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Book Review

Paolo Mancuso. *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. Oxford University Press, Oxford, 1996. viii + 275 pages.

Studies of the Scientific Revolution have been heavily weighted in favor of astronomy and physics, a fact often noted by historians of chemistry, biology, and medicine. Although mathematics was intimately connected with the former subjects, it too has been relatively neglected, at least in general surveys of the period. The seventeenth century was an exciting time in mathematics, as whole new subjects emerged and a fundamental shift in foundations occurred. Paolo Mancosu's book is an attempt to bridge the gap between the history of mathematics—where detailed specialist studies of these developments have been produced—and the history of philosophy, which despite the manifest importance of mathematics in early modern philosophy has tended to deal with this subject only in a rather general way.

There are really two parts to this book. In Chapters 1 and 4, Mancosu investigates a series of discussions initiated by Renaissance essentialist philosophers who were critical of the status of mathematics in traditional scholastic philosophy. The debate was framed in terms of Aristotle's conception of science as set forth in the *Posterior Analytics.* Writers in the sixteenth century such as Alessandro Piccolomini and Benedictus Pereyra argued that mathematics lacks the characteristics of a true science, that mathematical demonstrations are not causal in the sense required of a scientific syllogism. These thinkers were motivated by an opposition to the increasing mathematization of natural philosophy taking place at the time, although the arguments in question also appealed to later anti-Aristotelian skeptics such as Pierre Gassendi. One of the propositions they objected to was Euclid's *Elements* (I, 32) asserting that the sum of the angles of a triangle is two right angles. Their criticism was based not as one might suspect on concerns about the fifth postulate, but rather on what today seem plainly sterile considerations about the structure of the syllogism. Much of the fascination of this material derives from the fact that such matters could ever have been the subject of deep concern. Indeed, subsequent philosophers and mathematicians such as Guiseppe Biancani and Issac Barrow soundly rejected their arguments, although the debate did give rise to lasting criticism of two sorts of proof that appear in the *Elements,* proof by contradiction and proof by superposition.[1] The first type of proof was used not infrequently by Euclid, for example in (I, 6) to show that a triangle in which the two base angles are equal is isosceles, in (IX, 20) to show that there are infinitely many prime numbers, and in (XII, 2) in the method of exhaustion applied to the area of a circle. Superposition refers to the idea that figures may be moved around in space without distortion and was used by Euclid in (I, 4) to establish the basic side-angle-side congruency theorem for triangles.

One of Mancosu's main conclusions is that Aristotle's conception of scientific knowledge was invoked by both the detractors and supporters of mathematics, a situation in marked contrast to the new physics where a strong anti-Aristotelian bias prevailed. He also acknowledges that internal factors played a major role in the developing mathematics of the period, stating there was "a purely mathematical program . . . to overcome the intrinsic limitations of the Greek approach to geometry" (p. 93). In Chapter 4 he traces these issues, as well as the subject of genetic definition (i.e., definition involving generation through motion) in the work of Barrow and Antoine Arnauld in the 1660s. He continues on to Immanuel Kant one century later and proceeds to Bernhard Bolzano in the 1830s where he provides a comparative examination of the two authors' views on proof-by-contradiction in philosophy and mathematics. The appendix contains an English translation by Gyula Klima of Biancini's 1615 tract "A treatise on the nature of mathematics," one of the essays that attempted to respond to the Renaissance critics of mathematics.[2]

The remainder and majority of Mancosu's book is a collection of studies of major episodes in seventeenth-century mathematics. In Chapter 3 he examines Bonaventura Cavalieri's method of indivisibles and Paul Guldin's barycentric calculus, both works of the 1630s that dealt with problems that would later be handled by the integral calculus. Cavalieri's theory and Guldin's criticisms of it illustrate some of the technical and conceptual problems associated with the early development of the calculus. Guldin's work is interesting as an attempt to demonstrate some standard results on quadrature without using reductio ad absurdum, the basic logical form of the classical method of exhaustion. Chapter 3 is devoted to a study of René Descartes's analytic geometry, a subject that has received a great deal of historical attention, but is surveyed here with several fresh insights. Among other things Mancosu shows that Descartes's conception of geometrical construction was strongly influenced by the Jesuit mathematician Christoph Clavius. In Chapter 5, titled "Paradoxes of the infinite," Mancosu returns to the method of indivisibles as he examines the work in 1643 of Evangelista Torricelli on solids of revolution. Torricelli discovered the remarkable result that the solid of revolution obtained by rotating the hyperbola $y = 1/x$ about the x-axis has a finite volume and infinite surface area. This result "stretched the intuitive universe of geometric figures," and raised significant questions about the ontological status of mathematical objects. In the final chapter Mancosu examines Gottfried Wilhelm Leibniz's invention of the calculus and some of the contemporary controversies it gave rise to. It is worth noting that the latter considerably predated George Berkeley's famous assault of 1734.[3]

On the whole Mancosu has produced a satisfying and interesting book, one that explicates philosophical issues while remaining in contact with technical mathematical developments. Beyond the ostensible themes developed in the book there are numerous strands in the narrative that taken together point to profound differences between Greek and seventeenth-century mathematics. Points of contrast are found in the early modern understanding of mathematical method, in an expanded concept of generality, and in the role assigned to algebra and the analytic art in mathematical investigation.

