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MATHEMATICS

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Considered broadly, mathematical activity in the eighteenth century was characterized by a strong emphasis on analysis and mechanics. The great advances occurred in the development of calculus-related parts of mathematics and in the detailed elaboration of the program of inertial mechanics founded during the Scientific Revolution. There were other mathematical developments of note – in the theory of equations, number theory, probability and statistics, and geometry – but none of them reached anything like the depth and scope attained in analysis and mechanics.

The close relationship between mathematics and mechanics had a basis that extended deep into Enlightenment thought. In the Preliminary Discourse to the famous French *Encyclopédie,* Jean d'Alembert distinguished between "pure" mathematics (geometry, arithmetic, algebra, calculus) and "mixed" mathematics (mechanics, geometrical astronomy, optics, art of conjecturing). He classified mathematics more generally as a "science of nature" and separated it from logic, a "science of man." An internalized and critical spirit of inquiry, associated with the invention of new mathematical structures (for example, non-commutative algebra, non-Euclidean geometry, logic, set theory), represents characteristics of modern mathematics that would emerge only in the next century.

Although there were several notable British mathematicians of the period – Abraham De Moivre, James Stirling, Brook Taylor, and Colin Maclaurin among them – the major lines of mathematical production occurred on the Continent, a trend that intensified as the century developed.[1] Leadership was provided by a relatively small number of energetic figures: Jakob, Johann, and Daniel Bernoulli, Jakob Hermann, Leonhard Euler, Alexis Clairaut, Jean d'Alembert, Johann Heinrich Lambert, Joseph Louis Lagrange, Adrien Marie Legendre, and Pierre Simon Laplace. Research was coordinated by national

[1] For a study of British mathematics in the eighteenth century, see Niccolò Guicciardini, *The Development of Newtonian Calculus in Britain, 1700–1800* (Cambridge University Press, 1989).

and regional scientific academies, of which the most important were the academies of Paris, Berlin, and St. Petersburg. Roger Hahn has noted that the eighteenth-century academy allowed "the coupling of relative doctrinal freedom on scientific questions with their rigorous evaluations by professional peers," an important characteristic of modern professional science.[2] The academic system tended to promote a strongly individualistic approach to research. A determined individual such as Euler or Lagrange could emphasize a given program of research through his own work, the publications of the academy, and the setting of the prize competitions.

Although the academy as a social institution was inherently centralized and elitist, the writings of the academicians were more discursive, expository, and inclusive than would be the case in the specialized research journals of later science. The democratization of science that occurred in the nineteenth century, with the opening of scientific careers to a wide segment of society, was accompanied intellectually within each field by a rather narrow and proprietary specialization that was foreign to the spirit of inquiry in the age of Enlightenment. In comparing Euler's writings with those of a hundred or a hundred and fifty years later one is struck by the change in the way in which the audience is conceived from, in the first case, anyone in principle who is curious about mathematics to, in the second, a group of specialists who have already undergone considerable initiation and concerning whose knowledge many assumptions may be tacitly accepted.

This essay is devoted to major developmental trends in advanced theoretical mathematics during the eighteenth century. It is important nevertheless to call attention to the spread of mathematical methods and mentalities in a range of more practical subjects and pursuits. In navigation, experimental physics, engineering, botany, demography, government, and insurance there was an increasing emphasis on quantification and rational method. In the burgeoning industrial arts, instrument-makers achieved new levels of precision measurement. In French engineering schools, sophisticated mathematics – including the calculus – was introduced for the first time into the teaching curriculum, a practice that would be widely followed in later education. The operational, algebraic character of advanced theoretical analysis was reflected at a wider level in a pronounced instrumentalist understanding of the uses and nature of mathematics. In an overview of Enlightenment quantitative science John Heilbron writes as follows:

> [In the later eighteenth century] analysis and algebra, which, in contrast to geometry had an instrumentalist bias, became the exemplar of the mathematical method. . . . This instrumentalism was a key ingredient of the quantifying spirit after 1760. . . . Most of the leading proponents of the Standard

[2] Roger Hahn, *The Anatomy of a Scientific Institution: The Paris Academy of Science, 1666–1803* (Berkeley: University of California Press, 1971), p. 313.

> Model [i.e. Laplacian molecular physics] . . . made clear that they understood it in an instrumentalist sense. . . . They found themselves in agreement with the epistemologies of Hume and Kant, and perhaps also with Condillac's teaching that clear and simple language, not intuitions of truth, conduces to the advancement of science.[3]

The rational "quantifying spirit" of the Enlightenment would find a lasting and pervasive legacy in the adoption at the end of the century in France of the metric system, a development that took place under the direct supervision of prominent mathematical scientists of the time.[4]

THE CENTURY OF ANALYSIS

Euler and Lagrange were leading and representative practitioners of analytical mathematics in the eighteenth century. Together they dominated the subject from 1740 until early into the next century. Their writings, and more particularly their extensive contributions to analysis, defined advanced mathematical activity. What is fundamental to an understanding of the intellectual fabric of mathematics of the period is the distinctive conception of algebraic analysis that guided their work. They conceived of the metaphysics of the calculus in a way that is significantly different from our outlook today. Although we tend to take the modern foundation for granted, the older approach of algebraic analysis was based on a different point of view, a different conception of how generality is achieved in mathematics, and a rather different understanding of the relationship of analysis to geometry and physics. The interest of the eighteenth-century work lies in considerable part in providing an example of an alternative conceptual framework, one with great historical integrity and cohesion.[5]

LEONHARD EULER

Euler became established as a mathematician of note during the decade of the 1730s. He was a young man in his twenties, a member of the St. Petersburg

[3] Heilbron's remarks are contained in his introduction to the volume, T. Frängsmyr et al. (eds.), *The Quantifying Spirit in the 18th Century* (Berkeley: University of California Press, 1990), pp. 3, 5.

[4] For studies of quantitative applied science in the eighteenth century, see H. Gray Funkhouser, "Historical Development of the Graphical Representation of Statistical Data," *Osiris,* 3 (1937), 269–404, and Laura Tilling, "Early Experimental Graphs," *British Journal for the History of Science,* 8 (1975), 193–213.

[5] Although the emphasis of the present essay is on calculus-related parts of mathematics, a concern for symbolic methods was also evident in such subjects as the theory of equations and number theory. See L. Novy, *Origins of Modern Algebra* (Leiden: Noordhoff International Publishing, 1973). Progress in formal mathematics was evident in probability and statistics; see Stephen M. Stigler, *The History of Statistics: The Measurement of Uncertainty before 1900* (Cambridge, MA: Harvard University Press, 1986).

Academy of Sciences and a colleague of Hermann, Daniel Bernoulli, and Christian Goldbach. Euler's interest in analysis is evident in writings from this period, including his major treatise of 1736 on particle dynamics, *Mechanica sive Motus Scientia Analytice Exposita*. Although the theme of analysis was well established at the time, there was in his work something new: the beginning of an explicit awareness of the distinction between analytical and geometrical methods and an emphasis on the desirability of the former in proving theorems of the calculus.

