# *The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century*

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## Introduction

In his *History of the Calculus* CARL BOYER [1959, 242–3] noted a change of view that developed at the middle of the 18th century, a rejection of geometric conceptions and an emphasis on formal methods in the new analysis. The tendency noted by BOYER has since been documented in more detail in the literature.[1] The picture that now emerges of the development of the calculus on the Continent would divide advanced research in the subject into three broad periods: a geometric stage, in which geometric problems and conceptions predominate; an analytical or "algebraic" stage that begins in the 1740s in the writings of LEONHARD EULER and reaches it final expression in work of JOSEPH LOUIS LAGRANGE at the end of the century; and the period of classical analysis that begins in the early 19th century in the writings of AUGUSTIN LOUIS CAUCHY.[2]

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[1] See in particular Bos [1974] and ENGELSMAN [1984].

[2] This division of stages is not absolutely rigid; one can discern at different times and at different levels of research a varying mixture of geometric, algebraic and arith-

The first part of this paper presents some examples to illustrate in specific and selected detail the calculus of EULER and LAGRANGE. My intent is to identify as clearly as possible those elements that are *common* in their approach to analysis. My contention is that these elements constitute evidence of a shared conception significantly different from the modern one, with its origins in CAUCHY's early $19^{th}$-century work.

The second part elaborates this thesis by presenting a characterization of EULER and LAGRANGE's calculus and an account of how it differs from CAUCHY's arithmetical theory. The discussion is complemented by a consideration of philosophical differences between mathematics in the $18^{th}$ and $19^{th}$ centuries.

## Part One

The calculus today is the core of analysis, a subject whose basic concepts are domain, functional mapping, limit, continuity, differentiability and so on. At the elementary level the calculus is developed for real values of the variable. The derivative of a function is defined at each number of the domain by a limit process in terms of the values of the function in a neighborhood about the number. Conditions of continuity and differentiability enable one to connect the local behavior of the function, how it changes in the neighborhood of a number, to its behavior over the entire domain.

When the calculus, restricted to real values, is developed more fully, using terminology and methods borrowed from point-set topology, it becomes modern real analysis. A different sort of generalization is obtained when the domain is assumed to be a region of the complex plane. The earlier approach, involving the definition of the derivative using the concepts of neighborhood and limit, is also applicable here. Although complex analysis places special emphasis on the notion of analyticity and on the use of power series, it retains in its foundation the same concepts (neighborhood, limit) as the calculus.

The process that led to the modern calculus and classical real and complex analysis had its beginnings in CAUCHY's textbooks of the 1820s.[3] The work of CAUCHY constituted a major break with the then-established tradition, prevalent in Continental $18^{th}$-century work and presented in detail in the famous textbooks of EULER (middle of the century) and LAGRANGE (end of the century).[4] Although the careers of the two men spanned almost a century, and although they differed in their specific foundational proposals, their work taken broadly shares an

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metic elements. Specifically, it refers to tendencies in the $18^{th}$ century in the work of advanced researchers published in the memoirs of the three leading European academies, Paris, Berlin and St. Petersburg.

[3] CAUCHY [1821], [1823] and [1829]. For discussions of CAUCHY's foundation see [JOURDAIN 1913], [GRATTAN-GUINNESS 1970a, b], [FREUDENTHAL 1971], [GRABINER 1981], [SMITHIES 1986] and [BOTTAZZINI 1986].

[4] EULER [1748], [1755], [1768–70] and LAGRANGE [1797], [1801] and [1806]. CANTOR [1901, 699–721, 749–773] provides a summary of [EULER 1748 and 1755]. EULER's elimination [1755, Chapter 4] of higher-order differentials is discussed by BOS [1974]. LAGRANGE's textbooks are discussed in [OVAERT 1976], [GRABINER 1981] and [FRASER 1987].

explicit emphasis on the analytical or "algebraic" character of the differential and integral calculus, both as a foundational description and as a theme to unify the different branches of the subject; on the need to separate the calculus from geometry, while continuing to cultivate geometrical and mechanical applications; and on a belief in generality as a primary goal of mathematics.

It must be noted that EULER and LAGRANGE differed in their specific ideas concerning the foundation of the calculus. EULER retained as fundamental the differential, while LAGRANGE tried using TAYLOR's theorem and derived functions to rid analysis of infinitesimals. In presenting the following examples, selected to illustrate their calculus, I have emphasized the significant, more general similarities which I believe exist in their approach to the subject.

### (a) Theorem on the equality of mixed partial differentials

The theorem on the equality of mixed partial differentials was published by EULER in 1740 in the memoirs of the Academy of Sciences in St. Petersburg. EULER used the theorem to derive expressions for partial differentials that had arisen in problems involving the construction of orthogonal trajectories to families of curves.[5] The result itself was suggested by his experience in working with differential expressions, by the recognition that the result obtained in the successive application of the differential algorithm to expressions involving two variables was independent of the order of differentiation.