Perhaps the single greatest innovation of seventeenth-century researchers was the realization that mathematical understanding and demonstration must reflect the way in which results are discovered or constructed. This insight was expressed very clearly in the development of algebraic procedures for solving polynomial equations, procedures that indicated how the result was obtained and why it was true. These techniques replaced the purely synthetic propositions of Book II of the *Elements* (e.g., II, 11), where the answer was simply announced at the outset and then verified in a series of deductive steps. Early modern researchers correctly realized that the verificationist approach to mathematical exposition followed by Euclid had fundamental drawbacks.

The new insight into mathematical method was evident in remarks Descartes made critical of proof-by-contradiction, although the point in question transcends the particular issues raised in this instance. The background here was an earlier discussion involving most prominently Jacopo Zabarella concerning the methods of analysis (resolution) and synthesis (composition) in rational investigation. Mancosu observes, "Descartes claims that analytic methods, by showing how a result is obtained, also show why the result holds, and therefore analysis deserves to be considered as the paradigmatic form of a priori proof" (p. 84). An explicit statement of Descartes's position is contained in his reply to the objections made by Arnauld to the second mediation where he writes:

> Analysis reveals the true way in which something is methodically discovered and shows how effects depend on causes . . . . Synthesis by contrast demonstrates, in a quite different way by examining causes in terms of their effects (although the proof that it produces is often from causes to effects), the truth that is contained in its conclusions, and provides a long sequence of definitions, suppositions, axioms, theorems and problems, so that if one denies some of the consequences to which it leads, one is shown how they are contained in the antecedents, and the consent of the reader is compelled, however obstinate and opinionated he may be; but it does not give, as does the former, complete satisfaction to the minds of those who desire to learn, since it doesn't teach the method by which the thing was discovered.[4]

An appreciation of Descartes's insight has been somewhat obscured today by the influence of the deductive axiomatic method in modern mathematical philosophy. However, during the seventeenth century astute thinkers recognized the full significance of the point in question. Descartes's conception of analysis as consisting of demonstrations a priori was developed into an explicit epistemological theory by Arnauld in his celebrated *La Logique ou l'Art De Penser* (1762) written with Pierre Nicole. (Their work is better known as the *Port-Royal Logic.)* Although he admired the clarity and simple principles used by geometers, Arnauld was critical of geometry's reliance on synthesis for what he regarded as its failure to spread enlightenment. He wrote concerning geometry:

> *FIRST DEFECT. Paying more attention to certainty than to evidence, and to the conviction of the mind than to its enlightenment.*
> The geometers are worthy of all praise in seeking to advance only what is convincing: but it would appear that they have not sufficiently observed, that it does

> not suffice for the establishment of a perfect knowledge of any truth to be convinced that it is true, unless beyond this, we penetrate into the reasons, derived from the nature of the thing itself, why it is true. For until we arrive at this point, our mind is not fully satisfied, and still seeks greater knowledge than this, which marks that it has not yet true knowledge.[5]

Mancosu emphasizes the importance of Aristotle's conception of science for both Arnauld's theory of demonstration and the work of subsequent philosophers in this area up to the nineteenth century.[6]

On the plane of mathematical practice the new understanding was reflected in the emphasis placed on problems rather than theorems in Descartes's *Géométrie*. The guiding thematic in this work was provided by Pappus's locus problem and its generalizations, and the traditional definition-postulate-proposition style of deductive treatises in the exact sciences was conspicuously absent. The subsequent development of the calculus was based on the realization that powerful heuristic procedures lay at the foundation of this emergent subject. This point is clear when one considers the tangent methods developed during the period. Isaac Barrow was a tradionalist, "between ancients and moderns," who in his *Lectiones Geometricae* (1670) derived several elegant and beautiful theorems concerning the tangents to various curves.[7] In each case the result was stated and then proved using the skillful employment of properties of the curves in question in a virtuoso display of deductive inventiveness. By contrast the Fermat tangent method involving infinitesimals provided a procedure based on direct understanding of the problem that could be applied by anyone to any problem without any special knowledge or geometrical sophistication.[8] The great power of this method was indicated by its successful and immediate application to a broad range of curves, both geometrical (represented in terms of polynomials) and mechanical (represented in terms of transcendental expressions).