Euler's program of analysis would be launched in a series of comprehensive treatises on different branches of the calculus and celestial dynamics published between 1744 and 1766. During this period he was mathematics director of the Berlin Academy of Sciences. His capacity for calculation and tremendous output later led François Arago to confer on him the title of "Analysis Incarnate." In the last part of his career Euler returned to St. Petersburg where he continued to carry out research and to publish. In 1735 Euler lost the sight of his right eye, and shortly after his arrival in St. Petersburg, he lost the sight of his remaining eye. Despite working in conditions of near blindness he was able with the assistance of his family and servants to remain productive mathematically up to his death in 1783.

GRAPHICAL METHODS AND THE FUNCTION CONCEPT

The geometrical curve was an object of intensive mathematical and physical interest throughout the seventeenth and early eighteenth centuries. The study of the relations that subsist between the lengths of plane curves gave rise in 1718 in the writings of Count C. G. Fagnano to a theory of elliptic integrals. In the calculus of variations, a branch of mathematics pioneered by Jakob and Johann Bernoulli, classes of curves constituted the primary object of study; the goal of each problem was the selection of a curve from among a class of curves that rendered a given integral quantity a maximum or minimum. In analytical dynamics attention was concentrated on determining the relation between trajectories of particles moving in space and the forces that act on them. In the theory of elasticity, researchers studied the shape of static equilibrium assumed by an elastic lamina under various loadings, as well as the configurations of a vibrating string.

The curve also played a fundamental and very different role in the conceptual foundation of the calculus. By representing the relationship between two related variable magnitudes of a problem by means of a graphical curve the various mathematical methods that had been developed for the geometrical analysis of curves could be brought to bear on the problem. Graphical procedures had been employed by Galileo Galilei in his *Discorsi* of 1637 to relate the speed of a falling body to the time of its descent. They had become common in mathematical treatises by the late seventeenth century. Christiaan Huygens in his *Horologium Oscillatorium* (1673) and Isaac Barrow in his *Lec-*

tiones Geometricae (1670) represented quadrature relationships in this way. In his very first published paper in the calculus Gottfried Leibniz (1684) derived the optical law of refraction from the principle that light follows the path of least time. He considered two related magnitudes: the distance of the point of contact of the light ray along the interface, and the time of transit that corresponds to this distance. He represented this relationship graphically by means of a curve and proceeded to apply the differential algorithm, introduced earlier in the paper for the analysis of curves, to obtain the desired law. In his *Principia Mathematica* (1687) Isaac Newton investigated the inverse problem of central-force particle motion. In Propositions XXXIX and XLI of Book One he graphed the force as a function of the projection of position on the orbital axis and analyzed the resulting curve to arrive at expressions for the particle's trajectory. Jakob Bernoulli employed graphical methods throughout his researches of the 1690s. In his study of the elastica, the relation between the restoring force and the distance along the lamina was superimposed in graphical form on the diagram of the actual physical system.

Graphical methods played a role in the early calculus that would later be filled by the function concept. This point of view was formalized to some extent by Pierre Varignon in a 1706 memoir devoted to the study of spiral curves given in terms of polar variables.[6] Varignon considered a fixed reference circle ABYA with center C (Figure 13.1). A "courbe génératrice" HHV is given; a point H on this curve is specified by the perpendicular ordinate GH, where G is a point on the axis xCX of the circle. The line CX is conceived as a ruler that rotates with center C in a clockwise direction tracing out a spiral OEZAEK. Consider a point E on the spiral. With center C draw the arc EG. Let c = the circumference of the reference circle ABYA, x = arc AMB, CA = a, CE = y, GH = z and AD = b a constant line. The arc x is defined by the proportion c:x = b:z. Varignon wrote what he called the "équation générale de spirals à l'infini" as cz = bx. By substituting the value for z given by the nature of the generating curve into this equation, the character of the spiral was revealed. Depending on whether the generating curve was a parabola, a hyperbola, a logarithm, a circle, and so on, the corresponding spiral was called parabolic, hyperbolic, logarithmic, circular, and so on.

In Varignon's paper the equation of the spiral was formulated a priori in terms of Cartesian coordinates in the associated "generating curve." The latter embodied in graphical form the functional relationship between the polar variables and acted as a standard model to which this relationship can be referred.

From the very beginning of his mathematical career in the 1730s, Euler

[6] Pierre Varignon, "Nouvelle formation de spirales beaucoup plus différentes entr'elles que tout ce qu'on peut imaginer d'autres courbes quelconques à l'infini; avec les touchantes, les quadratures, les déroulemens, & les longueurs de quelques-unes de ces spirales qu'on donne seulement ici pour éxemples de cette formation générale," *Histoire de l'Académie royale des sciences avec les mémoires de mathématique et de physique tirés des registres de cette Académie 1704* (Paris, 1706), pp. 69–131.

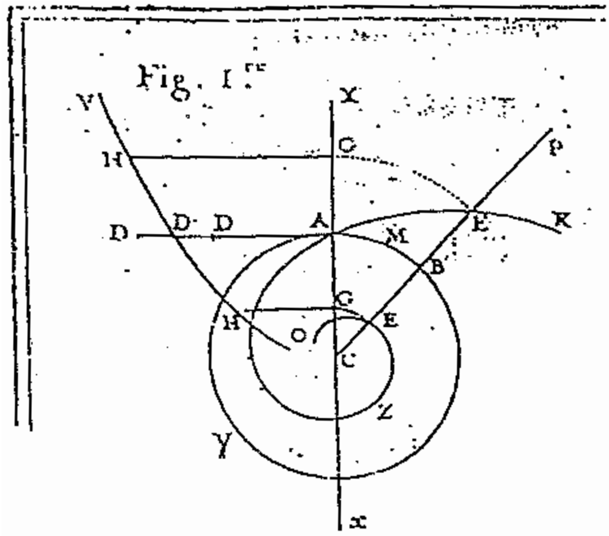

Figure 13.1. Varignon and the "Courbe généatrice."

imparted a new direction to the calculus by clearly emphasizing the importance of separating analysis from geometry. His program was evident in 1744 in his major treatise on the calculus of variations, *Methodus Inveniendi Lineas Curvas.*[7] A typical problem of the early calculus involved the determination of a magnitude associated in a specified way with a curve. To find the tangent to a curve at a point, it was necessary to determine the length of the subtangent there; to find the maximum or minimum of a curve, one needed to calculate the value of the abscissa that corresponded to an infinite subtangent; to find the area under a curve, it was necessary to calculate an integral; to determine the curvature at a point, one had to calculate the radius of curvature. The calculus of variations extended this paradigm to classes of curves.[8] In the fundamental problem of the *Methodus Inveniendi* it is required to select that curve from among a class of curves that makes a given magnitude expressing some property a maximum or minimum.

Near the beginning of his treatise (p. 13) Euler noted that a purely analytical interpretation of the theory is possible. Instead of seeking the curve that renders the given integral quantity an extremum, one seeks that "equation"

[7] Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solution problematis isoperimetrici lattisimo sensu accepti* (Lausanne, 1744). Reprinted in Euler's *Opera Omnia,* Ser. 1, V. 24.