EULER's derivation of the theorem originated out of his belief that a geometrical demonstration would be "drawn from an alien source", and that what was needed was an analytical argument based on "the nature of differentiation itself".[6]

A comparison of his derivation and today's proof provides an informative study of $18^{\text{th}}$-century and modern calculus. EULER [1740, 177–178] considers a quantity $z$ that is a function of the variables $x$ and $a$. If $dx$ and $da$ are the differentials of $x$ and $a$, let $e, f$, and $g$ denote the values of $z$ at $(x + dx, a)$, $(x, a + da)$ and $(x + dx, a + da)$. EULER differentiates $z$ holding $a$ constant to obtain

$$P\,dx = e - z. \tag{1}$$

Here $P$ denotes the differential coefficient, in later mathematics the partial derivative, of $z$ with respect to $x$. He differentiates $P\,dx$ holding $x$ constant:

$$B\,dx\,da = g - f - e + z. \tag{2}$$

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[5] Uses of the theorem by NICHOLAS I BERNOULLI and EULER are described in [ENGELSMAN 1984, Chapters Four and Five].

[6] These comments appear in "De differentiatione functionum duas pluresve variabiles quantitates involventium", which dates from the mid 1730s and is a draft of his [1740]. In this memoir EULER first considers a geometric justification of the theorem and then presents the analytical derivation that was later published. The draft is reproduced with English translation as Appendix Two of [ENGELSMAN 1984]. The relevant sentence reads: "Quia autem haec demonstratio ex alieno fonte est petita, aliam ex ipsius differentiationis natura derivabo."

He now differentiates $z$ holding $a$ constant to obtain

$$Q\,da = f - z. \tag{3}$$

Finally, he differentiates $Q\,da$ holding $x$ constant:

$$C\,da\,dx = g - e - f + z. \tag{4}$$

By rearrangement of terms the right sides of (2) and (4) are seen to be equal. Equating the left sides EULER obtains

$$B = C, \tag{5}$$

which is the desired result.

The modern proof of the theorem reformulates EULER's demonstration using the law of the mean and a limit argument.[7] Suppose $z = z(x, a)$ and its first and second partial derivatives are defined and continuous on a rectangular region in the $x - a$ plane. For $x$ and $a$ in this region we have by the law of the mean for small $h$ and $k$ the four equations:

$$\frac{\partial z}{\partial x}(x + \varepsilon_1 h, a)\,h = z(x + h, a) - z(x, a), 0 \leqq \varepsilon_1 \leqq 1, \tag{1$'$}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial a\,\partial x}(x + \varepsilon_1 h, a + \varepsilon_2 k)\,hk = {} & z(x + h, a + k) - z(x, a + k) \\ & - z(x + h, a) + z(x, a),\ 0 \leqq \varepsilon_2 \leqq 1,\end{aligned} \tag{2$'$}$$

$$\frac{\partial z}{\partial a}(x, a + \eta_1 k)\,k = z(x, a + k) - z(x, a),\ \ 0 \leqq \eta_1 \leqq 1, \tag{3$'$}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\,\partial a}(x + \eta_1 h, a + \eta_2 k)\,kh = {} & z(x + h, a + k) - z(x + h, a) \\ & - z(x, a + k) + z(x, a), 0 \leqq \eta_2 \leqq 1.\end{aligned} \tag{4$'$}$$

By rearrangement the right sides of (2)′ and (4)′ are equal. The left sides may therefore be equated:

$$\frac{\partial^2 z}{\partial a\,\partial x}(x + \varepsilon_1 h, a + \varepsilon_2 k) = \frac{\partial^2 z}{\partial x\,\partial a}(x + \eta_1 h, a + \eta_2 k).$$

Letting $h$ and $k$ tend to zero we obtain from the continuity of the second partial derivatives the desired result

$$\frac{\partial^2 z}{\partial a\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial a}. \tag{5$'$}$$

The formal demonstration of the theorem and its classical rehabilitation using the law of the mean are typical of many $18^{\text{th}}$ century arguments and their modern counterparts. The law of the mean introduces a distinguished value ($x + \varepsilon_1 h$, and so on), localizing at a particular number the analytical relation or property

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[7] The demonstration presented here is adapted from [TAYLOR 1955, 220–221].

in question.[8] The result is then deduced using conditions of continuity and differentiability by means of a limit argument.

In 18th-century analysis distinguished values were considered exceptional, special cases without mathematical significance.[9] EULER believed with some reason that the essential element in a demonstration was its generality, guaranteed by a formal algebraic argument. Thus the key step in his proof, the equality of the right sides of equations (2) and (4), was an algebraic identity that ensured the validity of the result. Given EULER's understanding of the calculus his demonstration was quite satisfactory, not at all incomplete or unrigorous.