Another feature of modern mathematics that emerged in the seventeenth century is the importance attached to generality. Mancosu's account of Cavalieri's *Geometria Indivisibilibus Continuorum* (1635) illustrates this development very well. Cavalieri's method of indivisibles provided a striking approach to various problems of quadrature and cubature and paved the way for the development of the integral calculus later in the century. An interesting example of his general approach is provided by his solution of the very first problem. The problem is a geometrical version of what is today called the mean-value theorem in the calculus. Given any curve ABC and straight line BC (where B and C are arbitrary points on the curve) one has to find the vertex of the curve with respect to BC, that is, the point A at which the tangent is parallel to BC. Cavalieri imagines that a line parallel to BC moves upward until it reaches a position KV outside of curve. At some point this moving line will touch the curve, defining the vertex A. Mancosu observes:

> What is striking about this result is the extreme generality of the statement. It is not a question of finding a tangent to a specific curve, say a cissoid or a parabola, as would have been the case with a Greek geometrical work. Rather, we are aiming at a general solution of the problem. But the increase in generality creates a loss in constructivity. In other words, the theorem functions more as an existence theorem than as a constructive rule for drawing a tangent to an arbitrary curve . . . . Guldin's objections [to Cavalieri's result] highlight an important aspect of the characteristic trend of seventeenth-century geometry—the passage from special constructions to general theorems. (p. 53)

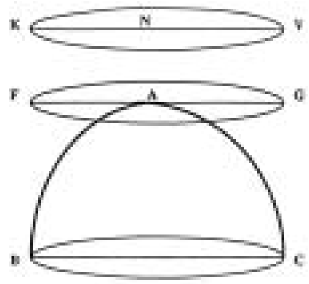

Figure 1: from Mancosu, p. 52

Cavalieri's propositions about "all the lines" or "all the squares" of a figure could be interpreted in different ways in different problems, for example, as giving the area under a curve or the volume of a solid of revolution. Although the method of indivisibles was a general and innovative heuristic tool it possessed several drawbacks that limited its potential for further mathematical development. The first was the concept of indivisible itself. As his older contemporary Guldin showed, it is possible to match in a one-to-one fashion the indivisibles of figures of different areas. In Figure 2 consider the triangles HAD and HGD. A given parallel IL to the base ADG defines the matched vertical lines IC and LE in the two triangles. There is then a correspondence to the indivisibles IC of triangle HAD and the indivisibles LE of triangle HGD. Hence "all the indivisibles of HAD" is equal to "all the indivisibles of HGD" although the two triangles evidently possess different areas. Cavalieri responded to this difficulty in various ad hoc ways, failing to realize that the problem lies with the very notion of an indivisible; instead one needs the concept of an infinitesimal, possessing in a well-defined sense a definite width, which can be connected analytically to the coordinates describing the figure.[9]

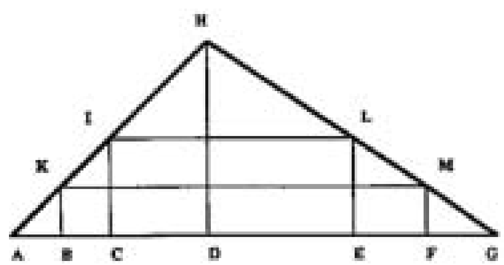

Figure 2: from Mancosu, p. 123

A larger issue concerned Cavalieri's exclusively geometrical outlook. Mancosu observes that "A very important problem . . . is the relationship between geometry and algebra, and the advantages and disadvantages of the algebraic and geometric approach in mathematical practice" (p. 227). It turned out that an effective method for determining quadratures must be formulated algebraically, with algorithms and uniform procedures for generating results. This insight provided the key to the work of Isaac Newton and Leibniz, both young researchers at the time they developed their respective versions of the calculus. Even in their work the new analysis remained firmly rooted in geometrical conceptions; the calculus was viewed not so much as an independent subject as an important tool in the investigation of "fine" geometry.

An indication of the conceptual character of the early calculus is provided by Leibniz's approach in his very first paper on the subject in 1684. He began with the analytical equation of a curve and showed how his differential algorithm could be applied to obtain the tangent to the curve at any point. Given any problem which depends on the analysis of some relationship between two quantities, Leibniz modeled this relation by a curve in Cartesian coordinates, so that his differential method could be introduced into the solution . He had evidently achieved a quite different kind of generality than Cavalieri, one that moved beyond the realm of geometric curves to include in principle any kind of relational dependence, although the resulting solution was still elaborated in a geometrical context.[10]