[8] For historical studies of Euler's calculus of variations, see Herman H. Goldstine, *A History of the Calculus of Variations from the 17th through the 19th Century* (New York: Springer-Verlag, 1980), chap. 3; and Craig Fraser, "The Origins of Euler's Variational Calculus," *Archive for History of Exact Sciences,* 47 (1994), 103–41; and Fraser, "The Background to and Early Emergence of Euler's Analysis," in M. Otte and M. Panza (eds.), *Analysis and Synthesis in Mathematics History and Philosophy,* Boston Studies in the Philosophy of Science, vol. 196 (Dordrecht: Kluwer, 1997).

between x and y that, among all such equations, renders the quantity a maximum or minimum. He wrote, "In this way questions in the doctrine of curved lines may be referred back to pure analysis. Conversely, if questions of this type in pure analysis be proposed, they may be referred to and solved by means of the doctrine of curved lines."

Euler's derivation of the basic equations and principles of the calculus of variations was formulated in terms of the detailed study of the properties of geometrical curves. Nevertheless, in Chapter 4 of his book he showed that a purely analytical interpretation of the theory was possible. He observed that "the method presented earlier may be applied widely to the determination of equations between the coordinates of a curve which render any given expression $\int Zdx$ a maximum or a minimum. Indeed it may be extended to any two variables, whether they involve an arbitrary curve, or are considered purely in analytical abstraction." He illustrated this claim by solving several examples using variables other than the usual rectangular Cartesian coordinates. In the first example he employed polar coordinates to find the curve of shortest length between two points. He was completely comfortable with these coordinates; gone was the Cartesian "generating curve" that Varignon had employed in his investigation of 1706 to introduce general polar curves. In the second example Euler displayed a further level of abstraction, employing variables that were not even coordinate variables in the usual sense.

A range of non-Cartesian coordinate systems had been employed in earlier mathematics but never with the same theoretical import as in Euler's variational analysis. Here one had a fully developed mathematical process, centered on the consideration of a given analytically expressed magnitude, in which a general equational form was seen to be valid independent of the particular interpretation conferred upon the variables of the problem.

Euler had succeeded in showing that the basic subject matter of the calculus – what in some ultimate sense the calculus is "about" – could be conceived independently of geometry in terms of abstract relations between continuously variable magnitudes. To develop this point of view systematically it was necessary to introduce formal concepts and principles. To do this Euler turned to the concept of a function, a concept that had appeared in earlier eighteenth-century work and that he made central in his mid-century treatises on the calculus.[9] His *Introductio in Analysin Infinitorum* of 1748 contained an explicit definition: "A function of a variable quantity is an analytical expression composed in any way from the variable and from numbers or constant quantities" (p. 4). Although he sometimes considered a more

[9] Carl Boyer observes that for Euler "analysis was not the application of algebra to geometry; it was a subject in its own right – the study of variables and functions – and graphs were but visual aids in this connection. . . . It now dealt with continuous variability based on the function concept . . . only with Euler did it [analysis] take on the status of conscious program." (*History of analytic geometry;* originally published as Numbers Six and Seven of The Scripta Mathematica Studies; republished 1988 by The Scholar's Bookshelf; the quoted passage appears on page 190 of the latter edition.)

general notion of a function, for example in the discussion of the solution to the problem of the vibrating string, the *Introductio* furnished the operative fundamental definition for eighteenth-century work in analysis.[10]

A notable example of Euler's functional approach is provided by his introduction of the sine and cosine functions. Tables of chords had existed since Ptolemy in antiquity, and the relations between sines and cosines were commonly used in navigation and mathematical astronomy. With the advent of the calculus, trigonometric relations were expressed in terms of geometrical infinitesimal elements contained in a standard reference circle. Euler, by contrast, defined the sine and cosine functions as formulas involving variables that were given independently of geometrical constructions or dimensional considerations. He also derived the standard power series for the trigonometrical functions, using multiangle formulas and techniques he had employed earlier in the treatise to obtain the exponential series. Although these expansions were not new, they had been derived by analytical principles: a function that was a solution to a definite differential equation had been expanded to yield the given series.[11]

DIFFERENTIATION

In the original Leibnizian calculus, the concept of differentiation possessed a dual character: algebraic/algorithmic on the one hand, and geometric on the other. The algebra comprised a set of rules that governed the use of the symbol d and was based on two postulates: $d(x + y) = dx + dy$ and $d(xy) = ydx + xdy$. Accompanying these rules there was also an order principle, according to which higher-order differentials in a given equation were to be neglected with respect to differentials of a lower order.

The differentials that appeared in a given problem could also be understood in another way: as the differences of values of a variable quantity at successive points in the geometrical configuration. The differential dx was set equal to the difference of the value of x at two consecutive points infinitely close together; higher-order differentials were set equal to the difference of successive lower-order differentials. Euclidean geometry was used to analyze the properties of the curve in terms of these differentials.

A good illustration of the dual character of differentiation is provided by

[10] For studies dealing with the history of the function concept, see Ivor Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (Cambridge, MA: MIT Press, 1970); A. P. Youschkevitch, "The Concept of the Function up to the Middle of the 19th Century," *Archive for History of Exact Sciences,* 16 (1976), 37–85; and Steven Engelsman, "D'Alembert et les Équations aux Dérivées Partielles," *Dix-Huitième Siècle,* 16 (1984), 27–37. In the secondary literature there has tended to be something of a historiographical divide. Authors such as Truesdell, Demidov, and Youschkevitch have emphasized Euler's modernism, whereas Grattan-Guinness and Fraser have in a less Whiggish vein called attention to historically particular features of his thought.

[11] For a historical account, see Victor J. Katz, "Calculus of the Trigonometric Functions," *Historia Mathematica,* 14 (1987), 311–24.

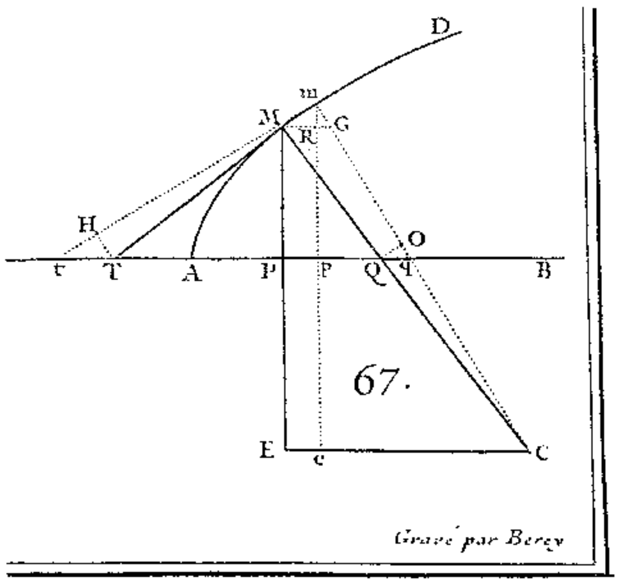

Figure 13.2. L'Hôpital and the center of curvature.

