Steuergleichungen"

$$\left.\begin{aligned} \frac{dx}{dt} &= k\cos\varphi + u(x,y,t) \\ \frac{dy}{dt} &= k\sin\varphi + v(x,y,t)\ , \end{aligned}\right\} \tag{1}$$

welche die Bewegung vollständig bestimmen, wenn φ als Funktion der Zeit gegeben ist. In Wirklichkeit soll aber diese Funktion so gewählt werden, daß die gestellte Minimumsbedingung erfüllt ist. Es handelt sich also um ein Va-