The mathematical events of the seventeenth century, so richly documented in Mancosu's book, were truly remarkable and there is consequently a danger of losing perspective on the overall epistemological setting within which mathematics developed during this period. Modern mathematics unfolded in several stages and the period 1500–1700 for all its revolutionary character was only one stage in this history. It would be wrong, as some authors have done, to read into any single achievement of the period a kind of overarching modernity, be it François Viète's analytic art, Descartes's putative notion of construction, or Leibniz's writings on formal logic. Mathematics and physics were viewed as related subjects that were part of the same intellectual domain; the calculus investigated the fine structure of curves and the function concept was barely adumbrated; the arithmetization of analysis lay in the distant future; subjects such as the theory of numbers and probability theory were in a very preliminary state; and the critical and logical underpinnings of mathematics remained largely unexplored. Most fundamentally, the early modern mathematical outlook lacked the sense of logical freedom that is basic in today's mathematics. A biographer of the Enlightenment, mathematician Jean d'Alembert, has noted his insistence on "the elementary truth that the scientist must always accept the essential 'giveness' of the situation in which he finds himself."[11] Each mathematical problem or theorem was understood as something that was given implicitly from without; there was no conception at the level of actual mathematical practice of a purely internal, self-generated theory. Although there were occasional anticipations the next great epistemological leap in the history of mathematics would occur in the nineteenth century. The deep intellectual transformation that took place after 1800 is succinctly expressed in Dedekind's famous comment of 1888 that "Numbers are free creations of the human mind," a statement which presupposes a spirit of logical freedom that would have been foreign to seventeenth- or eighteenth-century masters of the subject. The development of modern mathematics in this sense is rightly associated with major philosophical thinkers such as Bernhard Bolzano, Gottlob Frege, and Dedekind himself, but it may also be found in the mathematical practice of a range of working mathematicians. In his study of philosophy and mathematical practice in the seventeenth century, Mancosu has made a substantial contribution to our understanding of the Scientific Revolution and provided a model for how to investigate the interaction of these subjects in later periods.

NOTES

1. Biancini (1566–1624) is an obscure figure not listed in any of the standard reference sources in history and philosophy of science. He was a Jesuit who taught mathematics in Parma and was critical of Galileo's telescopic discoveries in astronomy. See also the next note.

2. In the biography Drake [1] writes (p. 441): "In 1615 he [Biancini] published a book on the mathematical passages in Aristotle that is still of value."

3. To explain why the method of fluxions produced the tangent to the curve Berkeley introduced the doctrine of compensation of errors. This explanation was quite noncausal in the Aristotelian sense and missed the point entirely. The reason the calculus produces the tangent is because the preliminary expression it yields for the slope approximates in some definite sense the actual value slope of the curve. Berkeley's critique seems to have limited intrinsic merit. Perhaps its greatest value was to have stimulated other mathematicians; for example, Jean d'Alembert in his article "Différentiel" in the *Encyclopédie* (1754) provided a lucid discussion of the point in terms of the notion of limit. The idea of compensation of errors was resurrected by Lazare Carnot in an unmemorable prize-competition essay of 1786 on the foundations of the calculus. See Youschkevitch [10].

4. Translated by the reviewer from the French passage on p. 226 of Mancosu's book. Mancosu notes that the "Latin uses *a priori* and *a posteriori* instead of 'from causes to effects' and 'from effects to causes'."

5. Mancuso, pp. 100–101.

6. Both Descartes and Arnauld disliked proof-by-contradiction, because they believed the deductive form of a proposition should reflect how the thing is discovered or derived. Consider Euclid (X11, 2) where *reductio ad absurdum* is used in the method of exhaution to establish the area result for circles, that is, the areas of two circles are as the squares on their diameters. (Euclid's proof is given in many books on the history of mathematics, for example, Katz ([7], p. 92). Why does this result hold? Because it is true for similar regular polygons, and becuase such polygons fill out the circle, that is, the circle is the limit of inscribed polygons as the number of sides increases to infinity. By recasting the proof in direct form we are led to the central concept of limit: the area of the circle is the limit of the area of the inscribed polygons. If this idea is expressed in terms of Leibniz's calculus we are led directly to the integral formula in polar coordinates for the area under a circle.

7. See Mahoney [8]. A good account of Barrow's results on tangents is contained in Fancy [2].

8. See Jensen [6].

9. Cavalieri regarded "all the lines" of a given figure as a magnitude, and such magnitudes were assumed to satisfy the rules of Eudoxan proportion theory. There is the implied suggestion in some of the histroical literature that Cavalieri was dealing with a kind of weighting that may be interpreted in a modern abstract measure-theoretic sense as a measure defined on plane figures. Guldin's counterexample indicates clearly that such an interpretation is in error.

10. For further discussion of these points see Fraser [4].

11. Grimsley [5]. For studies of the older eighteenth-century understanding of mathematics see Fraser [4], Panza [9], and Ferraro [3].

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Craig Fraser
Institute for History and Philosophy of Science and Technology
University of Toronto
Toronto ON M5S 1K7
CANADA
email: cfraser@chass.utoronto.ca