the derivations of the formula for the radius of curvature of a curve given by the Marquis de l'Hôpital in his textbook *Analyse des infiniment petits, pour l'intelligence des lignes courbes* (1696; second edition 1716). This formula was used in analytic geometry to calculate the evolute to a curve, that is the locus formed by the center of the radius of curvature. In mechanics it was known that the restoring force on an element of a stretched elastic string is proportional to the curvature (the reciprocal of the radius of curvature) of the string at the point where the element is located. The expression for the radius could be used to derive a differential equation to describe the string's motion.[12]

The first derivation of the formula that we shall consider was taken by l'Hôpital from a textbook published by Johann Bernoulli in 1691. Assume M is any point on the curve AMD (Figure 13.2). Let m be a point on the curve infinitely close to M. The normals to the curve at M and m intersect at the center of curvature C. The distance MC is the radius of curvature. Suppose AP = x and PM = y are the abscissa and ordinate of M. The lines MR and Rm parallel to AP and PM are the infinitesimal increments dx and dy of x and y. L'Hôpital calculated that PQ = ydy / dx. Let Q and q be the intersections of the normals MC and mC and the axis of the abscissae. L'Hôpital

[12] A detailed historical account of the theory of differentials from Leibniz to Euler, including a description of the calculation of the radius of curvature, is contained in Henk J. Bos, "Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus," *Archive for History of Exact Sciences,* (1974), 1–90.

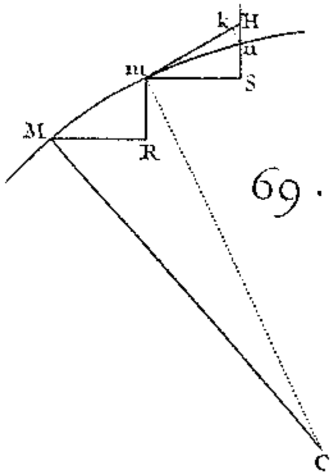

Figure 13.3. L'Hôpital and second-order differentials.

supposed that the quantity dx is constant, a step that corresponds from a modern perspective to the assumption that x is the independent variable in the problem. Since Qq = d(AQ) = dx + d(PQ), he applied the differential algorithm and obtained the expression $Qq = dx + (dy^2 + yddy) / dx$. Using similar triangles he proceeded to calculate the radius MC and obtained the formula

$$MC = \frac{(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{-dxddy} \qquad (1)$$

In a subsequent derivation l'Hôpital employed a different procedure, calculating the second differentials directly in terms of the elements of the geometrical configuration. Consider again the portion of the curve AMD containing Mm (Figure 13.3). Let n be a point on the curve infinitely close to m. L'Hôpital conceived of the portion Mmn as composed of the polygonal segments Mm and mn. The second differential of y, ddy, is given as ddy = nS − mR = nS − HS = −Hn. By means of similar triangles he arrived at an estimate for the radius of curvature that reduced to formula (1).

Another illustration of the dual character of differentiation is provided in mathematical dynamics in the calculation of differential equations of motion connecting the force to the spatial coordinates of a moving particle. The usual procedure during the period involved the comparison of the dynamical system at three successive instants in time. The second differentials appearing in the equations of motion were calculated in terms of the second differences arising in these configurations. In the 1740s and the 1750s, in the writings of Euler and d'Alembert, the second differentials were calculated directly in terms of the differential algorithmic procedures of the calculus.[13] This method, as-

[13] Both methods of calculating second differentials were employed by d'Alembert in his *Traité de Dy-*

sociated today with the differential-equation form of Newton's second law, soon became standard in classical mechanics.

In his mid-century treatises Euler, as part of his program of separating analysis from geometry, made the algebraic conception of differentiation fundamental. In so doing he made the concept of the *algorithm* primary in his understanding of the foundations of the calculus. Some of the issues that arise in this shift in viewpoint are illustrated by his theory of differential expressions set forth in Chapters 8 and 9 of the first part of his 1755 *Institutiones Calculi.* Consider any formula containing dx, ddx, dy, ddy, ... Because these quantities are no longer interpreted geometrically the meaning of the formula is unclear; its value will depend on whether dx or dy is held constant, an assumption that is not evident in the algebra. For example, the quantity ddy / dx^2 is zero if dy is constant; if dx is constant its value will vary according to the functional relation between x and y. Conversely, certain expressions, such as $(dyddx - dxddy) / dx^3$, may be shown to be invariant regardless of which variable is taken to be independent.

Euler's solution to the problem of indeterminacy in differential expressions was to introduce notation that made clear the relations of dependency among the variables. He did so by eliminating higher-order differentials as such, replacing them instead with differential coefficients. Rather than write ddy / dx^2 (dx constant) we define the differential coefficients p and q by the relations $dy = pdx$ and $dp = qdx$; ddy / dx^2 then becomes simply q. Euler provided rules and examples that showed how more complicated expressions can be reduced to ones containing only variables and differential coefficients. In addition to bringing order to the calculus, this emphasis on the differential coefficient was conceptually important in identifying the derivative as an independent object of mathematical study.[14]

INTEGRATION

Leibniz had regarded the integral as a kind of infinite summation carried out with reference to a sequence of values of one of the variables of the problem. He denoted integration using an elongated "S," which stood for the first letter of the Latin word "summa" for sum. Thus, the area under the curve $y = x^2$ was expressed as $\int x^2 dx$, where the limits of integration were understood to be given.

A significant modification of Leibniz's conception was introduced in the early 1690s by Johann Bernoulli, who replaced the concept of an integral as a sum with the quite different concept of the integral as an antiderivative.[15]

namique of 1744. See Craig Fraser, "D'Alembert's Principle: The Original Formulation and Application in Jean D'Alembert's *Traité de Dynamique* (1743)," *Centaurus,* 28 (1985), 31–61, 145–59.

[14] For a more detailed description of Euler's theory, see Bos, "Differentials, Higher-Order Differentials."

[15] Bernoulli's definition was contained in his *Die erste Integralrechnung,* a selection of his writings from

Taking the d-operation as logically primary, Bernoulli defined integration as the operational inverse of differentiation. The integral $\int x^2 dx$ was by definition equal to $x^3 / 3$, because the differential of the latter expression is equal to $x^2 dx$.

In his mid-century writings on analysis Euler adopted Johann Bernoulli's notion of the integral as an antiderivative, a point of view that Euler made fundamental in his two-volume *Institutiones Calculi Integralis* of 1768. It is clear that Euler held to this conception from a very early stage of his career. In the 1730s he had investigated the problem of determining orthogonal trajectories to families of curves, a subject that had been broached by Leibniz forty years earlier.[16] The latter had considered integrands consisting of expressions involving both a variable x and a parameter t. Leibniz showed that the partial derivative with respect to t of the integral is equal to the integral of the partial derivative of the expression itself with respect to t:

$$\frac{\partial}{\partial t}\int f(x, t)\,dx = \int \frac{\partial}{\partial t} f(x, t)\,dx \qquad (2)$$

To establish this result, known in modern calculus as Leibniz's rule, Leibniz used the fact that the differential of a sum of infinitesimal elements is equal to the sum of the differentials of each of the elements. In his studies of orthogonal trajectories Euler provided a quite different proof of the same result, a proof that rested on his understanding of the integral as an antiderivative.[17] To carry out the derivation Euler first established a preliminary theorem, showing that if f is a function of the two variables x and t then the second partial derivative of f is independent of the order of differentiation:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial x}\frac{\partial}{\partial t} f(x, t) \qquad (3)$$

With this result and his definition of the integral as an antiderivative Euler was able to deduce Leibniz's rule directly:

$$\begin{aligned}\frac{\partial}{\partial t}\int f(x, t)\,dx &= \int \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\int f(x, t)\,dx\right)dx = \\ &= \int \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\int f(x, t)\,dx\right)dx = \int \frac{\partial}{\partial t} f(x, t)\,dx\end{aligned} \qquad (4)$$

the years 1691 and 1692 published in 1914, p. 3. See Carl Boyer, *A History of the Calculus and Its Conceptual Development* (New York: Dover Publications, Inc., 1959; originally published by Hafner Publishing Company in 1949 under the title "The Concepts of the Calculus, A Critical and Historical Discussion of the Derivative and the Integral"), pp. 278–9.