### (b) Infinite Series

Research on infinite series was one of the most extensive subjects of 18th-century analysis. Infinite series were used in numerical approximation, in the integration of differential equations and in the foundations of the calculus.

From the vast researches of the period it is possible to isolate certain leading ideas that were characteristic of advanced views on the subject. Infinite series were never introduced arbitrarily; they were derived from expressions that were themselves formed in finitely many steps using the processes of ordinary algebra and the differential and integral calculus. This point is made clearly by EULER in his discussion of divergence, where he defends his method for assigning a sum to a divergent series:

> If therefore we change the accepted notion of sum to such a degree that we say the sum of any series is a finite expression out of whose development that series is formed, all difficulties vanish of their own accord. For first that expression from whose expansion a convergent series arises displays the sum, this word being taken in its ordinary sense; and if the series is divergent, the search cannot be thought absurd if we hunt for that finite expression which expanded produces the series according to the rules of analysis.
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> [1760, 211–212][10]

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[8] The mean value theorem is first stated in [LAGRANGE 1797], where it is introduced (without any of the modern conditions) in order to derive estimates for the remainder in the TAYLOR series. This part of LAGRANGE's theory, which stands somewhat apart from the rest of his treatise, is concerned with the numerical approximation of functions. From his own standpoint, it constitutes an application of the calculus. The theory would assume a new, different and fundamental significance in CAUCHY's later work. The history of the mean value theorem is presented by FLETT [1974], and the relation of LAGRANGE's theory to CAUCHY is discussed by GRABINER [1981].

[9] The treatment of exceptional values in EULER's calculus is discussed by ENGELSMAN [1984, 10–13], and in LAGRANGE's calculus, by FRASER [1987].

[10] "Si igitur receptam summae notionem ita tantum immutemus, ut dicamus cuisque seriei summam esse expressionem finitam, ex cuius evolutione illa ipsa series nascatur, omnes difficultates, quae ab utraque parte sunt commotae, sponte evanescent. Primo enim ea expressio, ex cuius evolutione nascitur series convergens, eius simul summam, voce hac vulgari sensu accepta, exhibet, neque, si series fuerit divergens, questio amplius

Elsewhere EULER observes [CANTOR 1901, 692] that it is "certain that the same series can never arise from the evolution of two genuinely different finite expressions."[11]

As the title indicates, infinite series are the subject of EULER's *Introductio in analysin infinitorum* (1748), the work that made the concept of function central to mathematics:

> A function of a variable quantity is an analytical expression composed in any way from the variable and from numbers or constant quantities. [1748, § 4][12]

Although infinite series, and more particularly power series, are a useful tool in the investigation of all functions, they are especially important for the study of transcendental expressions, the logarithmic, exponential and trigonometric functions:

> Moreover the nature of a transcendental Function is made more intelligible if it is in this form, although infinite, expressed. [1748, § 59][13]

Note that infinite series are not themselves regarded as functions, but serve only to render these objects "intelligible".

An example of how infinite series entered analysis is provided by the trigonometrical functions. In his *Traité de Dynamique* [1743, 100–1] D'ALEMBERT considered the differential equation in the variables $u$ and $t$

$$d^2u/dt^2 = -(2\sqrt{2}/T^2)\,u \qquad (T \text{ a constant})$$

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absurda reputari poterit, si eam indagemus expressionem finitam, quae secundum regulas analyticas evoluta illam ipsam seriem producat." EULER's work in summability is the subject of [BARBEAU & LEAH 1976], which also contains a partial English translation of [EULER 1760]. The above translation follows this source.

[11] „..ich glaube aber gewiss zu seyn, dass nimmer eben dieselbe series aus der Evolutionem zweyer wirklich verschiedener expressionum finitarum entstehen könne." EULER's comments appear in a letter to GOLDBACH dated August 7, 1745, published in [FUSS 1843 I 324].

[12] "Functio quantitas ergo variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili & numeris seu quantitatibus constantibus." Compare D'ALEMBERT [1757]: "on appelle *fonction* de $x$, ou en générale d'une quanitité algébrique composée de tant de termes qu'on voudra et dans laquelle $x$ se trouve d'une manière quelconque, melée, ou non, avec des constants; ainsi $x^2 + x^3$, $\sqrt{aa + xx}$, $\sqrt{(aa + x^3)/(bb + x^4)}$, $\int dx\sqrt{a^2 - x^2}$, etc. sont des fonctions de $x$."