16 For a historical survey of this subject, see Steven B. Engelsman, *Families of Curves and the Origins of Partial Differentiation* (Amsterdam: North-Holland, 1984).

17 The derivation is contained in Euler, "De infinitis curvis eiusdem generis seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis," *Commentarii Academiae Scientiarum Petropolitanae 7 1734–1735* (1740), 174–89, 180–9. (Pages 190–9 were incorrectly numbered as 180–9.) In Euler's *Opera* Ser. 1, V. 22, pp. 36–56.

In his later writings Euler followed the pattern here, obtaining first equation (3) and then proceeding to derive Leibniz's rule; the proof rested at base on Euler's concept of the integral as an antiderivative. In his *Institutiones Calculi Integralis* he expounded in some detail on his operational understanding of the integral. Integration understood as the inverse of differentiation was analogous to subtraction as the inverse of addition, division as the inverse of multiplication, and the taking of roots as the inverse of the taking of powers. When it is not possible to express the inverse of a given expression Xdx in terms of known algebraic functions, then it follows that the resulting integral must be transcendental. The situation is analogous to the one with respect to the three inverse algebraic operations. When subtraction leads to numbers that are not positive then we arrive at negative numbers; when division results in nonintegral numbers we arrive at fractions; when the taking of roots leads to nonintegral numbers then we arrive at radicals.

The definition of integration as the operational inverse of differentiation was widely adopted in late eighteenth-century mathematics. By taking integrals one obtained new functional objects, and by applying functional inversion to these objects one obtained a further class of functions. The domain of analysis was thereby enlarged greatly. In an early memoir on elliptic integrals, Lagrange had observed that the investigation of the integrability of rational polynomials opened "a vast field to the researches of the analysts."[18] It should be noted that in this conception a given transcendental integral and its various properties were understood to be a consequence of the algebraic nature of the differential process. In particular, the various considerations of existence that are so fundamental in modern theories of integration did not arise at all.

THEOREMS OF ANALYSIS

A fundamental difference between eighteenth-century and modern analysis is the absence in the former of what is known today as the mean value theorem or the law of the mean. This result, a basic part of the classical arithmetic foundation of the calculus, is used in theorem-proving to localize a given property or relation at a definite value of the numerical continuum. The proposition is established by showing its validity at each value of this continuum.

Euler's viewpoint was quite different. A relation between variables was regarded by him as a primitive of the theory; it was not further conceptualized in terms of the numerical continuum of values assumed by each variable. This notion of a primitive abstract relation in large part defined his approach to

[18] Lagrange, "Sur l'intégration de quelques équations différentielles dont les indéterminées sont séparées, mais dont chaque membre en particulier n'est point intégrable," *Miscellanea Taurinensia,* 4; in Lagrange's *Oeuvres de Lagrange* 2, pp. 5–33. The quote is on p. 33 of *Oeuvres* 2.

analysis, distinguishing his point of view both from that of the early pioneers, who made the geometrical curve the basic object of study, and that of the nineteenth-century researchers, for whom the numerical continuum constituted a fundamental object of study.

Euler's proof in 1740 of (3), the theorem on the equality of mixed partial differentials, was analytical in a formal, nongeometrical sense. He was motivated to develop such a proof by a belief that a geometrical demonstration would be "drawn from an alien source."[19] He considered a quantity z that is a function of the variables x and a. He expressed the relevant differentials in terms of differential coefficients and showed by a suitable rearrangement of terms that the two partial differentials are equal. In modern real-variable analysis, Euler's argument is reformulated using the law of the mean and a limit argument. Suppose $z = z(x,a)$ and its first and second partial derivatives are defined and continuous on a rectangular region in the x-a plane. The law of the mean is used to obtain expressions for the relevant partial derivatives, which by rearrangement and a limit argument are shown to be equal.[20]

This example is typical of eighteenth-century calculus theorems and their counterparts in modern analysis. (Other examples are the fundamental theorem of the calculus, the theorem on the change of variables in multiple integrals, and the fundamental lemma of the calculus of variations.) The law of the mean introduces a distinguished value, localizing at a particular number the analytical relation or property in question. The result is then deduced using conditions of continuity and differentiability by means of a limit argument. In Euler's formulation, by contrast, there was no consideration of distinguished or individual values as such. Euler believed that the essential element in the demonstration was its generality, which was guaranteed by a formal analytical or algebraic identity. Thus, the key step in his proof rested on an algebraic identity that ensured the validity of the result.

ANALYTICAL PHILOSOPHY

Although the leading analysts of the eighteenth century did not formulate an explicit mathematical philosophy, implicit philosophical attitudes were evident in their handling of issues such as generality and the relationship of pure and applied mathematics. For Enlightenment mathematicians, each part of mathematics was understood to be given in some objective sense; its range of application and certainty derived from this objective nature and were not consequences of the particular method or set of concepts adopted by the mathematician. The generality of mathematics was a consequence of the general

[19] See Engelsman, *Families of Curves,* p. 129.

[20] Euler's derivation is studied in more detail in Craig Fraser, "The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century," *Archive for History of Exact Sciences,* 39 (1989), 317–35, especially 319–21.

character of its objects, whether these be formulas of algebra or diagrams of geometry.

In the writings of the analysts the original problem of the calculus – to describe change along a curve – gave way to the study of formulas and relations. An analytic equation implied the existence of a relation that remained valid as the variables changed continuously in magnitude. Analytic algorithms and transformations presupposed a correspondence between local and global change, the basic consideration in the application of the calculus to the curve. The rules and procedures of the calculus were assumed to be generally valid. In a memoir published in 1751 Euler considered the rule d(logx) = dx/x.[21] He rejected an earlier suggestion of Leibniz that this rule was only valid for positive real values of x with the following observation:

> For, as this [differential] calculus concerns variable quantities, that is, quantities considered in general, if it were not generally true that d.lx = dx / x, whatever value we give to x, either positive, negative or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains. (pp. 143–4)

Eighteenth-century confidence in formal mathematics was almost unlimited. One historian has noted, "Sometimes it seems to have been assumed that if one could just write down something which was symbolically coherent, the truth of the statement was guaranteed," and another has commented on Euler's "naive faith in the infallibility of formulas and the results of manipulations upon them."[22] Functionality and operational efficacy were valued over deduction and logical verification. A belief in symbolic methods was supported by more general philosophical thinking about exact science. The writings of Nicolas Malebranche and his school had stressed the value of an arithmetical/algebraic approach to mathematics. Somewhat later, Étienne Condillac emphasized the importance of a well-constructed language in rational investigation, and he cited algebra as the paradigm of what could be achieved in this direction.[23]

That the problems of geometry and mechanics should conform to treatment by pure analysis was something that eighteenth-century authors accepted as a matter of philosophical principle. Sergei Demidov, writing of the failure

[21] Euler, "De la controverse entre Mrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires, *Mémoires de l'académie des sciences de Berlin 5 (1749),* (1751), 139–171; in the *Opera Omnia* Ser. 1, V. 17, 195–232.