[13] "Quin etiam natura Functionum transcendentium melius intelligi censetur, si per eiusmodi formam, etsi infinitam, exprimantur." The realization that plane curves are divided into two classes, corresponding to those that may be represented by a polynomial equation of finite degree in $x$ and $y$ and all others, seems to have appeared first in DESCARTES' *Géométrie* (1637). Non-algebraic, "mechanical" or transcendental curves appeared frequently in subsequent mathematics and are closely connected [MAHONEY 1984] to the invention and development of the calculus. GUISNÉE [1733] writes that in order to represent a "mechanical" curve by means of a polynomial equation "il faudrait qu'au moins une de ses inconnunes êut une infinité de dimensions, ce qui est impossible; et c'est pour cela ques Courbes sont aussi nommées transcendentes."

and using the substitution $v = du/dt$ integrated it to obtain

$$dt = -T\, du/\sqrt{A^2 - 2\sqrt{2}u^2},$$

where $A$ is a constant of integration. The solution of this equation is given by D'ALEMBERT as a geometrical construction connecting $u$ and $t$. His solution indicates that he recognized that $(\sqrt{2\sqrt{2}}\,t)/T$ is equal to the angle whose cosine is $(\sqrt{2\sqrt{2}}\,u)/A$, although no functional trigonometrical notation is employed. In this respect his treatment was entirely typical of the period. In the 1740s EULER, responding to the technical needs of mathematical astronomy, invented the calculus of the sine and cosine functions. Rejecting the special geometric constructions and dimensional considerations of earlier work, he developed a purely analytical theory.[14] In the *Introductio* [1748, Chapter 8] EULER derived the familiar power series for the trigonometrical functions, using multi-angle formulas and techniques he had employed earlier in the treatise to obtain the exponential series. Although the sine and cosine expansions were not new, they had now been derived by analytical principles: a function that was a solution to a definite differential equation had been expanded to yield the given series.

EULER's derivation illustrates I believe the 18th-century understanding of infinite series. While an infinite series could be generated recursively as a solution to a differential equation (using, for example, the method of undetermined coefficients [KLINE 1972, 488–499]), the series was not itself regarded as an independent mathematical object. A new transcendental function $y = f(x)$ defined as a solution to the given differential equation presented a relation between $x$ and $y$ whose permanence was ensured by the equation; $f(x)$ enjoyed definite analytical properties that might be regarded apart from its possible representation as an infinite polynomial.

The concept of function that EULER made central in the *Introductio* is also fundamental to LAGRANGE's algebraic calculus ([1797,] [1801], [1806]). This calculus is based on the assumption that every function $f(x)$ may be expanded in a power series

$$f(x+i) = f(x) + p(x)\,i + q(x)\,i^2 + r(x)\,i^3 + \dots,$$

except possibly at isolated values of $x$. LAGRANGE regards the possibility of forming such expansions as inherent in the notion of a function; he calculates the expansions explicitly for a few algebraic functions and the exponential, logarithmic and trigonometric functions. Although the subsequent theory uses series extensively, it is clear that LAGRANGE's concept of function does not itself include infinite expressions; a function is always a finite analytical formula.

Histories of mathematics have attributed the emphasis placed on convergence by CAUCHY, GAUSS and ABEL to the 19th-century movement to instill rigour in analysis. The significant change in the theory of infinite series, however, was not so much that classical analysis brought rigour to the subject by paying attention to

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[14] EULER's introduction of trigonometrical functions is discussed by WILSON [1985, 17–18] and KATZ [1987].

convergence, but that an arbitrary series whose individual terms were specified at will now became, subject to convergence over some domain, implicitly an object of mathematical study. The understanding of what an infinite series was had undergone a substantial transformation.

### (c) The calculus of variations

LAGRANGE'S first and arguably most significant contribution to mathematics was his discovery in 1755 at age nineteen of the $\delta$-algorithm for the formulation and derivation of the basic problems and equations of the calculus of variations.[15] LAGRANGE began his investigation with EULER'S *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* (1744). EULER had considered a curve that was represented by an analytical relation between the variables $x$ (abscissa) and $y$ (ordinate). The curve was assumed to be the one that maximized or minimized the definite integral of some given expression in $x$ and $y$ evaluated between specified endpoints. EULER took any $y$ and increased it by an infinitesimal "particle", thereby obtaining a second curve identical to the first except at $y$. Because the integral was an extremum the difference between its value along the two curves had to be zero. EULER used this condition and the methods of the calculus to derive the basic differential equation, known today as the EULER-LAGRANGE equation, that characterizes the extremalizing curve.

EULER'S derivation illustrated well the calculus as applied to the representation and investigation of the curve. The differential $dy$ of the variable $y$ was the difference of $y$ at two infinitesimally close values of $x$. These values belonged to a linear geometrical continuum composed of discrete, infinitesimal "particles". The comparison curve was obtained by increasing a given value of $y$ by an infinitesimal particle. Each step in the derivation possessed an explicit geometrical interpretation.