[22] See Judith V. Grabiner, "Is Mathematical Truth Time-Dependent?" *American Mathematical Monthly,* 81 (1974), 354–65, especially 356, and Rudolph E. Langer, "Fourier Series, The Evolution and Genesis of a Theory," *American Mathematical Monthly,* 54, pt. 2 (1947), 1–86, especially 17.

[23] Malebranche's mathematical philosophy is discussed in Craig Fraser, "Lagrange's Analytical Mathematics, Its Cartesian Origins and Reception in Comte's Positive Philosophy," *Studies in the History and Philosophy of Science,* 21 (1990), 243–56. An account of Condillac's thought is contained in Robert McRae, "Condillac: The Abridgement of All Knowledge in 'The Same is the Same,'" in *The Problem of the Unity of the Sciences: Bacon to Kant* (Toronto: University of Toronto Press, 1961), pp. 89–106.

of Euler and d'Alembert to understand each other's point of view in the discussion of the wave equation, observes:

> A cause no less important of this incomprehension rests, in our opinion, on the understanding of the notion of a solution of a mathematical problem. For d'Alembert as for Euler the notion of such a solution does not depend on the way in which it is defined . . . rather the solution represents a certain reality endowed with properties that are independent of the method of defining the solution. To reveal these properties diverse methods are acceptable, including the physical reasonings employed by d'Alembert and Euler.[24]

A biographer of d'Alembert has noted his insistence on "the elementary truth that the scientist must always accept the essential 'giveness' of the situation in which he finds himself."[25] The sense of logical freedom that developed in later mathematics – expressed, for example, in Richard Dedekind's famous statement of 1888 that numbers are free creations of the human mind and the belief that the essence of mathematics consists in its autonomous conceptual development – reflects aspects of the modern subject that were quite absent in the eighteenth century.

JOSEPH LOUIS LAGRANGE

Lagrange's professional career was an exceptionally long one, spanning from 1754, when he was eighteen, to his death in 1813. From his birth in 1736 until 1766 he lived in Turin, participating in the founding of the Turin Society in 1757 and then becoming one of its active members; from 1766 to 1787 he was mathematics director of the Berlin Academy of Sciences; from 1787 to his death he lived in France as a pensionnaire of the Paris Academy of Sciences.

Although Lagrange's analytical tendencies were apparent from the very beginning of his career, his distinctive mathematical style became consolidated only in the period 1770 to 1776, when he was in his late thirties and comfortably settled at the Berlin Academy. In these years the value of analysis became an explicit theme in his writings for the Academy on a range of subjects in pure and applied mathematics.[26] In a memoir of 1771 on Kepler's problem he distinguished three approaches to its solution: one involving numerical approximation, a second using geometrical or mechanical constructions, and a third that is algebraic, employing analytical expressions. The last he cited for its "continual and indispensable use in the theory of celestial bodies." In a paper the next year on the tautochrone, a problem first investigated geo-

[24] Sergei Demidov, "Création et développement de la théorie des équations différentielles aux dérivées partielles dans les travaux de J. d'Alembert," *Revue d'Histoire des Sciences* 35/1 (1982), 3–42, especially 37.
[25] R. G. Grimsley, *Jean d'Alembert (1717–1783)* (Oxford: Clarendon Press, 1963), p. 248.
[26] For references to the publications of Lagrange cited in this section, see René Taton, "Inventaire chronologique de l'oeuvre de Lagrange," *Revue d'Histoire des Sciences,* 26 (1974), 3–36.

metrically by Huygens, Lagrange took as a starting point "analytical solutions" that had been advanced by Johann Bernoulli and Euler. In 1775 several memoirs appeared in which the value of analysis is promoted. In his paper on the attraction of a spheroid Lagrange attempted to show that the method of "algebraic analysis" provides a more direct and general solution than the "synthetic" or geometrical approach followed by Maclaurin. (This appears, incidentally, to be the first explicit appearance in his writing of the term "algebraic analysis.") In his study of the rotation of a solid Lagrange advanced an alternative to the mechanical treatment of d'Alembert and Euler, one that was "purely analytic," whose merit consisted "solely in the analysis" that it employed, and which contained "different rather remarkable artifices of calculation." In a memoir on triangular pyramids Lagrange noted that his "solutions are purely analytic and can even be understood without figures"; he observed that independent of their actual utility they "show with how much facility and success the algebraic method can be employed in questions that would seem to lie deepest within the province of Geometry properly considered, and to be the least susceptible to treatment by calculation."

The theme of analysis recurs in Lagrange's writings of the late 1770s and 1780s. In a 1777 study of cubic equations he described a method due to Thomas Harriot that avoided the geometrical constructions that had been used by mathematicians to investigate expressions for roots. In a memoir submitted to the Paris Academy in 1778 on the subject of planetary perturbations Lagrange offered a method for transforming the equations of motion that would "take the place of the synthetic methods proposed until now for simplifying the calculation of perturbations in regions beyond the orbit" and that "has at the same time the advantage of conserving uniformity in the march of the calculus." In 1780 he published a memoir on a theorem of Johann Lambert's in particle dynamics. The result in question had been demonstrated synthetically, and Lagrange expressed concern that it might be regarded as one of "the small number [of theorems] in which geometric analysis seems to be superior to algebraic analysis." His purpose was to present a simple and direct analytical proof. In a study in 1781 of projection maps he offered a "research, equally interesting for the analytic artifices that it requires as well as for its utility in the perfection of geographical maps." In the preface to his famous *Traité de la méchanique analitique,* completed around 1783, he announced that in it "no figures would be found," that all would be "reduced to the uniform and general progress of analysis." In a memoir of 1788 he discussed successes and difficulties in treating analytically the various subjects of Newton's *Principia Mathematica* and offered a new analysis of the problem of the propagation of sound.

Directness, uniformity, and generality were qualities that Lagrange associated with analysis; he sometimes also mentioned simplicity. Analysis was cited not simply for the results to which it led but also for the methods that it offered. In the writings discussed earlier he was affirming the value of analysis in situations in which an alternative geometrical or mechanical treatment existed; it was the possibility of this alternative that led him to explicitly assert his own methodological preferences. One should also note the sheer preponderance of pure analysis in his work of the 1770s and 1780s in such topics as the theory of equations, diophantine arithmetic, number theory, probability, and the calculus, subjects in which explicit questions of approach or methodology did not arise.

THEORY OF ANALYTICAL FUNCTIONS

By the end of the century a more critical attitude began to develop both within mathematics and within general intellectual culture. As early as 1734 Bishop George Berkeley in his work *The Analyst* had called attention to what he perceived as logical weaknesses in the reasonings of the calculus arising from the employment of infinitely small quantities. Although his critique was somewhat lacking in mathematical cogency, it stimulated writers in Britain and the Continent to explain more carefully the basic rules of the calculus. In the 1780s a growing interest in the foundations of analysis was reflected in the decision of the academies of Berlin and St. Petersburg to devote prize competitions to the metaphysics of the calculus and the nature of the infinite. In philosophy, Immanuel Kant's *Kritik der reinen Vernunft* (1788) set forth a penetrating study of mathematical knowledge and initiated a new critical conceptual movement in the exact sciences.

The most detailed attempt to provide a systematic foundation of the calculus was contained in two treatises by Lagrange published at the end of the century: the *Théorie des fonctions analytiques* (1797) and *Leçons sur le calcul des fonctions* (1801; rev. ed. 1806). The full title of the first work explains its purpose: "Theory of analytical functions containing the principles of the differential calculus disengaged from all consideration of infinitesimals, vanishing limits or fluxions and reduced to the algebraic analysis of finite quantities." Lagrange's goal was to develop an algebraic basis for the calculus that made no reference to infinitely small magnitudes or intuitive geometrical notions. In a treatise on numerical equations published in 1798 he set forth clearly his conception of algebra:

> [Algebra's] object is not to find particular values of the quantities that are sought, but the system of operations to be performed on the given quantities in order to derive from them the values of the quantities that are sought. The tableau of these operations represented by algebraic characters is what in algebra is called a *formula,* and when one quantity depends on other quantities, in such a way that it can be expressed by a formula which contains these quantities, we say then that it is a *function* of these same quantities.[27]

[27] Lagrange, *Traité de la résolution des équations numériques de tous les degrés* (Paris, 1798). The second edition was published in 1808 and was reprinted as Lagrange's *Oeuvres* 8. The quoted passage appears on pp. 14–15 of the latter volume.

Lagrange used the term "algebraic analysis" to designate the part of mathematics that results when algebra is enlarged to include calculus-related methods and functions. The central object here was the concept of an analytical function. Such a function y = f(x) is given by a single analytical expression that is constructed from variables and constants using the operations of analysis. The relation between y and x is indicated by the series of operations schematized in f(x). The latter possesses a well-defined, unchanging algebraic form that distinguishes it from other functions and determines its properties.

The idea behind Lagrange's theory was to take any function f(x) and expand it in a Taylor power series:

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + si^4 + \cdots \quad (5)$$

The "derived function" or derivative f′(x) of f(x) is defined to be the coefficient p(x) of the linear term in this expansion. f′(x) is a new function of x with a well-defined algebraic form; it is different from but related to the form of the original function f(x). Note that this conception is very different from that of the modern calculus, in which the derivative of f(x) is defined at each real value of x by a limit process. In the modern calculus the relationship of the derivative to its parent function is specified in terms of correspondences that are defined in a definite way on the numerical continuum.

Lagrange's understanding of derived functions was revealed in his discussion in the eighteenth lesson of the method of finite increments. This method was of historical interest in the background to his program. Brook Taylor's original derivation in 1715 of Taylor's theorem was based on a passage to the limit of an interpolation formula involving finite increments. Lagrange wished to distinguish clearly between an approach to the foundation of the calculus that uses finite increments and his own quite different theory of derived functions. In taking finite increments, he noted, one considers the difference $f(x_{n+1}) - f(x_n)$ of the same function f(x) at two successive values of the independent argument. In the differential calculus the object Lagrange referred to as the derived function was traditionally obtained by letting $dx = x_{n+1} - x_n$ be infinitesimal, setting $dy = f(x_{n+1}) - f(x_n)$, dividing dy by dx, and neglecting infinitesimal quantities in the resulting reduced expression for dy/dx. Although this process leads to the same result as Lagrange's theory, the connection it presumes between the method of finite increments and the calculus obscures a more fundamental difference between these subjects: in taking $\Delta y = f(x_{n+1}) - f(x_n)$ we are dealing with one and the same function f(x); in taking the derived function we are passing to a new function f′(x) with a new algebraic form. Lagrange explained this point as follows:

> The passage from the finite to the infinite requires always a sort of leap, more or less forced, which breaks the law of continuity and changes the form of functions. (*Leçons* 1806, p. 270)
>
> In the supposed passage from the finite to the infinitely small, functions actually change in nature, and . . . dy/dx, which is used in the differential

Calculus, is essentially a different function from the function y, whereas as long as the difference dx has any given value, as small as we may wish, this quantity is only the difference of two functions of the same form; from this we see that, if the passage from the finite to the infinitely small may be admitted as a mechanical means of calculation, it is unable to make known the nature of differential equations, which consists in the relations they furnish between primitive functions and their derivatives. (*Leçons* 1806, p. 279)

Lagrange's *Théorie* and *Leçons,* written when he was in his sixties, were notable for their success in developing the entire differential and integral calculus on the basis of the concept of an analytical function.[28] They contained several quite important technical advances. Lagrange introduced inequality methods to obtain numerical estimates of the values of functions, thereby providing a source of techniques that Augustin Cauchy was later able to use in his arithmetical development of the calculus. Another significant contribution was contained in Lagrange's exposition of the calculus of variations. To obtain the variational equations he modeled the derivation after an earlier argument in the theory of integrability. Although his derivation never quite achieved acceptance among later researchers, it remains historically noteworthy as an example of advanced reasoning in algebraic analysis. Lagrange also introduced the multiplier rule in both the calculus and the calculus of variations, a powerful method that allows one to solve a range of problems in the theory of constrained optimization.[29]

REFLECTIONS ON ALGEBRAIC ANALYSIS

It is important to appreciate the distinctive philosophical character of eighteenth-century algebraic analysis, understood within the larger historical and intellectual evolution of mathematical analysis. The algebraic calculus of Euler and Lagrange was rooted in the formal study of functional equations, algorithms, and operations on variables. The values that these variables received, their numerical or geometrical interpretation, was logically of secondary concern. Such a conception, strongly operational and instrumentalist in

[28] For a more detailed study of these works, see J. L. Ovaert, "La thèse de Lagrange et la transformation de l'analyse," in Christian Houzel et al. (eds.), *Philosophie et Calcul de l'Infini* (Paris: Francois Maspero, 1976), pp. 122–157; Judith V. Grabiner, *The Origins of Cauchy's Rigorous Calculus* (Cambridge, MA: MIT Press, 1981); and Craig Fraser, "Joseph Louis Lagrange's Algebraic Vision of the Calculus," *Historia Mathematica,* 14 (1987), 38–53. Grabiner and Fraser emphasize somewhat different aspects of the historiography. Grabiner calls attention to the origins of Cauchy's technical methods in Lagrange's writings and makes the concept of rigor central to understanding Cauchy's achievement. Fraser is concerned with highlighting the conceptual differences between the viewpoints of Lagrange and Cauchy and sees the latter's central accomplishment as having made the numerical continuum a fundamental object of concern.