LAGRANGE'S "beautiful idea" [GOLDSTINE 1980, 110] was to introduce a second symbol $\delta$ to distinguish the variation in $y$ required to obtain the comparison curve from the usual differential $dy$ of $y$ with respect to $x$. He experimented with the resulting calculus until he had devised an algorithm that yielded the variational equations. Mathematically quite distinct from EULER'S procedure, his derivation required no reference to the geometrical configuration. When interpreted geometrically it could be seen to require the simultaneous variation of all the values of the ordinate $y$. LAGRANGE'S method included all the cases considered by EULER and, in addition, facilitated the treatment of problems involving variable endpoints.

LAGRANGE'S idea was immediately adopted by EULER and made the basis of his own presentation of the calculus of variations. The invention of the $\delta$-algorithm showed that technical, formal innovations could lead to very significant results when interpreted in the geometry of curves. It suggested that the substantial content of the variational calculus was contained in its formal aspects. LAGRANGE himself seems to have held such a view, and in subsequent published treatises he tended increasingly to present its principles in terms of formal algorithms and rules.

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[15] Recent studies of LAGRANGE'S calculus of variations are [GOLDSTINE 1980] and [FRASER 1985b]. These sources contain references to the original papers of EULER and LAGRANGE and provide a survey of the older secondary literature.

LAGRANGE'S derivation of the variational equations requires for its full justification concepts and techniques that belong to classical real analysis. In particular, one needs the concept of arithmetical continuity as well as a more general notion of function than the one he employed. In his writings LAGRANGE presented different arguments to justify the steps in the derivation. Perhaps the most interesting is contained in the second edition [LAGRANGE 1806] of his lessons on the calculus of functions. In this treatise he developed the calculus as an algebra of finite quantities, defining the derivatives of a function in terms of the coefficients of its TAYLOR expansion. Assuming the general possibility of such expansions, LAGRANGE worked out a rather complete theory of the differential and integral calculus. To obtain the variational equations he modelled the derivation after an earlier argument in the theory of integrability. Although his derivation never achieved acceptance among later researchers, it remains noteworthy [FRASER 1985b and 1987] as an example of advanced reasoning in algebraic analysis.

## (d) "Discontinuous" Functions

In textbooks of EULER and LAGRANGE a function is given by a single analytical expression, a formula constructed from variables and constants in finitely many steps using algebraic and transcendental operations and the composition of functions. In the 18th century such functions were termed "continuous", in opposition to "discontinuous" functions, expressions defined piecewise over more than one interval of real numbers.

The issue of the nature of a function arose openly in the debate over the vibrating string, probably the most interesting and best documented mathematical controversy of the 18th century.[16] Although the debate engaged several mathematicians and involved a range of issues, the points of relevant interest here are illustrated by the disagreement between D'ALEMBERT and EULER.

In 1747 D'ALEMBERT derived and integrated a partial differential equation, the wave equation, to describe the motion of a stretched elastic string. D'ALEMBERT'S achievement was a major one, both mathematically and in its use of dynamical principles. His derivation was immediately adopted by EULER who reinterpreted the solution to permit a broader class of curves acceptable as initial shapes of the string. EULER considered this reinterpretation a substantial addition to D'ALEMBERT'S analysis, and reacted strongly when the latter dismissed his work. The subsequent debate, which was never resolved to the participants' satisfaction, concerned the mathematical question of how to interpret the initial solution of the wave equation.

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[16] BURKHARDT [1908], TRUESDELL [1960], LANGER [1947] and RAVETZ [1961] provide an account of physical and mathematical issues in the debate as well as references to original papers. Further observations are contained in [GRATTAN-GUINESS 1970b] and [LÜTZEN 1983].

The basic issue was that the integration of partial differential equations required the introduction of "arbitrary" functions. D'ALEMBERT insisted that only an equation connecting the variables $x$ and $y$ was acceptable as an initial solution; hence he would permit only functions that were given by single analytical expressions. Since the initial solution to the vibrating string had been shown to be periodic, one could allow only those functions that were, by virtue of their algebraic form, periodic.

EULER could not accept the restrictions D'ALEMBERT had imposed on the possible initial solutions. According to D'ALEMBERT, an initial shape of the string that was given by an arc of the parabola $y = x(1 - x)$ would be unacceptable because the expression $x(1 - x)$ was non-periodic. EULER saw no reason why one could not translate the arc of the parabola along the horizontal axis and thus define a new periodic curve. The curve obtained would be given by a periodic function that was defined piecewise over each interval of length 1. In the solution $y = f(x)$ the function symbol $f(x)$ would now refer to different algebraic expressions depending on the interval of real numbers to which $x$ belonged.

In the subsequent debate D'ALEMBERT and EULER defended vigorously their respective positions. On grounds of physical plausibility and mathematical generality EULER advocated the acceptance of "discontinuous" functions as initial solutions to the wave equation. D'ALEMBERT maintained that EULER'S conception was artificial, that the curves in question were not produced by any natural mode of generation. The calculus studied definite, given analytical expressions corresponding to such natural curves and the inclusion of EULER'S more general functions violated the basic principles of the subject.

Although the debate over the concept of function touched the very foundation of the calculus, the issues at question remained curiously isolated from the mainstream of contemporary mathematical analysis. The general functions advocated by EULER raised foundational problems that could not be resolved given the current directions of research. D'ALEMBERT'S opposition may have seemed obstinate, but it also displayed a clear sense for the spirit of the calculus. The distinguishing property of a "discontinuous" function, that its algebraic form depended on the interval of real numbers to which the independent variable belonged, undermined the basis of the calculus as a subject explicable by formal analytical principles. To defend the introduction of these objects EULER appealed to the physical model, without any clear specification of the necessary mathematical conditions; his reasoning here has been aptly termed a "return to geometry" [GRATTAN-GUINNESS 1970b, 11]. His notion of a general function was never incorporated into the analytical theory presented in his mid-century textbooks, and indeed was at odds with its basic direction. At best, EULER'S conception of a general function constituted an unrealized "vision" [LÜTZEN 1983] of a future mathematics.

The theory of partial differential equations would wait until the $19^{th}$ century for the resolution of the issues in the vibrating-string debate. CAUCHY'S definition of the derivative, in which $f'(x)$ is a function obtained from $f(x)$ at each numerical value of $x$ by a limit process, logically presupposed a new understanding of a function, as an arithmetical object that is specified (in whatever way) at each value of the independent argument. The issue of whether a function is given by a single expression or defined piecewise disappeared.

### (e) Complex Analysis

As the examples discussed thus far suggest, the calculus in the 18th century was developed in systematic detail for functions of a real variable. For reasons discussed presently, complex analysis remained by and large undeveloped during the period.

In 18th-century analysis no restriction was assumed to hold for the values taken by a variable or "universal quantity". EULER observes in the *Introductio* of 1748

> A variable quantity includes all numbers, positive and negative, whole and fractional, rational, irrational and transcendental. Even zero and imaginary numbers are not excluded in the meaning of a variable quantity.
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> [1748, § 2][17]

One could by the rules of algebra manipulate expressions containing imaginaries, and it was clear in specific problems, *e.g.*, determination of the roots of unity, that the formalism possessed a geometrical interpretation. EULER showed how to define the elementary algebraic and transcendental functions for complex values of the independent variable. Both EULER and D'ALEMBERT attempted to show that the general polynomial with real coefficients had a root of the form $a + b\sqrt{-1}$ ($a$ and $b$ real).

Mathematicians in the 18th century never developed a full geometric representation for complex numbers. Applied analysis and mechanics, so important during the period in suggesting lines of investigation, generated here no new problems. The algebraic understanding of the calculus reinforced the implicit assumption that the extension of the calculus to the complex domain raised no new issues.[18] The idea that one must distinguish two theories, for functions of a real and a complex variable, never arose in the 18th century.[19]

CAUCHY's contributions to complex analysis were developed over thirty years in a series of memoirs devoted to integration in the complex domain.[20] Complex analysis emerged in his work as a subject with its own theorems, problems and

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[17] "Quantitas ergo variabilis in se complecitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur."

[18] The view is sometimes expressed that LAGRANGE's power-series approach to the theory of functions was vindicated in later mathematics by the conception of "analytical" function in complex analysis. It is worth noting that problems in this subject are conspicuously absent in LAGRANGE's *oeuvre*, and that his understanding of how and under what conditions a function may be expanded in terms of its derivatives is different from the later theory.

[19] A survey of 18th-century analytical work that involves imaginaries is presented by STÄCKEL [1900].

[20] Studies of CAUCHY's work in complex analysis, with references to the original papers, are provided by STÄCKEL [1900], JOURDAIN [1905] and GRATTAN-GUINESS [1970b, Chapter Two].

applications. Although he never presented a foundation for the theory in the style of his textbooks on the calculus of a real variable, his earlier approach was evidently valid here. ARGAND and GAUSS had shown that the field of complex numbers could be interpreted as points in a two-dimensional continuum. A function of a complex variable became a relation among variables connecting points in two such continua. The framework that CAUCHY had developed for real analysis, involving neighborhoods, limits and pointwise definition of the derivative, was immediately generalizable to the complex domain.

## Part Two

### (a) Mathematical Reflections

For reasons discussed above, the calculus in the $18^{th}$ century was developed systematically for functions of a real variable. It is therefore appropriate to restrict comparisons between this calculus and the modern subject to consideration of a theory of a real variable.

In the analytical approach of EULER and LAGRANGE the calculus is an instrument for investigating geometrical curves, a distinct class of mathematical objects. It is evident (in a particular but definite sense) that this separation of the formalism of the calculus from geometry does not arise in the modern subject. A theorem about a function defined on some interval of real numbers under specified conditions of differentiability has a geometrical interpetation implicit in its very formulation.