[29] For a discussion of Lagrange's calculus of variations, see Craig Fraser, "J. L. Lagrange's Changing Approach to the Foundations of the Calculus of Variations," *Archive for History of Exact Sciences,* 32 (1985), 151–91, and Fraser, "Isoperimetric Problems in the Variational Calculus of Euler and Lagrange," *Historia Mathematica,* 19 (1992), 4–23.

character, should be contrasted with the geometrical approach of the early calculus, which relied heavily on diagrammatic representations and intuitions of spatial continuity. The geometrical emphasis of the early calculus conditioned how the subject was understood, allowing it to be experienced intellectually as an interpreted, meaningful body of mathematics.

Lagrange's algebraic analysis should also be contrasted with the much more conceptual and intensional mode of reasoning that was characteristic of classical real analysis, the field that developed in the nineteenth century and became the foundation of the modern subject. Although real analysis is logically independent of geometry, it continues to posit objects – defined using the concept of arithmetical continuity – that constitute its subject matter and define its point of view as a mathematical theory. A proposition about a function defined on some interval of real numbers under specified conditions of differentiability has a geometrical interpretation implicit in its very formulation. On a foundational level the algorithmic character of differentiation in real analysis is irrelevant to a conceptual understanding of this process; in algebraic analysis, by contrast, the notion of algorithm is fundamental to the whole approach.[30]

ROBERT WOODHOUSE AND GEORGE PEACOCK

The algebraic program of Enlightenment mathematics was taken up and extended by several English figures of the early nineteenth century.[31] Although these researches fall somewhat outside the period of this essay, they are worthy of note here as a direct continuation of what was primarily an eighteenth-century development. The appeal of algebraic analysis to the English was due in considerable part to a reaction against the prevalent geometric synthetic spirit of British mathematics. In his 1802 memoir "On the Independence of the analytical and geometrical Methods of Investigation; and on the Advantages to be derived from their Separation" Cambridge fellow Robert Woodhouse recommended the removal from analysis of all notation of geometrical

[30] For a more detailed discussion of this subject, see Fraser, "The Calculus as Algebraic Analysis," and Marco Panza, "Concept of Function, between Quantity and Form, in the 18th Century," in H. Niels Jahnke et al. (eds.), *History of Mathematics and Education: Ideas and Experiences* (Göttingen: Vandenhoeck & Ruprecht, 1996), pp. 241–74. For a social-intellectual study that deals with the place of algebraic analysis in nineteenth-century German mathematics, see H. Niels Jahnke, *Mathematik und Bildung in der Humboldtschen Reform,* volume 8 of the series *Studien zur Wissenschafts-, Sozial- und Bildungsgeschichte der Mathematik,*" eds. Michael Otte, Ivo Schneider, and Hans-Georg Steiner (Göttingen: Vandenhoeck & Ruprecht, 1990).

[31] For historical studies of this subject, see Joan L. Richards, "The Art and the Science of British Algebra: A Study in the Perception of Mathematical Truth," *Historia Mathematica,* 7 (1980), 343–65; Helena Pycior, "George Peacock and the British Origins of Symbolical Algebra," *Historia Mathematica,* 8 (1981), 23–45; and Menachem Fisch, "'The Emergency Which Has Arrived': The Problematic History of Nineteenth-Century British Algebra – a Programmatic Outline," *British Journal for the History of Science,* 27 (1994), 247–76.

origin. He urged, for example, that instead of writing sin x, a term whose etymology involved graphical associations, we employ the expression $(2\sqrt{-1})^{-1}(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}})$. He also began to move toward a more careful explanation of the symbols of formal analysis. Thus he wrote the following concerning the symbol "=":

> It is true that its signification entirely depends on definition; but, if the definition given of it in elementary treatises be adhered to, I believe it will be impossible to show the justness and legitimacy of most mathematical processes. It scarcely ever denotes numerical equality. In its general and extended meaning, it denotes the result of certain operations. (p. 103)

Woodhouse illustrated this point with the inverse sine series

$$z = x + \frac{x^3}{3 \cdot 2} + \frac{3x^5}{5 \cdot 8} + \cdots,$$

in which

> nothing is affirmed concerning a numerical equality; and all that is to be understood is, that
>
> $$z = x + \frac{x^3}{3 \cdot 2} + \frac{3x^5}{5 \cdot 8} + \textit{etc.},$$
>
> is the result of a certain operation performed [on the series for sin x]
>
> $$x = z - \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \textit{etc.}$$

Woodhouse's formal viewpoint was developed into a complete theoretical system by another Cambridge mathematician, George Peacock. In his "Report on the Recent Progress and Present State of certain Branches of Analysis," which was delivered to the British Association for the Advancement of Science in 1833, Peacock defined analytical science to include algebra, the application of algebra to geometry, the differential and integral calculus, and the theory of series. The first part of the report was devoted to an outline of his theory of algebra, which he based on something that he called the principle of the permanence of equivalent forms. An equivalent form is any relation that expresses the result of an operation of algebra: $(a + b)c = ac + bc$, $a^n \cdot a^m = a^{n+m}$, and so on. The principle of equivalent forms asserts as follows:

> Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as their form. (p. 199)

Because the relation $a^n \cdot a^m = a^{n+m}$ is an equivalent form when n and m are integers, it is by the principle also an equivalent form as a purely symbolic relation. Peacock regarded this fact as justification for extending the range of validity of $a^n \cdot a^m = a^{n+m}$ to non-integral values of n and m. In other branches of analysis – for example the theory of infinite series – the principle plays a similar role. Thus, because the relation $1 / (1 - x) = 1 + x + x^2 + x^3 + \ldots$ is valid for $x < 1$, it possesses by virtue of its form a general symbolical validity. The relation therefore remains valid, or at least meaningful, when $x > 1$, although in this case it is no longer interpretable in the usual sense in arithmetic.

The principle of equivalent forms is the formal statement of the idea contained in Euler's assertion of the universal validity of the relation $d(\log x) = dx / x$. In Peacock's system of analysis the principle had a dual purpose. It made legitimate the use of general symbolic relations and allowed one to assume an extended domain of validity for the variables contained in these relations. In addition, it ensured that the algebraic relations have at least a partial interpretation in arithmetic, and it thereby restricted the proliferation of purely abstract symbolical systems.

CONCLUSION

Eighteenth-century analysis achieved a theoretical completeness and sophistication not attained by other parts of mathematics. From a historiographical viewpoint, algebraic analysis provides an interesting example of a mature mathematical paradigm that would be replaced by a quite different paradigm in the later development of the subject. The transition from Euler and Lagrange to Cauchy and Weierstrass constituted a profound intellectual transformation in conceptual thought. The sort of relativism of viewpoint documented by Thomas Kuhn in the history of the physical sciences is also present in mathematics, albeit at a more purely conceptual level.[32] The case of mathematics is even in some important respects more striking, because the point of view embodied in the older paradigm retains a certain intellectual interest and validity not found in quite the same way in the discarded theories of older physics.

[32] For discussion of the relevance of Kuhn's ideas to mathematics, see Donald Gillies (ed.), *Revolutions in Mathematics* (New York: Oxford University Press, 1992).