The algebraic calculus studies functional relations, algorithms and operations on variables. The values that these variables receive, their arithmetic or geometric interpretation, are of secondary concern. In real analysis, by contrast, the basic object of study is the numerical continuum. The formalism of the modern subject is in a fundamental sense interpreted—given meaning—as a theory of functions defined on domains of real numbers. In this respect classical real analysis resembles the calculus of the early $18^{th}$ century, when the formalism of the subject was regarded as a way of representing and investigating the curve. The early calculus was based on a concept of a particulate geometrical continuum, something that is quite different from the numerical continuum of classical analysis. Nevertheless, the early calculus was interpreted in the geometry of curves in the same way that the modern calculus is interpreted in real analysis.

The calculus of EULER and LAGRANGE differs from later analysis in its assumptions about mathematical existence. The relation of this calculus to geometry or arithmetic is one of correspondence rather than representation. Its objects are formulas constructed from variables and constants using elementary and transcendental operations and the composition of functions. When EULER and LAGRANGE use the term "continuous" function they are referring to a function given by a single analytical expression; "continuity" means continuity of algebraic form. A theorem is often regarded as demonstrated if verified for several examples, the assumption being that the reasoning in question could be adapted to any other example one chose to consider. The problem of establishing the *a priori* existence of a general solution to a given class of differential equations does not arise within such a framework.

In the modern calculus attention is focussed locally, on a curve near a point or on a neighborhood about a number. By contrast, the algebraic viewpoint of EULER and LAGRANGE is global. The existence of an equation among variables implies the global validity of the relation in question. An analytical algorithm or technique implies a uniform and general mode of operation. In EULER's or LAGRANGE's presentation of a theorem of the calculus, no attention is paid to considerations of domain. The idea behind the proof is always algebraic. It is invariably understood that the theorem in question is generally correct, true everywhere except possibly at isolated exceptional values. The failure of the theorem at such values is not considered significant. The primary fact, the meaning of the theorem, derives always from the underlying algebra.

### (b) Philosophical Reflections

In the Preface to the *Traité de Dynamique* (1743) JEAN D'ALEMBERT discusses the philosophy of mathematics as background to the presentation of his dynamics. D'ALEMBERT's comments express clearly the $18^{th}$-century understanding of mathematics:

> The certitude of Mathematics is an advantage that these Sciences owe principally to the simplicity of their object. [1743, i][21]

By the "object" of mathematics D'ALEMBERT is referring to

> the calculation of magnitudes and the general properties of extent, that is Algebra, Geometry and Mechanics, ... [1743, i][22]

He comments on the object of the mathematical sciences:

> The more the object they embrace is extended and considered in a general and abstract manner, the more also their principles are exempt from obscurity and easy to grasp. [1743, ii][23]

He proceeds to describe the correct method in any science:

> It results from these reflections that in order to treat according to the best possible Method any given part of Mathematics whatever (we could even say

[21] "La certitude des Mathématiques est un avantage que ces Sciences doivent principalement à la simplicité de leur objet."

[22] "calcul des grandeurs, & des propriétés générales de l'étendue, c'est-à-dire l'Algébre, la Géométrie & la Méchanique, ..."

[23] "Plus l'objet qu'elles embrassent est étendu, & considéré d'une manière générale & abstraite, plus aussi leurs Principes sont exempts de nuages, & faciles à saisir."

any Science) it is necessary not only to introduce and apply there ideas derived in the more abstract and consequently more simple Sciences, but moreover to envisage in the manner most abstract and most simple as is possible the particular object of this Science; to assume nothing and to admit nothing in this object except the properties that the Science itself there supposes. [1743, ii–iii)[24]

It is important to note that in this conception the generality of the method of any part of mathematics derives from its "object". Thus it is the generality of the formulas of algebra or the diagrams of geometry that assures the generality of the associated method and hence the generality of the mathematics itself. In particular, the range of application and the certainty of a given branch of mathematics derives, not from the inherent logically prior character of its method, but from the simplicity and abstraction of its object.

According to D'ALEMBERT the "object" of mathematics is given to us as algebra, geometry or mechanics. Mathematical concepts are idealizations derived from physical reality and distinguished by their exceptional abstraction and generality. Although mathematical knowledge consists of necessary truths, the mathematics itself is not made up of *a priori* constructs. Formulas, spatial configurations and dynamical interactions are given "objectively" as part of algebra, geometry and mechanics, and derive their meaning as part of these subjects.[25]

D'ALEMBERT's philosophy reflected the prevailing $18^{\text{th}}$-century understanding of mathematics. The placement of mathematics in its "object" and the emphasis on generality had implications for the differential and integral calculus, the most advanced mathematical science of the period. The calculus or infinitesimal analysis was an extension of ordinary algebra used to investigate the geometry of curves. The problems and theorems of the subject arose in the course of these investigations; they did not appear as arbitrarily formulated propositions.

The movement at the middle of the century to separate the calculus from geometry transformed the prevailing philosophy into a version of mathematical formalism. The original problem of the calculus, to describe mathematically change along a curve, gave way to the study of formulas as the defining characteristic of the subject. The rules and procedures of the calculus were assumed to be generally valid. In a memoir published in 1751 EULER considers the rule $d(\log x)$

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[24] "Il résulte de ces réflexions, que pour traiter suivant la meilleure Méthode possible quelque partie des Mathématiques que ce soit (nous pourrions même dire quelque Science que ce puisse être) il est nécessaire nonseulement d'y introduire & d'y appliquer autant qu'il se peut, des connoissances puisées dans des Sciences plus abstraites, & par conséquent plus simples, mais encore d'envisager de la manière la plus abstraite & la plus simple qu'il se puisse, l'objet particulier de cette Science; de ne rien admettre dans cet objet, que les propriétés que la Science même qu'on traite y suppose."

[25] It is interesting here to note the places of mathematics and logic in the organizational chart of knowledge presented at the beginning of the preliminary discourse to the *Encyclopédie*. D'ALEMBERT divides understanding into the three general categories of memory, reason and imagination. Although both mathematics and logic are listed under reason, mathematics is classified as a "science of nature" while logic is regarded as a "science of man."

$= dx/x$. He rejects an earlier suggestion of LEIBNIZ that this rule is only valid for positive real values of $x$ with the observation

> For, as this [differential] calculus concerns variable quantities, that is, quantities considered in general, if it were not generally true that $d \cdot lx = dx/x$, whatever value we give to $x$, either positive, negative or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains. [EULER 1751, 143–144][26]

EULER'S confidence in formalism derived from the evident success of analytical methods, but it was also supported by his philosophy. The calculus enjoys an extended range of application because of the generality of the analytical relations that comprise it. A formal demonstration of a relation, one involving no assumptions concerning the individual values of the variables, ensures its global validity. Since the truth of the calculus is grounded in generality isolated exceptional values at which the relation fails are not significant.

The calculus of EULER and LAGRANGE is composed independently of arithmetic and geometry of analytical relations and formulas. In addition, the principles that govern this calculus are not formulated as part of any logical method; they are in some unspecified sense given as part of the subject. A problem not resolved by EULER and LAGRANGE is to explain precisely how, given this independence of the calculus from arithmetic and geometry, its "object" is constituted. The 18th-century faith in formalism, which seems today rather puzzling, was reinforced in practice by the success of analytical methods.[27] At base it rested on what was essentially a philosophical conviction.[28]

## Conclusion

Historians of mathematics have noted traditional elements in CAUCHY'S calculus, his retention in practice at least partially of the older concept of function and his failure to distinguish between continuity and uniform continuity. The

[26] "Car, comme ce calcul roule sur des quantités variables, c. à d. sur des quantités considérées en général, s'il n'etoit pas vrai généralement qu'il fût d. $lx = dx/x$, quelque quantité qu'on donne à $x$, soit positive ou negative, ou même imaginaire, on ne pourrait jamais se servir de cette régle, la verité du calcul differentiel étant fondée sur la généralité des régles qu'il renferme."

[27] LANGER [1947, 17] writes "[EULER] combined with a phenomenal ingenuity an almost naive faith in the infallibility of mathematical formulas and the results of manipulations upon them." GRABINER [1974, 356] notes "Trust in symbolism in the eighteenth century is somewhat anomalous in the history of mathematics, and needs to be accounted for."

[28] Discussing the famous passage at the beginning of the *Cours d'analyse* in which CAUCHY rejects the "generality of algebra", FREUDENTHAL [1971, 377] refers to "CAUCHY'S own, much broader appreciation, by which *all* metaphysics are to be barred from mathematics."

present paper nevertheless indicates the radical character of his break with received practice. Apart from how he understood his theory, there were mathematical and logical consequences implicit in his new approach. Mathematically, CAUCHY rejected the algebraic viewpoint and returned the calculus to its original concern with the curve, where in his arithmetical theory the line was replaced by the numerical continuum and the curve by a functional relationship between numbers. In his development of complex analysis he showed—if only implicitly—that his approach was generalizable to any field of numbers that possessed a suitable topological structure.

Assumed in CAUCHY's theory was the logical realization that generality must be sought internally in the methods of mathematics, not somehow in the "objective" character of its subject. It is I believe in this sense that the traditional picture of CAUCHY as a mathematician concerned with rigour should be understood. His work may be viewed as an extremely significant contribution to the development in the 19th century of pure mathematics and to the corresponding explicit logical separation of mathematics and theoretical physics that occurred during this period.

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